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TWO CLASSES OF MEASURES

BY

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This paper* is concerned with two classes of measures: perfect measures ($=$ quasi-compact measures of Ryll-Nardzewski [13]) and semicompact measures ($=$ compact measures of Marczewski [7]).

These two classes are related to two constructions in probability theory: (indirect) product and disintegration. As long as we insist that probabilities be countably additive, it is inconvenient to allow arbitrary countably additive measures as probabilities; the reason is that the set functions produced as products and disintegrations of countably additive measures need not be countably additive, while intuitively they should be probabilities. The results surveyed in this paper suggest that the most natural property to start with is perfectness when constructing products, and semicompactness when constructing disintegrations.

Our characterization of semicompact measures in terms of disintegrations allows us to show that these measures have reasonable stability properties.

1. Preliminaries. A *measurable space* is a pair $(X, \mathscr{A})$, where X is a nonempty set and $\mathscr{A}$ is a σ-algebra of subsets of X; if μ is a *measure* (that is, a nonnegative finite countably additive set function) on $\mathscr{A}$, then we say that $(X, \mathscr{A}, \mu)$ is a *measure space*. When $\mathscr{L}$ is a class of subsets of a fixed set, denote by $a(\mathscr{L})$ and $\sigma(\mathscr{L})$ the algebra and the σ-algebra generated by $\mathscr{L}$. A *lattice* on X is a class of subsets of X that contains $\emptyset$ and X and is closed under finite unions and finite intersections.

A class $\mathscr{K}$ of sets is *semicompact* if every countable class $\mathscr{K}_0 \subset \mathscr{K}$ such that $\bigcap \mathscr{K}_0 = \emptyset$ contains a finite class $\mathscr{K}_{00} \subset \mathscr{K}_0$ such that $\bigcap \mathscr{K}_{00} = \emptyset$.

A "regular" measure will mean "inner regular": If $(X, \mathscr{A}, \mu)$ is a measure space and $\mathscr{K} \subset \mathscr{A}$, then μ is *regular* with respect to $\mathscr{K}$, or *$\mathscr{K}$-regular*, when

$$\mu E = \sup\{\mu K \mid E \supset K \in \mathscr{K}\} \quad \text{for each } E \in \mathscr{A}.$$

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A measure is called *semicompact* when it is regular with respect to a semicompact lattice. Semicompact measures are called *compact* by Marczewski [7] (his definition is slightly different, but equivalent in view of 4 (iv) in [7]).

For a real-valued function β on a class $\mathscr{K}$ of subsets of X, the *inner measure* β_* is defined by

$$\beta_* E = \sup\{\beta K \mid E \supset K \in \mathscr{K}\} \quad \text{for } E \subset X.$$

If $(X, \mathscr{A}, \mu)$ is a measure space and $Z \subset X$, then we denote by $\mathscr{A}|Z$ the σ-algebra $\{E \cap Z \mid E \in \mathscr{A}\}$; the measure $\mu|Z$ on $\mathscr{A}|Z$ is defined by

$$(\mu|Z)G = \inf\{\mu E \mid E \in \mathscr{A} \text{ and } G = E \cap Z\}.$$

The set Z is called *μ-thick* when $\mu_*(X \setminus Z) = 0$; an equivalent condition is that $i[\mu|Z] = \mu$, where $i: Z \to X$ is the inclusion map.

Given two measure spaces $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$, we denote by $\mathscr{A} \otimes \mathscr{B}$ the set of "rectangles" $E \times F$ with $E \in \mathscr{A}$ and $F \in \mathscr{B}$. Thus $a(\mathscr{A} \otimes \mathscr{B})$ is the product algebra on $X \times Y$, i.e. the smallest algebra on $X \times Y$ making both projections $\mathrm{pr}_X: X \times Y \to X$ and $\mathrm{pr}_Y: X \times Y \to Y$ measurable. A nonnegative finite finitely additive set function λ on $a(\mathscr{A} \otimes \mathscr{B})$ is a *subproduct* of μ and ν when $\mathrm{pr}_X[\lambda] \leqslant \mu$ and $\mathrm{pr}_Y[\lambda] \leqslant \nu$. A subproduct λ of μ and ν is a *product* of μ and ν when $\mathrm{pr}_X[\lambda] = \mu$ and $\mathrm{pr}_Y[\lambda] = \nu$. (In Section 3 we adapt this notion to infinite products: a product of infinitely many measures is a nonnegative finite finitely additive set function on the product algebra whose projections coincide with the given measures.)

PROPOSITION 1 ([8], 1 (i)). *If μ is semicompact and ν is an arbitrary measure, then every product of μ and ν is countably additive.*

From this result it is easy to deduce that even every subproduct is then countably additive.

LEMMA. *Let $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be two measure spaces, μ being given by the sum of a sequence of measures μ_n; that is*

$$\mu E = \sum_{n=1}^{\infty} \mu_n E \quad \textit{for each } E \in \mathscr{A}.$$

If λ is a countably additive subproduct of μ and ν, then for every n there is a countably additive subproduct λ_n of μ_n and ν such that

$$\lambda G = \sum_{n=1}^{\infty} \lambda_n G \quad \textit{for each } G \in a(\mathscr{A} \otimes \mathscr{B}).$$

Proof. Each μ_n is μ-absolutely continuous; take an $\mathscr{A}$-measurable function h_n with $\mu_n = \int h_n d\mu$. Put $\lambda_n = \int (h_n \cdot \mathrm{pr}_X) d\lambda$.

2. Disintegration and semicompactness. Let $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be two measure spaces and let λ be a subproduct of μ and ν. A *ν-disintegration* of λ is a family $\{(\mathscr{A}_y, \mu_y)\}_{y\in Y}$ such that:

(a) $\mathscr{A}_y$ is a σ-algebra on X and μ_y is a measure on $\mathscr{A}_y$ with $\mu_y X \leqslant 1$ for each $y \in Y$;

(b) for each $E \in \mathscr{A}$ there is a set $Q \in \mathscr{B}$ such that $\nu Q = 0$, $E \in \mathscr{A}_y$ for all $y \in Y\setminus Q$ and the function $y \to \mu_y E$, $y \in Y\setminus Q$, is $(\mathscr{B}\,|\,Y\setminus Q)$-measurable;

(c) if $E \in \mathscr{A}$ and $F \in \mathscr{B}$, then

$$\int_F \mu_y E d\nu(y) = \lambda(E\times F)$$

(in view of (b) the integral is well defined).

Note that if there exists a ν-disintegration of λ, then λ is countably additive.

THEOREM 1. *A measure space $(X, \mathscr{A}, \mu)$ is semicompact if and only if for every complete measure space $(Y, \mathscr{B}, \nu)$ and for every countably additive subproduct λ of μ and ν there exists a ν-disintegration of λ.*

Proof. "If" follows from [12], 2.2. To prove the "only if" part, we repeat the proof of 3.5 in [12]:

Choose a lifting ϱ on $(Y, \mathscr{B}, \nu)$. By the Radon-Nikodým theorem, for each $E \in \mathscr{A}$ there is a bounded $\mathscr{B}$-measurable function h_E such that

$$\int_F h_E d\nu = \lambda(E\times F) \qquad \text{for every } F \in \mathscr{B}.$$

Take a semicompact lattice $\mathscr{K}$ such that μ is $\mathscr{K}$-regular and for every $y \in Y$ define a function β_y on $\mathscr{K}$ by $\beta_y K = \varrho h_K(y)$, $K \in \mathscr{K}$.

From the properties of lifting it follows that β_y is monotone and modular and $\beta_y X \leqslant 1$. Apply 3.4 in [12] to obtain a monotone modular function γ_y on $\mathscr{K}$ such that $\gamma_y \geqslant \beta_y$, $\gamma_y X = \beta_y X$ and

$$\gamma_y K + (\gamma_y)_*(X\setminus K) = \gamma_y X \qquad \text{for every } K \in \mathscr{K}.$$

Define $\mathscr{A}_y$ as the class of the sets $E \subset X$ satisfying

$$(\gamma_y)_* E + (\gamma_y)_*(X\setminus E) = \gamma_y X$$

and for $E \in \mathscr{A}_y$ put $\mu_y E = (\gamma_y)_* E$.

COROLLARIES. 1. *The restriction of a semicompact measure to a sub-σ-algebra is semicompact* (cf. [12]).

2. *Let $(X, \mathscr{A}, \mu)$ be a measure space and suppose that there exists a μ-thick set $Z \subset X$ such that $\mu\,|\,Z$ is semicompact. Then μ is semicompact* (cf. [11]).

3. *Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of semicompact measures on a measurable space $(X, \mathscr{A})$ with*

$$\sum_{n=1}^{\infty} \mu_n X < \infty.$$

Then the measure $\mu = \sum \mu_n$ is semicompact.

4. *Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of semicompact measures. Then there is a semicompact lattice $\mathscr{K}$ such that each μ_n is $\mathscr{K}$-regular.*

5. *If M is a weakly compact set of semicompact measures, then there is a semicompact lattice $\mathscr{K}$ such that each $\mu \in M$ is $\mathscr{K}$-regular.*

Proof. 1. See [12], 4.1.

2. Apply Corollary 1 to the σ-algebra $\sigma(\mathscr{A} \cup \{Z\})$ and its sub-σ-algebra $\mathscr{A}$.

3. This follows from Theorem 1 and the Lemma if we write $\lambda = \sum \lambda_n$ as in the Lemma, take a ν-disintegration of each λ_n and sum these to get a ν-disintegration of λ.

4. Assume that $\mu_n X > 0$ for each n and apply Corollary 3 to $\mu_n' = (1/2^n \mu_n X)\mu_n$.

5. If M is weakly compact, then ([2], IV.9) there is a sequence $\{\mu_n\}_{n=1}^{\infty}$ in M and real numbers τ_n such that

$$\sum_{n=1}^{\infty} \tau_n \mu_n X < \infty$$

and every $\mu \in M$ is absolutely continuous with respect to $\sum_{n=1}^{\infty} \tau_n \mu_n$. Thus the result follows from Corollary 3.

THEOREM 2. *Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be two measurable spaces and let $f\colon X \to Y$ be a measurable map. If μ is a semicompact measure on $\mathscr{A}$, then the image measure $f[\mu]$ is semicompact.*

Proof. Combine Corollaries 1 and 2.

PROPOSITION 2. *Every measure μ can be (uniquely) written as $\mu = \mu_s + \mu_0$, where μ_s is a semicompact measure and μ_0 dominates no nonzero semicompact measure.*

Proof is standard (cf. [2], III.7.8). Put namely

$$\tau = \sup\{\tilde{\mu}X \mid \tilde{\mu} \leqslant \mu \text{ and } \tilde{\mu} \text{ is semicompact}\}$$

and choose a sequence of semicompact measures $\mu_n \leqslant \mu$ such that

$$\lim_n \mu_n X = \tau.$$

If $\tilde{\mu}$ and $\tilde{\tilde{\mu}}$ are two semicompact measures, then $\tilde{\mu} \vee \tilde{\tilde{\mu}} \leqslant \tilde{\mu} + \tilde{\tilde{\mu}}$ is semicompact by Corollary 3. Thus we may assume that $\mu_1 \leqslant \mu_2 \leqslant \ldots$; this being the case, $\mu_s = \lim_n \mu_n$ has the desired properties.

Two basic examples of semicompact measures are a Radon Borel measure (or its completion) and a 2-valued (or, more generally, atomic) measure. Other semicompact measures can be produced by passing to a measurable image (Theorem 2) or product [7]. Notice that, in particular, the restriction of a Radon Borel measure to the Baire σ-algebra is semicompact.

The results in Corollaries 4 and 5 suggest the following question: does there exist a set of semicompact measures which are not all regular with respect to a common semicompact lattice? (**P 1147**)

We prove in Theorem 4 below that if $\mathscr{A}$ is the Borel σ-algebra in a metric $2D$-space, then all semicompact measures on $\mathscr{A}$ are regular with respect to a single lattice, namely the lattice of compact sets. This phenomenon is caused by the peculiar structure of the Borel σ-algebra in a metric space, and can hardly be expected to hold true in general. However, no counterexample is known to the author.

3. Perfect measures. Recall that a measure is called *perfect* if it has one of the equivalent properties in the following theorem:

THEOREM 3. *The following properties of a measure space* $(X, \mathscr{A}, \mu)$ *are equivalent*:

(a) *if* $f: X \to \boldsymbol{R}$ *is a Borel-measurable real-valued function and* $H \subset \boldsymbol{R}$, $f^{-1}H \in \mathscr{A}$, *then there is a Borel set* $B \subset \boldsymbol{R}$ *such that* $B \subset H$ *and* $\mu(f^{-1}H) = \mu(f^{-1}B)$;

(b) *if* $f: X \to \boldsymbol{R}$ *is a Borel-measurable real-valued function, then there is a Borel set* $B \subset \boldsymbol{R}$ *such that* $B \subset fX$ *and* $\mu(f^{-1}B) = \mu X$;

(c) *if* $\varepsilon > 0$ *and* $E_n \in \mathscr{A}$, $n = 1, 2, \ldots$, *then there is a set* $E \in \mathscr{A}$ *such that* $\mu E \geqslant \mu X - \varepsilon$ *and the class* $\{E_n \cap E \mid n = 1, 2, \ldots\}$ *is semicompact*;

(d) *the restriction of* μ *to any countably generated sub-*σ*-algebra of* $\mathscr{A}$ *is semicompact*;

(e) *if* $f: X \to \mathscr{S}$ *is a Borel-measurable map into a separable metric space* $\mathscr{S}$, *then the image measure* $f[\mu]$ *is Radon* (*i.e. regular with respect to the class of compact subsets of* $\mathscr{S}$).

Proof and further properties of perfect measures may be found in [13] and [14]. Here we only recall two results:

1. *Every semicompact measure is perfect* ([13], Theorem II).

There exists a perfect measure that is not semicompact. The first example is in [17]; a more detailed account is given in [10].

2. *Any product of an arbitrary family of perfect measures is countably additive and its countably additive extension to the product* σ*-algebra is perfect* ([13], Theorem VIII).

This together with the forthcoming Proposition 3 suggests that perfect measures provide a natural environment for countably additive products.

The next proposition makes explicit two more characterizations of perfectness which are implicitly contained in [8].

PROPOSITION 3. *Let $(X, \mathscr{A}, \mu)$ be a measure space. The following properties are equivalent*:

(a) *μ is perfect*;

(b) *any product of μ with any measure is countably additive*;

(c) *if $(Y, \mathscr{B}, \nu)$ is a measure space and λ is a measure on $\sigma(\mathscr{A}\otimes\mathscr{B})$ such that* $\mathrm{pr}_X[\lambda] = \mu$ *and* $\mathrm{pr}_Y[\lambda] = \nu$ *(that is, the restriction of λ to $a(\mathscr{A}\otimes\mathscr{B})$ is a product of μ and ν), then*

$$\lambda_*(X\times F) = \nu_* F \quad \textit{for every set } F\subset Y.$$

Proof. By Proposition 1, (b) holds for every semicompact measure μ. However, it is clear that μ satisfies (b) whenever the restriction of μ to every countably generated sub-σ-algebra of $\mathscr{A}$ satisfies (b). Thus Proposition 1 and condition (d) in Theorem 3 show that (a) implies (b).

We are going to show that non(c) implies non(b). Assume that there s a space $(Y, \mathscr{B}, \nu)$ and a measure λ on $\sigma(\mathscr{A}\otimes\mathscr{B})$ such that $\mathrm{pr}_X[\lambda] = \mu$, $\mathrm{pr}_Y[\lambda] = \nu$ and $\lambda_*(X\times F_0) > \nu_* F_0$ for some $F_0\subset Y$. Pick an $F_1\in\mathscr{B}$ such that $F_1\subset F_0$ and $\nu F_1 = \nu_* F_0$. Put

$$\tilde{Y} = Y\setminus(F_0\setminus F_1), \quad \tilde{\mathscr{B}} = \mathscr{B}|\tilde{Y} \quad \text{and} \quad \tilde{\nu} = \nu|\tilde{Y}.$$

Define a product $\tilde{\lambda}$ of μ and $\tilde{\nu}$ as follows: when $\tilde{G}\in a(\mathscr{A}\otimes\tilde{\mathscr{B}})$, choose a $G\in a(\mathscr{A}\otimes\mathscr{B})$ such that $\tilde{G} = G\cap(X\times\tilde{Y})$ and put $\tilde{\lambda}\tilde{G} = \lambda G$. Since $\tilde{Y}$ is ν-thick, $\tilde{\lambda}$ is well defined. It remains to be shown that $\tilde{\lambda}$ is not countably additive. The measure λ on $\sigma(\mathscr{A}\otimes\mathscr{B})$ is regular with respect to $\big(a(\mathscr{A}\otimes\mathscr{B})\big)_\delta$. Therefore there is a set $G\in\big(a(\mathscr{A}\otimes\mathscr{B})\big)_\delta$ such that $G\subset X\times F_0$ and $\lambda G > \nu_* F_0$.

Write

$$G\setminus(X\times F_1) = \bigcap_{n=1}^{\infty} G_n, \quad \text{where } G_n\in a(\mathscr{A}\otimes\mathscr{B}) \text{ and } G_1\supset G_2\supset\ldots$$

Put

$$\tilde{G}_n = G_n\cap(X\times\tilde{Y}).$$

We have $\tilde{G}_n\searrow\emptyset$, but

$$\tilde{\lambda}\tilde{G}_n = \lambda G_n \geqslant \lambda G - \lambda(X\times F_1) = \lambda G - \nu_* F_0 > 0.$$

Thus $\tilde{\lambda}$ is not countably additive.

Finally, let us prove that non(a) implies non(c). Assuming that μ is not perfect, we find a Borel measurable function $f\colon X\to \boldsymbol{R}$ such that

$$\sup\{\mu(f^{-1}B)\,|\,B \text{ is a Borel set and } B\subset fX\} < \mu X.$$

Denote by $\mathscr{B}$ the Borel σ-algebra in $\boldsymbol{R}$ and put $\nu F = \mu(f^{-1}F)$ for $F\in\mathscr{B}$.

Define a measure λ on $\sigma(\mathscr{A}\otimes\mathscr{B})$ by

$$\lambda G = \mu\{x\in X \mid (x, f(x))\in G\} \quad \text{for } G\in\sigma(\mathscr{A}\otimes\mathscr{B}).$$

Obviously, $\mathrm{pr}_X[\lambda] = \mu$, $\mathrm{pr}_Y[\lambda] = \nu$ and

$$\nu_*(fX) = \sup\{\mu(f^{-1}B) \mid B \text{ is Borel and } B\subset fX\}.$$

On the other hand,

$$\lambda_*(X\times fX) = \mu X,$$

because $\{(x, f(x)) \mid x\in X\}\in\sigma(\mathscr{A}\otimes\mathscr{B})$ (see [1], 2.1), and

$$\lambda\{(x, f(x)) \mid x\in X\} = \mu X.$$

The proof is complete.

The following example shows that property (b) in Proposition 3 cannot be replaced by the weaker property "any product of μ with μ is countably additive". In fact, we construct a measure μ such that every product of an arbitrary family of measures which are isomorphic with μ is countably additive, and yet μ is not perfect.

Example. Denote by $\mathscr{B}$ the Borel σ-algebra in $[0,1]$ and by ν the Lebesgue measure on $\mathscr{B}$. Let Ω be the first ordinal of cardinality $2^{\aleph_0}$, and let $D_1, D_2, \ldots, D_\eta, \ldots$, $\eta<\Omega$, be all the Borel subsets of $[0,1]^N$ whose images under the canonical projections $[0,1]^N\to[0,1]$ are uncountable (these images are analytic sets, hence they are either countable or of cardinality $2^{\aleph_0}$ ([6], §39.I)). Similarly, let $F_1, F_2, \ldots, F_\eta, \ldots, \eta<\Omega$, be all uncountable Borel subsets of $[0,1]$. Choose $v_1\in F_1$, $w_1\in F_1$ and $y_1^{(n)}\in[0,1]$, $n = 1, 2, \ldots$, such that $w_1\neq v_1$, $\{y_1^{(n)}\}_{n\in N}\in D_1$ and $y_1^{(n)}\neq v_1$ for $n = 1, 2, \ldots$

By transfinite induction construct v_η, w_η and $y_\eta^{(n)}$, $n = 1, 2, \ldots$, $\eta<\Omega$, such that for each $\eta<\Omega$ we have

$$v_\eta, w_\eta\in F_\eta, \qquad \{y_\eta^{(n)}\}_{n\in N}\in D_\eta,$$

$$v_\eta\notin\{w_\xi \mid \xi<\eta\}\cup\{y_\xi^{(n)} \mid \xi<\eta, n = 1, 2, \ldots\},$$

and

$$y_\eta^{(n)}, w_\eta\notin\{v_\xi \mid \xi\leqslant\eta\}.$$

Put

$$X = \{w_\eta \mid \eta<\Omega\}\cup\{y_\eta^{(n)} \mid \eta<\Omega, n = 1, 2, \ldots\},$$

$$\mathscr{A} = \mathscr{B}|X, \quad \mu = \nu|X.$$

The set X intersects every uncountable Borel subset of $[0,1]$, hence X is ν-thick. For the same reason the set $[0,1]\setminus X\supset\{v_\eta \mid \eta<\Omega\}$ is ν-thick, hence X is not ν-measurable. Thus μ is not perfect.

In order to prove that an arbitrary product of any (finite or infinite, countable or uncountable) number of copies of μ is countably additive, it is enough to prove that any product of an *infinite countable* family of copies of μ is countably additive.

Thus let $\mathscr{E} = a(\otimes\mathscr{B})$ be the product algebra in $[0,1]^N$ and $\tilde{\mathscr{E}} = a(\otimes\mathscr{A})$ the product algebra in X^N. Obviously $\tilde{\mathscr{E}} = \mathscr{E}|X^N$. Let $\tilde{\lambda}$ on $\tilde{\mathscr{E}}$ be a product of μ's. Put $\lambda E = \tilde{\lambda}(E \cap X^N)$ for $E \in \mathscr{E}$. This λ is a product of ν's, hence countably additive by the result ([13], Theorem VIII) mentioned above. Suppose that $\tilde{E}_n \in \tilde{\mathscr{E}}$, $n = 1, 2, \ldots$, $\tilde{E}_1 \supset \tilde{E}_2 \supset \ldots$ and $\lim_n \tilde{\lambda}\tilde{E}_n > 0$. Find sets $E_n \in \mathscr{E}$ such that $\tilde{E}_n = E_n \cap X^N$ and $E_1 \supset E_2 \supset \ldots$ We have

$$\lim_n \lambda E_n = \lim_n \tilde{\lambda}\tilde{E}_n > 0$$

and λ is countably additive; this implies that all the canonical projections in $[0,1]$ of the Borel set $\bigcap_n E_n$ are uncountable. Thus $\bigcap_n E_n = D_\eta$ for some $\eta < \Omega$ and

$$\{y_\eta^{(n)}\}_{n\in N} \in \bigcap_n E_n \cap X^N = \bigcap_n \tilde{E}_n.$$

We conclude that $\tilde{\lambda}$ is countably additive.

4. Measures in metric spaces. In this section we extend Sazonov's theorem ([14], Corollary 2 to Theorem 11) to a large class of metric spaces, called $2D$-spaces below. The assumption that every metric space is a $2D$-space is consistent with the usual axioms of set theory.

When Z is a set, we denote by $\exp Z$ the σ-algebra of all subsets of Z; a measure μ on $\exp Z$ is called *nontrivial* if $\mu(\{z\}) = 0$ for each $z \in Z$ and $\mu Z = 1$. A cardinal m is *real-measurable* if for a set Z of cardinality m there exists a nontrivial measure on the σ-algebra $\exp Z$; if, in addition, μ assumes only two values (0 and 1), then m is called *2-measurable*. Thus every 2-measurable cardinal is real-measurable. (Sometimes, non-real-measurable cardinals are called *cardinals of measure zero* [5] and 2-measurable cardinals just *measurable* [3].) A detailed discussion of the two notions may be found in [15].

A metric space X will be called a *D-space* [5] (resp. a *2D-space*) if no closed discrete subspace of X has real-measurable (resp. 2-measurable) cardinal.

The forthcoming Theorem 4 strengthens Corollary 2 to Theorem 11 in [14]; to prove it, we combine Sazonov's method with a trick due to Ulam [16].

PROPOSITION 4. *If the cardinal of a set Z is not 2-measurable, then there is no nontrivial perfect measure on* $\exp Z$.

Proof. Let μ be a nontrivial measure on $\exp Z$. Since $\operatorname{card} Z$ is not 2-measurable, the atomic part of μ is zero. Thus μ is nonatomic and we can find a map $f\colon Z \to [0,1]$ such that the image measure $f[\mu]$ on $\exp[0,1]$ extends the Lebesgue measure. Now pick an $H \subset [0,1]$ with inner Lebesgue measure zero and outer Lebesgue measure one. We cannot have simultaneously $\mu(f^{-1}H) = 0$ and $\mu\big(f^{-1}([0,1]\setminus H)\big) = 0$. Hence H or $[0,1]\setminus H$ does not fulfill (a) in Theorem 3. Thus μ is not perfect.

THEOREM 4 ([1]). *Let μ be a measure on the Borel σ-algebra in a metric $2D$-space. The following conditions are equivalent:*

(a) *μ is a Radon measure (i.e. μ is regular with respect to the class of compact sets);*

(b) *μ is semicompact;*

(c) *μ is perfect.*

Proof. The implication (a) $\Rightarrow$ (b) is trivial and (b) $\Rightarrow$ (c) holds always (see p. 335). To prove (c) $\Rightarrow$ (a), proceed as in [14]: use a σ-discrete basis in X and Proposition 4 to show that μ is supported by a separable subset, and apply condition (e) in Theorem 3.

5. Existence of disintegrations and strong measure-compactness. Let X be a metric space and $\mathscr{A}$ its Borel σ-algebra. Assume that $(X, \mathscr{A})$ has the following property:

($*$) If μ is a measure on $\mathscr{A}$, $(Y, \mathscr{B}, \nu)$ is a complete measure space and λ is a product of μ and ν, then there is a ν-disintegration $\{(\mathscr{A}_y, \mu_y)\}_{y\in Y}$ of λ such that $\mathscr{A}_y \supset \mathscr{A}$ for each $y \in Y$.

Does it follow that every measure on $\mathscr{A}$ is Radon? (The spaces in which every Baire measure is Radon are called *strongly measure-compact* [9].) Goldman ([4], Theorem 4.7) shows that this is the case where X is separable. The answer for general (metric) X depends on the existence of measurable cardinals in the following way:

PROPOSITION 5. (a) *If X is a metric $2D$-space satisfying ($*$), then every measure on $\mathscr{A}$ is Radon.*

(b) *If the cardinal $2^{\aleph_0}$ is real-measurable and X is an arbitrary metric space satisfying ($*$), then every measure on $\mathscr{A}$ is Radon.*

Proof. (a) follows from Theorems 1 and 4.

(b) follows from (a) and from the following fact: if $2^{\aleph_0}$ is real-measurable and X satisfies ($*$), then X is a D-space (and hence also $2D$-space). Indeed, assume that X contains a discrete closed subset Z of cardinality $2^{\aleph_0}$ and that there is a nontrivial measure ν on $\exp Z$. Since $2^{\aleph_0}$ is not 2-measurable ([3], 12.5), the measure μ on $\mathscr{A}$ defined by $\mu E = \nu(E\cap Z)$, $E \in \mathscr{A}$, is not semicompact by Proposition 4. Hence, by Theorem 1, X does not satisfy ($*$).

([1]) Note added in proof. This theorem has been obtained also by G. Koumoullis (*On perfect measures*, Notices of the American Mathematical Society 26 (1979), p. A-274). A stronger result has been proved by A. Goldman and M. Talagrand (*Propriétés de Radon-Nikodym pour les cônes positifs de mesures*, to appear).

The cardinal assumptions in Proposition 5 cannot be omitted in view of the following

Remark. Assume that $2^{\aleph_0}$ is not real-measurable and there exists a 2-measurable cardinal. Take a discrete space X of 2-measurable cardinality. There exists a (2-valued) measure in X which is not Radon; nevertheless, X satisfies $(*)$ because any Borel measure in X is atomic.

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