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THE MARTIN COMPACTIFICATION OF THE POLYDISC AT THE BOTTOM OF THE POSITIVE SPECTRUM

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Introduction. The l-polydisc D, considered as a product of l hyperbolic discs D_i, carries a product Riemannian structure and associated Laplace–Beltrami operator L, which is the sum of the hyperbolic Laplacians on each of the unit discs D_i. In this article it is shown that the closure $\overline{D}$ of D in $\mathbf{C}^l$ is the Martin compactification of D relative to the potential theory associated with the operator $L + l\mathrm{Id}$, where $-l$ is the bottom of the positive spectrum of L, i.e. the smallest number λ for which the equation $Lf = \lambda f$ has a positive global solution.

Note that the Martin compactification associated with L remains to be made explicit (the general determination was announced by Ol'shanetskiĭ [12]), even though by the work of Karpelevich [6] (cf. Guivarc'h [4]) all the minimal functions are known and the way in which the corresponding ideal points are attached to D is also known. These minimal functions are all associated with points of the distinguished boundary T of D, with many minimal points associated to one point of T.

For the Martin compactification associated with the bottom of the positive spectrum, the minimal points are in one-to-one correspondence with the points of T and the corresponding minimal functions are the square roots of the products of the Poisson kernels in the individual discs. The rest of the topological boundary consists of non-minimal points.

The polydisc D is a bounded symmetric domain and its closure is also a so-called Satake compactification (in fact the maximal one). Satake [13] defined compactifications of symmetric spaces of non-compact type by embedding them in a suitable projective space and taking the closure. These compactifications were also studied by Moore [11] using ideas of Furstenberg [3].

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It is clear that the methods used in this article carry over to describe the corresponding Martin compactification for the general symmetric space of non-compact type once they are expressed using Lie algebras and root systems. In a subsequent article, the relation will be discussed between the Martin compactification at the bottom of the positive spectrum for a symmetric space of non-compact type and Satake compactifications.

1. Description of the boundary of the polydisc. Let $D = D_1 \times \ldots \times D_l$ denote the l-polydisc, where each D_i is the unit disc. Let $\overline{D}$ denote its closure in $\mathbf{C}^l$. The topological boundary ∂D is the disjoint union of the following sets, where E runs over the proper subsets of $\{1,\ldots,l\}$: for each subset E let $D_E \times T_{E'}$ denote those points $\overline{\mathbf{w}} = (w_1,\ldots,w_l) \in \overline{D}$ such that $|w_i| < 1 \Leftrightarrow i \in E$. Let such a point be denoted by $(\mathbf{w},\mathbf{b})$—i.e. $\mathbf{w} = (w_{i_1},\ldots,w_{i_k})$ if $E = \{i_1 < \ldots < i_k\}$ and $\mathbf{b} = (w_{j_1},\ldots,w_{j_m})$ where the complement E' of E is $\{j_1 < \ldots < j_m\}$ and $l = k + m$.

DEFINITION 1.1. Let E be a proper subset of $\{1,\ldots,n\}$. A sequence $(\mathbf{w}_n)_{n\geq 0}$ of points $\mathbf{w}_n = (w_1(n),\ldots,w_l(n))$ in D will be said to be *E-bounded* if

(1) $i \in E \Rightarrow |w_i(n)| \leq r_i < 1$ for all n;
(2) $j \in E' \Rightarrow |w_j(n)| \to 1$ as $n \to \infty$.

If in addition, $i \in E \Rightarrow |w_i(n)| = 0$ for all n, then the sequence will be said to be *E-canonical*.

It is clear that a boundary point of D is in $D_E \times T_{E'}$ if and only if it is the limit of an E-bounded sequence in D.

2. Potential theory on the polydisc. Let each unit disc be equipped with the Poincaré metric $|ds|^2 = (1-|z|^2)^{-2}|dz|^2$, i.e. each unit disc is to be viewed as a hyperbolic disc. It is very well known (cf. [5]) that the hyperbolic disc can be viewed as $\mathrm{SU}(1,1)/\mathrm{SU}(1)$, where the action of $g = \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix}$ on z is given by $g \cdot z = \frac{\alpha z+\beta}{\overline{\beta} z+\overline{\alpha}}$, and $\mathrm{SU}(1)$ is the subgroup of matrices for which $\beta = 0$. This subgroup is isomorphic to $\mathrm{SO}(2)$ and also acts on the boundary of the unit disc sending $b = e^{i\alpha}$ to $k \cdot b = e^{i(\alpha+\theta)}$ if $k = \begin{bmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{bmatrix}$.

The unit disc may be conformally transformed onto the upper half plane H by the map $z \mapsto w = \gamma(z) = i\frac{1+z}{1-z}$. Conjugating the subgroup $\mathrm{SU}(1,1)$ of $\mathrm{SL}(2,\mathbf{C})$—the 2×2 complex matrices of determinant one—with $g_0 = \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix}$ shows that $\mathrm{SL}(2,\mathbf{R})$ acts on H in the same way as fractional linear transformations. Let $\widetilde{A}$ be the subgroup of $\mathrm{SL}(2,\mathbf{R})$ of matrices $\widetilde{a}_t =$

$\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$, and let $\widetilde{N}$ be the subgroup of $\mathrm{SL}(2,\mathbf{R})$ of matrices $\widetilde{n}_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$. Then if $w = u + iv$, we have $\widetilde{a}_t \cdot w = e^{2t} w$ and $\widetilde{n}_x \cdot w = u + i(v + x)$. The set $\widetilde{S} = \widetilde{N}\widetilde{A}$ is a group as $\widetilde{n}_x \widetilde{a}_t = \widetilde{a}_t \widetilde{a}_{-t} \widetilde{n}_x \widetilde{a}_t = \widetilde{a}_t \widetilde{n}_{e^{-2t}x}$, and $H = \widetilde{S} \cdot i = \widetilde{S} \cdot \gamma(o)$.

Let A and N be the subgroups of $\mathrm{SU}(1,1)$ that correspond to $\widetilde{A}$ and $\widetilde{N}$ under the conjugation by g_0^{-1} with corresponding elements a_t and n_x. Then $S = NA$ is a group and $D = S \cdot o = NA \cdot o$. Note that a function f on D is N-invariant if and only if it is of the form $f(z) = f(na_t \cdot o) = F(v) = F(e^{2t}) = \psi(t)$, $u + iv = w = \gamma(z)$.

Remark 2.1. The conjugation formula $a_{-t} n_x a_t = n_{e^{-2t}x}$ implies that, for any $n \in N$, $a_{-t} n a_t \to e$ as $t \to \infty$.

Note also that $D = KA \cdot o$, $K = \mathrm{SU}(1)$, where the K-component in this polar decomposition is unique if the A-component is a_t, $t > 0$.

The Laplace–Beltrami operator for the hyperbolic disc is $Lf(z) = (1 - |z|^2)^2 \Delta f(z)$, where Δ denotes the euclidean Laplacian. It is well known that there are positive solutions to the equation $Lf = \lambda f$ if and only if $\lambda \geq -1$. This is a consequence of Lemma 4.1 and Theorem 4.3 in the Introduction of [5]. This number is the bottom of the positive spectrum for the hyperbolic disc. Note that by Proposition 1.2 of [8], $L + \mathrm{Id}$ has a Green function.

The Poisson kernel $P_b(z) = \frac{1-|z|^2}{|b-z|^2}$ is harmonic relative to L and its powers P_b^β satisfy $Lf = 4\beta(\beta - 1)f$ (cf. [7]). If $\beta = \alpha + 1/2$, with $\alpha \geq 0$, then P_b^β is a minimal solution of $Lf = 4\beta(\beta - 1)f$ and every minimal solution has this form (cf. Theorem 4.3 in the Introduction of [5] and also [9], Proposition 3.2, for a simple proof based on the asymptotics of the Green function).

As a result, there is a unique positive SU(1)-invariant—i.e. rotation invariant—solution ϕ_0 of the equation $Lf = -f$ such that $\phi_0(o) = 1$. It is

$$\phi_0(z) = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{\frac{1 - |z|^2}{|e^{i\theta} - z|^2}}\, d\theta .$$

PROPOSITION 2.2. *$\phi_0(z) = 1$ implies $z = o$.*

Proof. The Cauchy–Schwarz inequality implies that

$$\frac{1}{2\pi} \int_0^{2\pi} \sqrt{P_{e^{i\theta}}(z)}\, d\theta \leq \left[\frac{1}{2\pi} \int_0^{2\pi} P_{e^{i\theta}}(z)\, d\theta \right]^{1/2} = 1 = \phi_0(o) .$$

Further, equality holds if and only if the function $\theta \mapsto \sqrt{P_{e^{i\theta}}(z)}$ is constant, i.e. $z = o$. ∎

Under the transformation γ, the Laplacian transforms to $4v^2\Delta$, and so if f is an N-invariant function on D, and $f(z) = f(na_t \cdot o) = \psi(t) = e^t\varphi(t)$, it follows that

$$(*) \qquad Lf(na_t \cdot o) = e^t\{\varphi''(t) - \varphi(t)\}.$$

This has the following important consequence:

PROPOSITION 2.3. *Let h be a positive function on D such that*

(1) *h is N-invariant;*
(2) *$h(o) = 1$; and*
(3) *$Lh = -h$.*

Then

$$h(z) = \sqrt{P_1(z)} = \sqrt{\frac{1 - |z|^2}{|1 - z|^2}}.$$

Proof. If $\gamma(na_t) = w = u + iv$, then $P_1(z) = v = e^{2t}$. Furthermore, $h(o) = 1$ and $(*)$ imply that $Lh(na_t \cdot o) = e^t$. ∎

The polydisc carries the product Riemannian metric and the product G of l copies of $\mathrm{SU}(1,1)$ operates on D in the obvious way as an isometry group. Let K denote the subgroup of G that is the product of l copies of $\mathrm{SU}(1)$ and let A and N denote the products of l copies of the corresponding group for the unit disc. Clearly, $D = NA \cdot \mathbf{o}$, $\mathbf{o} = (o, \dots, o)$. Also, $D = KA \cdot \mathbf{o}$, and K acts on the distinguished boundary T in the obvious way: $\mathbf{k} \cdot \mathbf{b} = (k_1 \cdot b_1, \dots, k_l \cdot b_l)$ where the action of $\mathrm{SU}(1)$ on the components was given earlier.

If $\mathbf{a} \in A$, then $\mathbf{a} = (a_1, \dots, a_l)$. If $a_i = a_{t_i}$ for each i and $\mathbf{t} = (t_1, \dots, t_l)$, let $\mathbf{a} = \mathbf{a_t}$. Set $\rho(\mathbf{t}) = \sum_{i=1}^{l} t_i$.

A function f on the polydisc D is N-invariant if $f(\mathbf{z}) = f(\mathbf{na_t} \cdot \mathbf{o}) = F(\mathbf{v}) = \psi(\mathbf{t})$, where $\mathbf{v} = (v_1, \dots, v_n)$ and $v_i = e^{2t_i}$, $\gamma(z_i) = u_i + \sqrt{-1}v_i$.

The Laplace–Beltrami operator L for the polydisc is the sum of the hyperbolic Laplacians:

$$\begin{aligned} Lf(\mathbf{z}) = &\{L_1 f(\cdot, z_2, \dots, z_l)\}(z_1) \\ &+ \{L_2 f(z_1, \cdot, \dots, z_l)\}(z_2) + \dots + \{L_l f(z_1, z_2, \dots, \cdot)\}(z_l), \end{aligned}$$

where L_i denotes the Laplace–Beltrami operator on the disc D_i. It follows that the equation $Lf = \lambda f$ on D has positive solutions if and only if $\lambda \geq -l$. This is because if such functions exist, then minimal or extremal solutions exist and (cf. Freire [2]) such a function is minimal if and only if it is a product of minimal functions f_i on D_i where $L_i f_i = \lambda_i$, and $\lambda_1 + \dots + \lambda_l = \lambda$.

To each point $\mathbf{b}$ in the distinguished boundary $T = T_\emptyset$ of an l-polydisc D, let $P_\mathbf{b}(\mathbf{z}) = \prod_{i=1}^{l} P_{b_i}(z_i)$ denote the product of the Poisson kernels $P_{b_i}(z_i) = \frac{1-|z_i|^2}{|b_i - z_i|^2}$ on each of the component discs D_i. Then $P_\mathbf{b}$ satisfies $Lf = 0$.

Further, if $\lambda_i = 4\beta_i(\beta_i - 1)$, $\beta_i \geq 1/2$, then $\prod_{i=1}^{l} P_{b_i}^{\beta_i}(z_i)$ is a minimal solution of the equation $Lf = (\sum_{i=1}^{l} \lambda_i)f$ and every minimal solution has this form.

In particular, if $\lambda = -l$, the minimal solutions of the equation $Lf = -lf$ are the functions $\sqrt{P_{\mathbf{b}}}$, $\mathbf{b} \in T = T_{\emptyset}$.

Consequently, every non-negative solution h of the equation $Lf = -lf$ has a representation as

$$h(\mathbf{z}) = \int_T \sqrt{P_{\mathbf{b}}(\mathbf{z})}\,\mu(db),$$

for a unique Borel measure μ on the torus T. The normalized Haar measure $d\mathbf{b}$ on T represents the unique positive K-invariant solution Φ_0 of $Lf = -lf$ on D for which $f(\mathbf{o}) = 1$. From this it follows that $\Phi_0(\mathbf{z}) = \prod_{i=1}^{l} \phi_0(z_i)$. The next result is an immediate consequence of Proposition 2.2.

PROPOSITION 2.4. *Let Φ_0 be the unique positive solution of the equation $Lf = -lf$ for which $f(\mathbf{o}) = 1$. Then $\Phi_0(\mathbf{z}) = 1$ implies $\mathbf{z} = \mathbf{o}$.*

The formula for the Laplacian implies that if f is N-invariant on D, and $f(\mathbf{na_t} \cdot \mathbf{o}) = F(\mathbf{v}) = \psi(\mathbf{t}) = e^{\rho(\mathbf{t})}\varphi(t)$, then

$$Lf(\mathbf{na_t} \cdot \mathbf{o}) = e^{\rho(\mathbf{t})}\{\Delta\varphi(\mathbf{t}) - l\varphi(t)\}. \tag{$**$}$$

As in the case of the unit disc this has the following important consequence.

PROPOSITION 2.5. *Let h be a positive function on the polydisc D such that*

(1) *h is N-invariant;*
(2) *$h(\mathbf{o}) = 1$; and*
(3) *$Lh = -lh$.*

Then, if $\mathbf{1} = (1, \ldots, 1)$,

$$h(\mathbf{z}) = \sqrt{P_{\mathbf{1}}(\mathbf{z})} = \prod_{i=1}^{l} \sqrt{P_1(z_i)}.$$

Proof. $P_{\mathbf{1}}(\mathbf{na_t} \cdot \mathbf{o}) = e^{2\rho(\mathbf{t})}$. Furthermore, $(**)$ implies that $Lh(\mathbf{na_t} \cdot \mathbf{o}) = e^{\rho(\mathbf{t})}$ as $h(\mathbf{o}) = 1$. ∎

3. The Martin compactification. Since for the hyperbolic disc, $L + \mathrm{Id}$ has a Green function, it follows that one also exists for $L + l\mathrm{Id}$ on D. In other words, there exists a function $G(\mathbf{z}, \mathbf{w})$ such that if $G\varphi(\mathbf{z}) = \int G(\mathbf{z}, \mathbf{w})\varphi(\mathbf{w})\,dv(\mathbf{w})$ where v is the volume measure, then $G(L + l\mathrm{Id})\varphi = -\varphi$, for all smooth functions φ with compact support.

The *Martin compactification* $\widetilde{D}$ of the polydisc at the bottom of the positive spectrum is the compactification $\widetilde{D}$ of D for which

(1) the functions $\mathbf{w} \mapsto G(\mathbf{z},\mathbf{w})/G(\mathbf{o},\mathbf{w})$, $\mathbf{z} \in D$, extend continuously;

(2) the extended functions separate the points of the ideal boundary $\widetilde{D}\backslash D$ (see [10], also [14]).

The Martin compactification is metrizable (the proof in [10] applies here).

DEFINITION 3.1. A sequence $(\mathbf{w}_n)_{n\geq 0}$ of points $\mathbf{w}_n$ in D will be said to be *fundamental* if for every $\mathbf{z} \in D$, $h(\mathbf{z}) = \lim_{n\to\infty} G(\mathbf{z},\mathbf{w}_n)/G(\mathbf{o},\mathbf{w}_n)$ exists.

The following elementary lemma will be useful.

LEMMA 3.2. *Let K_1 and K_2 be two metrizable compactifications of D. Denote by i_1 and i_2 the respective embeddings of D onto dense subsets of K_1 and K_2. The following conditions are equivalent:*

(1) *there is a continuous map $\pi : K_2 \to K_1$ such that $\pi \circ i_2 = i_1$; and*

(2) *if a sequence $(x_n)_{n\geq 0}$ in D converges in K_2 then it converges in K_1.*

Proof. Clearly (1) implies (2). Assume (2), where a sequence $(x_n)_{n\geq 0}$ in D converges to $b_2 \in K_2$ means that $b_2 = \lim_{n\to\infty} i_2(x_n)$.

Define $\pi : K_2 \to K_1$ by setting $\pi(i_2(x)) = i_1(x)$ for all $x \in D$ and $\pi(b_2) = b_1$ if b_1 is the limit in K_1 of a sequence in D that converges to b_2. Interlacing two sequences that converge to b_2 shows that π is well defined. It is clearly onto.

Since D is locally compact it is open in each compactification and so π is continuous if $V = \pi^{-1}U$ is a neighborhood of b_2 in K_2 whenever U is a neighborhood in K_1 of $b_1 = \pi(b_2)$. If V is not a neighborhood of b_2, then there is a sequence in D that converges to b_2 but not to b_1. This contradicts the definition of π. ∎

To prove that the Martin compactification is $\overline{D}$, it therefore suffices to prove that (1) every fundamental sequence of points in D converges to a boundary point, and (2) every sequence of points in D that converges to a boundary point is fundamental. This will be done in the next section (see Theorem 4.7) by showing that convergent E-bounded sequences are fundamental and by computing the limit functions for these sequences.

Since the operator $L + l\mathrm{Id}$ is self-adjoint with respect to the volume measure $d\nu$ and invariant under isometries, it follows that for all $\mathbf{z},\mathbf{w}$:

(1) $G(\mathbf{z},\mathbf{w}) = G(\mathbf{w},\mathbf{z})$;

(2) $G(\mathbf{g}\cdot\mathbf{z},\mathbf{g}\cdot\mathbf{w}) = G(\mathbf{z},\mathbf{w})$ for all $\mathbf{g} \in G$; and

(3) $G(-\mathbf{z},-\mathbf{w}) = G(\mathbf{z},\mathbf{w})$.

From these properties of the Green function and the fact that $D = NA\cdot o$ one obtains the following important formula, due to Dynkin in [1] in another but similar context.

PROPOSITION 3.3. *Let* $\mathbf{a}, \mathbf{a_t} \in A$ *and* $\mathbf{n} \in N$. *Then*

$$G(\mathbf{na} \cdot \mathbf{o}, \mathbf{a_t} \cdot \mathbf{o}) = G(-(\mathbf{a}^{-1}\mathbf{n}_t^{-1} \cdot \mathbf{o}), \mathbf{a_t} \cdot \mathbf{o}), \quad \textit{where} \quad \mathbf{n_t} = \mathbf{a}_t^{-1}\mathbf{n}\mathbf{a_t}.$$

Proof.

$$\begin{aligned} G(\mathbf{na} \cdot \mathbf{o}, \mathbf{a_t} \cdot \mathbf{o}) &= G(\mathbf{a}_t^{-1}\mathbf{na} \cdot \mathbf{o}, \mathbf{o}) = G(\mathbf{n_t}\mathbf{a}\mathbf{a}_t^{-1} \cdot \mathbf{o}, \mathbf{o}) \\ &= G(\mathbf{a}_t^{-1} \cdot \mathbf{o}, \mathbf{a}^{-1}\mathbf{n}_t^{-1} \cdot \mathbf{o}) = G(-(\mathbf{a}_t^{-1} \cdot \mathbf{o}), -(\mathbf{a}^{-1}\mathbf{n}_t^{-1} \cdot \mathbf{o})) \\ &= G(\mathbf{a_t} \cdot \mathbf{o}, -(\mathbf{a}^{-1}\mathbf{n}_t^{-1} \cdot \mathbf{o})) = G(-(\mathbf{a}^{-1}\mathbf{n}_t^{-1} \cdot \mathbf{o}), \mathbf{a_t} \cdot \mathbf{o}), \end{aligned}$$

since $\mathbf{a}_t^{-1} \cdot \mathbf{o} = \mathbf{a}_{-t} \cdot \mathbf{o} = -(\mathbf{a_t} \cdot \mathbf{o})$ in view of the fact that in the unit disc, $a_t \cdot o = \tanh t$. ■

4. Computation of the limit functions

DEFINITION 4.1. If h is a limit function and $\mathbf{g} \in G$, define $S_{\mathbf{g}}h$ to be the function

$$S_{\mathbf{g}}h(\mathbf{z}) = h(\mathbf{g} \cdot \mathbf{z})/h(\mathbf{g} \cdot \mathbf{o}), \quad \mathbf{z} \in D.$$

PROPOSITION 4.2 (cf. Dynkin [1]). *Let h be the limit function determined by a fundamental sequence $(\mathbf{w}_n)_{n\geq 0}$ and let $(\mathbf{g}_n)_{n\geq 0}$ be a sequence in G that converges to $\mathbf{g}$. Then $(\mathbf{g}_n^{-1} \cdot \mathbf{w}_n)_{n\geq 0}$ is a fundamental sequence and $S_{\mathbf{g}}h$ is the corresponding limit function.*

Proof. This is obvious when $\mathbf{g}_n = \mathbf{g}$ for all n as

$$h(\mathbf{g} \cdot \mathbf{z}) = \lim_{n\to\infty} \frac{G(\mathbf{g} \cdot \mathbf{z}, \mathbf{w}_n)}{G(\mathbf{o}, \mathbf{w}_n)} = \lim_{n\to\infty} \frac{G(\mathbf{z}, \mathbf{g}^{-1} \cdot \mathbf{w}_n)}{G(\mathbf{o}, \mathbf{w}_n)}.$$

For any $\mathbf{z} \in D$, the set $\{\mathbf{g}_n \cdot \mathbf{z} \,|\, n \geq 0\} \cup \{\mathbf{g} \cdot \mathbf{z}\}$ is compact. The limit in Definition 3.1 is uniform on compact sets by Harnack's inequality and so

$$h(\mathbf{g} \cdot \mathbf{z}) = \lim_{n\to\infty} \frac{G(\mathbf{g}_n \cdot \mathbf{z}, \mathbf{w}_n)}{G(\mathbf{o}, \mathbf{w}_n)} = \lim_{n\to\infty} \frac{G(\mathbf{z}, \mathbf{g}_n^{-1} \cdot \mathbf{w}_n)}{G(\mathbf{o}, \mathbf{w}_n)}. \ ■$$

Note that for $\mathbf{k} \in K$, $S_{\mathbf{k}}h(\mathbf{z}) = h(\mathbf{k} \cdot \mathbf{z})$ and so if $\mathbf{k} \cdot (1, \ldots, 1) = \mathbf{k} \cdot \mathbf{1} = \mathbf{b} \in T$, then $S_{\mathbf{k}}\sqrt{P_{\mathbf{b}}}(\mathbf{z}) = \sqrt{P_{\mathbf{b}}}(\mathbf{k} \cdot \mathbf{z}) = \sqrt{P_{\mathbf{1}}}(\mathbf{z})$, since on the hyperbolic disc, $P_{e^{i\theta}}(e^{i\theta}z) = P_1(z)$.

PROPOSITION 4.3. *Let $(\mathbf{w}_n)_{n\geq 0}$ be a $\emptyset$-canonical sequence with limit point $\mathbf{b} \in T$. Then it is a fundamental sequence and the limit function h is $\sqrt{P_{\mathbf{b}}}$.*

Proof. In view of the action of K on the limit functions, it suffices to consider the case where $\mathbf{b} = \mathbf{1}$. Furthermore, if $\mathbf{w}_n = \mathbf{k}_n\mathbf{a}_n \cdot \mathbf{o}$ and $\mathbf{k}_n \to \mathbf{e}$, the sequence $(\mathbf{w}_n)_{n\geq 0}$ is fundamental if and only if the sequence $(\mathbf{a}_n \cdot \mathbf{o})_{n\geq 0}$ is fundamental.

Let $(\mathbf{a}_{n_k} \cdot \mathbf{o})_{k\geq 0}$ be a fundamental subsequence of $(\mathbf{a}_n \cdot \mathbf{o})_{n\geq 0}$. If $\mathbf{a}_{n_k} = \mathbf{a}_{\mathbf{t}_k}$, then $t_i(k) \to \infty$, $1 \leq i \leq l$. It follows from Remark 2.1 and Proposition 3.3 that the limit function is N-invariant since the conjugation in N takes place component by component. Since $h(0) = 1$, $h = \sqrt{P_1}$ by Proposition 2.5. The result follows from this identification of the limit of a fundamental subsequence. ∎

PROPOSITION 4.4. *Let* $(\mathbf{w}_n)_{n\geq 0}$ *be a convergent* E*-canonical sequence with limit point* $(\mathbf{o}, \mathbf{b})$*. Then* $(\mathbf{w}_n)_{n\geq 0}$ *is a fundamental sequence, and if* h *is the limit function, then*

$$h(\mathbf{z}) = \Phi_0(\mathbf{z}_1)\sqrt{P_{\mathbf{b}}(\mathbf{z}_2)},$$

where $\mathbf{z}_1 = (z_{i_1}, \ldots, z_{i_k})$, $\mathbf{z}_2 = (z_{j_1}, \ldots, z_{j_m})$ *and the functions* $\Phi_0(\mathbf{z}_1) = \Phi_0^E(\mathbf{z}_1)$ *and* $P_{\mathbf{b}}(\mathbf{z}_2)$ *are defined relative to the two polydiscs* D_E *and* $D_{E'}$ *determined by* E.

Proof. It suffices to consider the case where $\mathbf{b} = \mathbf{1}$.

View D as the product of D_E and $D_{E'}$ and the group G—and its relevant subgroups—as the product $G(1) \times G(2)$. Then $\mathbf{w}_n = (\mathbf{o}, \mathbf{k}_n(2)\mathbf{a}_n(2)\cdot\mathbf{o})$. As $\mathbf{k}_n(2) \to \mathbf{e}$, the sequence $(\mathbf{w}_n)_{n\geq 0}$ is fundamental if and only if the sequence $(\mathbf{o}, \mathbf{a}_n(2)\cdot\mathbf{o})_{n\geq 0}$ is fundamental.

Let

$$h(\mathbf{z}_1, \mathbf{z}_2) = \lim_{k\to\infty} \frac{G((\mathbf{z}_1, \mathbf{z}_2), (\mathbf{o}, \mathbf{a}_{n_k}(2)\cdot\mathbf{o}))}{G((\mathbf{o}, \mathbf{o}), (\mathbf{o}, \mathbf{a}_{n_k}(2)\cdot\mathbf{o}))}$$

be the limit of a fundamental subsequence.

As pointed out in §2, the limit function h is represented by the Poisson kernel $\sqrt{P_{\mathbf{b}}}$ and a unique measure μ on the torus T. This measure is $K(1)$-invariant because h has this property. Hence, μ is the product of the normalized Haar measure on the distinguished boundary T_E of D_E and a unique measure η on $T_{E'}$.

Consequently, $h(\mathbf{z}_1, \mathbf{z}_2) = \Phi_0(\mathbf{z}_1)f(\mathbf{z}_2)$, where $\Phi_0(\mathbf{z}_1)$ is the unique positive $K(1)$-invariant solution of the equation $L_E f = -|E|f$ on D_E for which $\Phi_0(o) = 1 - |E|$ is the cardinality of E. To show that $f(\mathbf{z}_2) = \sqrt{P_1(\mathbf{z}_2)}$, it suffices in view of Proposition 2.5 to show that f is N_2-invariant. This follows from Remark 2.1 and an obvious modification of Proposition 3.3, since if $\mathbf{a}_{n_k}(2) = \mathbf{a}_{t_k}$, then $\mathbf{t} = (t_{j_1}, \ldots, t_{j_m})$ and $t_{j_q} \to \infty$, $1 \leq q \leq m$, where $E' = \{j_1 < \ldots < j_m\}$. As in the previous argument, this establishes the result. ∎

Remark. The above argument is a simple illustration of an argument used extensively by Dynkin [1] and Karpelevich [6].

COROLLARY 4.5. *Let* $(\mathbf{w}_n)_{n\geq 0}$ *be a convergent* E*-bounded sequence with limit point* $(\mathbf{w}, \mathbf{b})$*. Let* $\Phi_{\mathbf{w}}(\mathbf{z}_1) = S_{\mathbf{g}(1)}\Phi_0^E(\mathbf{z}_1)$*, where* $\mathbf{g}(1) \in G(1)$ *satisfies* $\mathbf{g}(1)\cdot\mathbf{w} = \mathbf{o}$*. Then* $(\mathbf{w}_n)_{n\geq 0}$ *is a fundamental sequence and the limit function* h *is*

$$h(\mathbf{z}) = \Phi_{\mathbf{w}}(\mathbf{z}_1)\sqrt{P_{\mathbf{b}}(\mathbf{z}_2)},$$

where $\mathbf{z}_1 = (z_{i_1}, \dots, z_{i_k})$ *and* $\mathbf{z}_2 = (z_{j_1}, \dots, z_{j_m})$.

Proof. If $\mathbf{w}_n = (\mathbf{w}_n(1), \mathbf{w}_n(2))$ and $\mathbf{g}(1) \cdot \mathbf{w} = \mathbf{o}$, then the sequence $(\mathbf{o}, \mathbf{w}_n(2))_{n \geq 0}$ is E-canonical with limit $(\mathbf{o}, \mathbf{b})$. Consequently, by Proposition 4.2, the original sequence is fundamental with limit function $h(\mathbf{z}_1, \mathbf{z}_2) = S_{\mathbf{g}}(\Phi_0^E(\mathbf{z}_1)\sqrt{P_{\mathbf{b}}(\mathbf{z}_2)}) = (S_{\mathbf{g}(1)}\Phi_0^E)(\mathbf{z}_1)\sqrt{P_{\mathbf{b}}(\mathbf{z}_2)}$, $\mathbf{g} = (\mathbf{g}(1), \mathbf{e})$. ∎

PROPOSITION 4.6. *The correspondence* $(\mathbf{w}, \mathbf{b}) \mapsto \Phi_{\mathbf{w}}(\mathbf{z}_1)\sqrt{P_{\mathbf{b}}(\mathbf{z}_2)}$ *is injective.*

Proof. Proposition 2.4 implies that $\mathbf{w}$ is the unique point in the polydisc where $\Phi_{\mathbf{w}}$ takes the value 1. The result then follows from the fact that the Poisson kernel of the unit disc is parametrized by the unit circle. ∎

THEOREM 4.7. *Every fundamental sequence converges to a unique boundary point* $(\mathbf{w}, \mathbf{b})$. *Conversely, every sequence convergent to a boundary point is fundamental. Hence,* $\widetilde{D} = \overline{D}$.

Proof. The sequences that converge to a boundary point are the convergent E-bounded sequences, for a suitable proper subset E of $\{1, \dots, l\}$. By Corollary 4.5 they are all fundamental.

Conversely, it follows from this and Proposition 4.6 that the subsequences of a fundamental sequence that converge in $\overline{D}$ all converge to the same boundary point. ∎

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