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SUBALGEBRA OF $L_1(G)$ ASSOCIATED WITH LAPLACIAN ON A LIE GROUP

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In [18] Stein has considered one-parameter semigroups of operators $\{T^t\}_{t\geqslant 0}$ defined simultaneously on all spaces $L_p(M)$, $1\leqslant p\leqslant\infty$, for a measure space M, which satisfy the following properties: $\|T^tf\|_p\leqslant\|f\|_p$; T^t is a self-adjoint operator on $L_2(M)$; $T^tf\geqslant 0$ for $f\geqslant 0$; $T^t1=1$.

One of the leading examples of such a semigroup is that whose infinitesimal generator is a laplacian on a Lie group G. It is called the *heat-diffusion semigroup*. Another semigroup of this type is the Poisson semigroup closely related to the previous one, to which the major portion of [18] is devoted.

The aim of this paper is to study spectral properties of operators T^t simultaneously on all L_p-spaces in the case where T^t is the heat-diffusion semigroup or the Poisson semigroup and the underlying Lie group G is of polynomial growth, as well as some tauberian properties of the solutions of the heat equation and the Laplace equation on G.

The basic tool for that purpose is a commutative Banach *-subalgebra $\boldsymbol{A}$ of $L_1(G)$ generated by the fundamental solution of the heat equation on G. For a Lie group G of polynomial growth we prove first that $\boldsymbol{A}$ is symmetric and that for an r large enough the functions $C_c^r(\boldsymbol{R})$ operate on a certain dense subalgebra of $\boldsymbol{A}$ into $\boldsymbol{A}$. This leads to the proof that $\boldsymbol{A}$ is regular and allows to establish the Tauber-Wiener property for $\boldsymbol{A}$, which, in turn, yields tauberian theorems for the solutions of the heat equation (and of the Laplace equation) on G.

Suppose that for an f in $L_\infty(G)$ the function $u(x,t)$ satisfies

$$(*)\quad \Delta_x u(x,t)=\frac{\partial}{\partial t}u(x,t),\qquad u(x,0)=f(x)\quad\text{ and }\quad \|u(\cdot,t)\|_\infty\leqslant\|f\|_\infty.$$

Hence if, for a t_0, $\lim_{x\to\infty} u(x, t_0)$ exists and is equal to a, then for every g in $L_1(G)$

$$\lim_{x\to\infty} g*f(x) = a\int g.$$

(The same holds if the right-hand side of $(*)$ is replaced by $-\partial^2 u(x,t)/\partial t^2$.)

This paper contains essentially the material presented in a series of lectures given during the spring of 1973 at the University of Nancy. We have included here much of the preparatory material and, therefore, the better part of the paper has an expository character. It presents general properties of the laplacian on a Lie group and the fundamental solution of the heat equation on it. All the theorems presented in sections 1-3 are known and are due to Nelson [14] and to Nelson and Stinespring [15]. The proofs given here avoid the theory of elliptic operators on which the original proofs were based, the main tool being simply the Sobolev inequality. In many points the dependence of our presentation on Gårding [4] and Stein [18] is apparent.

Perhaps some novelty could be claimed in section 4. The original proof of proposition 4.1 due to Nelson [14] is based on the theory of Markov processes and is fairly involved. This, for the purpose it served (i. e., a construction of analytic vectors for a representation) was greatly simplified by Gårding [4]. Here we give still another proof of Nelson's lemma, which seems to apply to a wider class of operators and directly gives the original L_1 version of Nelson, which is vital for our study.

1. Submultiplicative functions on a Lie group. Let G be a connected Lie group. By LG we denote the Lie algebra of differential operators of the first order acting on $C_c^\infty(G)$ and commuting with the right translations, i. e., if $X \in LG$ and $f \in C_c^\infty(G)$, then

$$X(f_x) = (Xf)_x, \quad \text{where } f_x(y) = f(yx), \ x, y \in G.$$

LG is a finite-dimensional linear space over the reals. Every element X of LG defines a homomorphism

$$\boldsymbol{R} \ni t \to \exp tX \in G$$

in a way such that for an f in $C^\infty(G)$

$$(Xf)(x) = \frac{d}{dt} f(\exp tX\cdot x)|_{t=0}.$$

For an arbitrary function f on G we write

$$(1.2)\qquad |Xf|(x) = \limsup_{t\to 0} |t|^{-1} |f(\exp tX\cdot x) - f(x)|.$$

Let dx denote the differential of the left-invariant Haar measure on G. We write

$$|U| = \int_U dx$$

for every Borel set U in G.

We write

$$\langle f, g\rangle = \int f(x)\overline{g(x)}\,dx$$

for $f\in L_p(G)$, $g\in L_q(G)$. For f, g in $C_c^\infty(G)$ we have

$$\langle Xf, g\rangle = -\langle f, Xg\rangle \tag{1.3}$$

because, since dx is left-invariant,

$$\int f(\exp tX\cdot x)\bar{g}(x)\,dx = \int f(x)\bar{g}(\exp[-tX]\cdot x)\,dx,$$

whence, by (1.1), (1.3) follows.

Let $X_1, \ldots, X_n$ be a fixed basis in LG. For every f in $C^\infty(G)$ we define a vector-valued function

$$\nabla f(x) = \big(X_1 f(x), \ldots, X_n f(x)\big)$$

and we write

$$|\nabla f|(x) = \Big(\sum_j^n |X_j f|(x)^2\Big)^{1/2}. \tag{1.4}$$

For non-differentiable functions, the right-hand side of (1.4) also makes sense, if only $|X_j f|(x)$ is understood as in (1.2).

For a fixed x in G we define a linear mapping

$$LG \ni X \to X_x \in T_x(G), \tag{1.5}$$

where $T_x(G)$ is the tangent space to G at x, by the following usual formula

$$X_x f = (Xf)(x).$$

Of course, (1.5) is a linear isomorphism and thus the selection of the basis $X_1, \ldots, X_n$ in LG defines a riemannian (right-invariant) structure on G

$$\boldsymbol{g}(X_x, Y_x) = \sum_j^n a_j b_j \quad \text{if } X = \sum_j^n a_j X_j,\ Y = \sum_j^n b_j X_j. \tag{1.6}$$

This, in turn, defines a riemannian metric $d(x, y)$ on G by

$$d(x, y) = \inf \int_0^1 \boldsymbol{g}\big(\gamma'(t), \gamma'(t)\big)^{1/2}\,dt,$$

where the infimum is taken over all C^1-curves γ: $[0, 1] \to G$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $\gamma'(t_0)$ is the element of $T_{\gamma(t_0)}(G)$ defined by

$$\gamma'(t_0)f = \frac{d}{dt} f\big(\gamma(t)\big)\big|_{t=t_0}.$$

Since the riemannian form g is right-invariant, the metric d is right-invariant, i. e.,

$$(1.7)\qquad d(xz, yz) = d(x, y) \quad \text{for all } x, y, z \text{ in } G.$$

We also have

$$(1.8)\qquad d(x, y) \leqslant d(x, z) + d(z, y), \quad x, y, z \text{ in } G.$$

Let

$$(1.9)\qquad \tau(x) = d(x, e).$$

Then (1.7) and (1.8) imply

$$\tau(xy) \leqslant \tau(x) + \tau(y) \quad \text{for all } x, y \text{ in } G.$$

It is clear that the riemannian measure associated with the riemannian structure g is the right-invariant Haar measure on G.

Let for an X in LG

$$\|X\| = \Big(\sum_j^n a_j^2\Big)^{1/2} \quad \text{if } X = \sum_j^n a_j X_j.$$

Then, of course,

$$\|X\|^2 = g(X_x, X_x) \quad \text{for all } x \text{ in } G.$$

PROPOSITION 1.1. *For every X in LG we have*

$$|X\tau|(e) < \|X\|.$$

Proof. In virtue of the definition of $d(x, y)$ we have

$$\tau(\exp tX) = d(e, \exp tX) \leqslant \int_0^1 \|X\| ds = \|X\| t,$$

whence

$$|X\tau|(e) = \limsup_{t\to 0} |t|^{-1}\tau(\exp tX) \leqslant \|X\|.$$

(1.10) A function ϱ on G is called *subadditive*, if

(a) ϱ is Borel and bounded on compact sets,

(b) $\varrho(x) \geqslant 0$,

(c) $\varrho(xy) \leqslant \varrho(x) + \varrho(y)$ for all x, y in G,

(d) $\varrho(x) = \varrho(x^{-1})$, $x \in G$.

PROPOSITION 1.2. *For every subadditive function ϱ on G there is a constant C such that*

$$\varrho(x) \leqslant C\tau(x) + C.$$

Proof. Suppose

$$n \leqslant \tau(x) \leqslant n+1.$$

Then, by the definition of τ, there exist points $x_1, \ldots, x_n$ such that

$$\tau(x_1) \leqslant 1, \quad \tau(x_2 x_1^{-1}) \leqslant 1, \quad \ldots, \quad \tau(x x_n^{-1}) \leqslant 1.$$

But the riemannian structure defined by (1.6) admits a transitive group of isometries and it is complete, consequently the balls

$$\{x\colon \tau(x) \leqslant a\}$$

are compact. Let

$$C = \sup\{\varrho(x)\colon \tau(x) \leqslant 1\}.$$

Then $C < \infty$ and

$$\begin{aligned}\varrho(x) &= \varrho(x x_n^{-1} \ldots x_2^{-1} x_1) \leqslant \varrho(x x_n^{-1}) + \ldots + \varrho(x_1)\\ &\leqslant (n+1)C \leqslant C\tau(x) + C.\end{aligned}$$

(1.11) A function φ on G is called *submultiplicative* if
(a) φ is Borel and bounded on compact sets,
(b) $\varphi(x) \geqslant 1$, $x \in G$,
(c) $\varphi(xy) \leqslant \varphi(x)\varphi(y)$ for all x, y in G,
(d) $\varphi(x) = \varphi(x^{-1})$, $x \in G$.

If ϱ is subadditive, then $1+\varrho$ and e^{ϱ} are submultiplicative, every non-negative power of a submultiplicative function is a submultiplicative function, and a product of submultiplicative functions is submultiplicative. If φ is submultiplicative, then $\log\varphi$ is subadditive.

PROPOSITION 1.3. *If U is a relatively compact set with non-void interior containing e and such that $U = U^{-1}$, then*

$$\tau_U(x) = \inf\{n\colon x \in U^n\}$$

is a subadditive function.

PROPOSITION 1.4. *If U is an open set with compact closure, then*

$$|U^n| < C^{n-1}|U| \quad \textit{for a constant } C.$$

Proof. Since U^2 has compact closure, there exists a finite set F such that $U^2 \subset FU$. Hence $U^n \subset F^{n-1}U$ and, consequently, $|U^n| < C^{n-1}|U|$, where $C = \operatorname{card} F$.

PROPOSITION 1.5. *For C large enough*

$$\int e^{-C\tau(x)}dx < \infty.$$

Proof. Let U be a relatively compact set such that

$$G = \bigcup_n U^n.$$

Let k be such that

$$\tau_U(x) \leqslant k\tau(x) + k,$$

whence

$$e^{-\tau_U(x)} \geqslant e^{-k\tau(x)-k}.$$

But, by proposition 1.4, for constants a and b, we have

$$|U^n \setminus U^{n-1}| \leqslant a^n = e^{bn}.$$

Hence, for x in $U^n \setminus U^{n-1}$ and a C,

$$|U^n \setminus U^{n-1}| e^{-C\tau_U(x)} \leqslant e^{bn-Cn},$$

and so

$$\int e^{-C\tau_U(x)} dx \leqslant \sum_n^{\infty} e^{-cn} |U^n \setminus U^{n-1}| \leqslant \sum_n^{\infty} e^{-(C-b)} < \infty$$

for C large enough.

PROPOSITION 1.6. *Let ϱ be a subadditive function on G such that $\varrho(e) = 0$. Then for each X in LG we have*

$$|X\varrho|(x) \leqslant |X\varrho|(e).$$

Proof. By (1.10) (c) and (d) we have $\varrho(y) = \varrho(x^{-1}xy) \leqslant \varrho(x^{-1}) + \varrho(xy) = \varrho(x) + \varrho(xy)$, whence

$$|\varrho(xy) - \varrho(y)| \leqslant \varrho(x).$$

Thus, for every X in LG, we have

$$|X\varrho|(x) \leqslant \limsup_{t\to 0} |t|^{-1} \varrho(\exp tX) = |X\varrho|(e).$$

2. The laplacian. First let us recall few definitions concerning unbounded operators in Banach spaces and at the same time establish notation (cf., e. g., [3] and [19]).

Let E and F be two Banach spaces. The *domain* of a linear mapping

$$A: E \to F$$

is denoted by $\boldsymbol{D}(A)$. We assume that for an operator A the domain $\boldsymbol{D}(A)$ is dense in E. The *adjoint* A' of A is an operator

$$A': F' \to E'$$

defined as follows:

$$\begin{aligned}\boldsymbol{D}(A') &= \{y' \in F' : \langle Ax, y' \rangle \leqslant C\|x\| \text{ for all } x \text{ in } \boldsymbol{D}(A)\} \\ &= \{y' \in F' : \text{there is an } x' \text{ in } E' \text{ s. t. } \langle Ax, y' \rangle = \langle x, x' \rangle\}.\end{aligned}$$

Since $\boldsymbol{D}(A)$ is dense in E, x' is uniquely defined for each y' in $\boldsymbol{D}(A')$ and we put

$$A'y' = x'.$$

An operator A is called *closed*, if the graph

$$\{(x, Ax): x \in \boldsymbol{D}(A)\}$$

is closed in $E \times F$.

(2.1) We have:
(a) A' is a closed operator.
(b) A has a closed extension if and only if $\boldsymbol{D}(A')$ is dense in F'.
(c) If E and F are reflexive and A has a closed extension, then $(A')'$ is the smallest closed extension of A.

If H is a Hilbert space, then we identify H and H' and we say that A is *symmetric* if

$$\langle Ax, y\rangle = \langle x, Ay\rangle \quad \text{for all } x, y \text{ in } \boldsymbol{D}(A), \tag{2.2}$$

or, in other words, if

$$A \subset A'. \tag{2.3}$$

An operator A on a Hilbert space is *self-adjoint* if $\boldsymbol{D}(A) = \boldsymbol{D}(A')$ and (2.2) holds, i. e., if $A = A'$.

In what follows we shall use the following form of the spectral theorem:

SPECTRAL THEOREM [3]. *Let A be a self-adjoint operator in a Hilbert space H. Then the spectrum of A is real and there exists a uniquely determined regular spectral measure E defined on the Borel subsets of $\boldsymbol{R}$ vanishing on the complement of the spectrum such that*

$$\boldsymbol{D}(A) = \Big\{x \in H: \int_{-\infty}^{+\infty} \lambda^2 d\langle E(\lambda)x, x\rangle < \infty\Big\},$$

$$Ax = \lim_{n\to\infty} \int_{-n}^{n} \lambda dE(\lambda)x.$$

If F is a Borel measurable function on $\boldsymbol{R}$, we denote by F_n the function

$$F_n(\lambda) = \begin{cases} F(\lambda) & \text{if } |F(\lambda)| \leqslant n, \\ 0 & \text{if } |F(\lambda)| > n. \end{cases}$$

We put

$$F(A)x = \lim_{n\to\infty} \int_{-\infty}^{+\infty} F_n(\lambda) dE(\lambda)x \tag{2.4}$$

and we define $\boldsymbol{D}\big(F(A)\big)$ as the set of the x's in H for which the limit (2.4) exists.

If F is real-valued, then $F(A)$ is self-adjoint and

$$\mathrm{Sp}\big(F(A)\big) \subset \overline{F\big(\mathrm{Sp}(A)\big)};$$

if F is continuous, then also

$$\mathrm{Sp}(F(A)) = \overline{F(\mathrm{Sp}(A))}. \tag{2.5}$$

We say that a bounded operator T defined on a Banach space E *commutes* with an operator $A\colon E\to E$, if $T(\boldsymbol{D}(A)) \subset \boldsymbol{D}(A)$ and $ATx = TAx$ for all x in $\boldsymbol{D}(A)$.

If A is a self-adjoint operator in a Hilbert space, then A commutes with T, if and only if $E(M)$ commutes with T for all Borel subsets M of $\boldsymbol{R}$. Consequently, if T commutes with A, then T commutes with $F(A)$ for every Borel function F on $\boldsymbol{R}$.

Now for a connected Lie group G and a basis $X_1, \ldots, X_n$ of the Lie algebra LG of G we write

$$L = X_1^2+\ldots+X_n^2. \tag{2.6}$$

We consider L as an operator on $L_2(G)$ with the domain $C_c^\infty(G)$ which is dense in $L_2(G)$. Let

$$\Delta = L' \quad \text{and} \quad D = \boldsymbol{D}(\Delta). \tag{2.7}$$

We call Δ a *laplacian* on G. By (1.3) we have

$$\langle Lf, g\rangle = \langle f, Lg\rangle \quad \text{for all } f, g \text{ in } C_c^\infty(G). \tag{2.8}$$

Consequently, $L \subset L'$ and so

$$\Delta' \subset \Delta. \tag{2.9}$$

We introduce the following notation:

$$(\nabla f, \nabla g) = \sum_j^n \langle X_j f, X_j g\rangle = -\langle Lf, g\rangle \quad \text{for all } f, g \in C_c^\infty(G).$$

Similarly, we write

$$(|\nabla f|, |\nabla g|) = \sum_j^n \langle |X_j f|, |X_j g|\rangle$$

whenever $|X_j f|$ and $|X_j g|$ are in $L_2(G)$, $j = 1, \ldots, n$. Accordingly,

$$\|\nabla f\|_2^2 = (\nabla f, \nabla f) \quad \text{and} \quad \||\nabla f|\|_2^2 = (|\nabla f|, |\nabla f|).$$

Let

$$\|f\|^0 = (\|f\|_2^2 + \|\nabla f\|_2^2)^{1/2}, \quad f \in C_c^\infty(G).$$

The completion of $C_c^\infty(G)$ in the norm $\|\cdot\|^0$ is a Hilbert space which we denote by H^0 and which consists of functions f such that $\|f\|_2$ and $\||\nabla f|\|_2$ are both finite.

Now we are going to prove that Δ, as defined in (2.7), is a self-adjoint operator. Our proof follows that of Gårding [4] which is an adaptation of a proof by Carleman [1].

PROPOSITION 2.1. *We have*

$$-\langle \Delta f, f\rangle = \||\nabla f|\|^2 \quad \text{for all } f \text{ in } D. \tag{2.10}$$

Moreover,

$$\Delta' = \Delta. \tag{2.11}$$

Proof. First we introduce two regularizations: local and at infinity, by defining two sequences of functions e_n and ξ_n $(n = 1, 2, \ldots)$, where

$$\begin{gathered} e_n = f_n * f_n^*, \quad f_n \in C_c^\infty(G), \quad f_n(x) \geqslant 0, \\ \int f_n(x)\,dx = 1, \quad \operatorname{supp} f_n \to e, \end{gathered} \tag{2.12}$$

and

$$\xi_n = e_1 * \chi_n,$$

where χ_n is the characteristic function of an open set U_n with $U_n \subset U_{n+1}$ and $\bigcup_n U_n = G$. We have

$$|X\xi_n(x)| \leqslant \int |Xe_1(x)|\,dx \tag{2.13}$$

and

$$\xi_n \to 1 \text{ and } X\xi_n \to 0 \text{ uniformly on compact sets.} \tag{2.14}$$

Let E_n be the operator defined by

$$E_n f = f * e_n.$$

Of course,

$$LE_n f = E_n L f \quad \text{for all } f \text{ in } C_c^\infty(G)$$

and so, for every $n = 1, 2, \ldots,$ the operators Δ and E_n commute.

Consequently, since $\Delta E_n f = E_n \Delta f$ implies

$$\lim_{n\to\infty} \Delta E_n f = \lim_{n\to\infty} E_n \Delta f = \Delta f \quad \text{for } f \text{ in } D,$$

we have

$$\text{if } f \in D, \text{ then } \Delta E_n f \to \Delta f \text{ and } E_n f \to f. \tag{2.15}$$

Now to prove (2.10), we take an f in D and for a fixed $n = 1, 2, \ldots$ we write $f_n = E_n f$. Since e_n has compact support and is infinitely differentiable, we have

$$Xf_n \in L_\infty(G) \quad \text{for all } X \text{ in } LG. \tag{2.16}$$

Thus

$$\langle \Delta f_n, f_n\rangle = \lim_{m\to\infty}\langle \Delta f_n, \xi_m f_n\rangle$$
$$= -\lim_{m\to\infty}\Big(\sum_i \langle X_i f_n, \xi_m X_i f_n\rangle + \sum_i \langle X_i f_n, (X_i\xi_m) f_n\rangle\Big).$$

Hence, in virtue of (2.14) and (2.16), we get

$$\langle \Delta f_n, f_n\rangle = -\|\nabla f_n\|_2^2. \tag{2.17}$$

But (2.15) shows that the left-hand side of (2.17) is convergent to $\langle \Delta f, f\rangle$ and, consequently, f_n is a fundamental sequence in H^0, so $f\in H^0$ and (2.10) follows.

To complete the proof of proposition 2.1, we take f, g in $D\cap C^\infty$ and write

$$\langle \xi_n Lf, g\rangle - \langle \xi_n f, Lg\rangle = (\nabla f, g\nabla\xi_n) - (f\nabla\xi_n, \nabla g).$$

Hence, as before, if n tends to infinity, we obtain

$$\langle Lf, g\rangle = \langle f, Lg\rangle \quad \text{for all } f, g \text{ in } D\cap C^\infty,$$

which, in virtue of (2.14), gives

$$\langle \Delta f, g\rangle = \langle f, \Delta g\rangle \quad \text{for all } f, g \text{ in } D,$$

i. e., $\Delta\subset\Delta'$ and so, by (2.9), equality (2.11) follows.

It follows immediately from (2.10) that $-\Delta$ is a non-negative operator, whence

$$\mathrm{Sp}\Delta \subset (-\infty, 0]. \tag{2.18}$$

Let

$$F_t\colon \boldsymbol{R}\ni x \to e^{tx}\in \boldsymbol{R}^+.$$

Then, by (2.4),

$$F_t(\Delta) = e^{t\Delta} = T^t \tag{2.19}$$

is a hermitian operator in $L_2(G)$. Since the spectral measure of Δ is concentrated on the non-positive half-line, we have

$$T^t f = \int_{-\infty}^{0} e^{t\lambda}\, dE(\lambda) f$$

and so T^t is a bounded operator in $L_2(G)$ whose norm is equal to

$$\|T^t\| = \sup\{e^{t\lambda}\colon \lambda\in \mathrm{Sp}\Delta\} \leqslant 1. \tag{2.20}$$

On the other hand, since Δ and hence $E(M)$ commute with right translations, so does T^t. Consequently,

$$T^t \text{ is a multiplier on } L_2(G). \tag{2.21}$$

3. The heat equation on G. Let Δ be a laplacian on a Lie group G as defined by (2.7) and let $\lambda > 0$. By (2.18), λ is not in the spectrum of Δ, i. e.,

(3.1) $$R_\lambda = (\lambda - \Delta)^{-1} \text{ is a bounded operator on } L_2(G).$$

We prove the following

PROPOSITION 3.1. (a) $R_\lambda f = \mu_\lambda * f$, *where* μ_λ *is a non-negative measure on* G.

(b) $\lambda\mu_\lambda(G) = 1$.

Remark. As we shall see later on, the measure μ_λ is absolutely continuous with respect to the Haar measure.

We prove first the following lemmas.

LEMMA 3.2. *For every complex number λ which is not in the non-positive half line of the real axis, the set*

$$\{(\lambda - L)\varphi\colon \varphi \in C_c^\infty(G)\}$$

is dense in $C_0(G)$ — the space of continuous functions on G vanishing at infinity, equipped with the L_∞-norm.

Proof. Suppose μ is a bounded measure on G such that for all φ and ψ in C_c^∞ we have

$$\langle (\lambda - L)(\varphi * \tilde{\psi}), \mu\rangle = 0.$$

Then, since $\lambda - L$ commutes with the right translations,

$$\langle (\lambda - L)\varphi, \mu * \psi\rangle = 0 \quad \text{for all } \varphi \in C_c^\infty(G).$$

But since ψ belongs to $L_2(G)$, so does $\mu * \psi = f$. Hence

$$|\langle L\varphi, f\rangle| = |\lambda|\,|\langle \varphi, f\rangle| \leqslant |\lambda|\,\|\varphi\|_2 \|f\|_2,$$

which shows that $f \in \boldsymbol{D}(L') = \boldsymbol{D}(\Delta)$. Consequently,

$$\langle \varphi, (\bar{\lambda} - \Delta) f\rangle = 0 \qquad \text{for all } \varphi \text{ in } C_c^\infty(G)$$

and so

$$\bar{\lambda} f = \Delta f \quad \text{and} \quad \bar{\lambda}\langle f, f\rangle = \langle \Delta f, f\rangle,$$

which, in virtue of (2.10), proves that $0 = f = \mu * \psi$. Since ψ is an arbitrary function in $C_c^\infty(G)$, we have $\mu = 0$ and the proof of the lemma is completed.

LEMMA 3.3. *For $\lambda > 0$ we have*

$$\|(\lambda - L)\varphi\|_\infty \geqslant \lambda \|\varphi\|_\infty \quad \textit{for all } \varphi \textit{ in } C_c^\infty(G).$$

Proof. Suppose $|\varphi|(x)$ assumes the maximum at a point x_0. Then for a complex number θ of modulus 1 we have

$$|\varphi|(x_0) = \mathscr{R}(\theta\varphi)(x_0) \quad \text{and} \quad \|\mathscr{R}(\theta\varphi)\|_\infty = \mathscr{R}\theta\varphi(x_0).$$

Consequently, since for a basis $Y_1, \ldots, Y_n$ of $T_{x_0}(G)$

$$(Y_1^2+\ldots+Y_n^2)f(x_0) = Lf(x_0), \quad f \in C^\infty(G),$$

we have

$$\mathscr{R}(L\theta\varphi)(x_0) = L[\mathscr{R}(\theta\varphi)](x_0) \leqslant 0$$

and so

$$\|(\lambda - L)\varphi\|_\infty \geqslant \|\mathscr{R}\lambda\theta\varphi - L\mathscr{R}(\theta\varphi)\|_\infty$$
$$\geqslant \lambda\mathscr{R}\big(\theta\varphi(x_0)\big) - L\mathscr{R}(\theta\varphi)(x_0) \geqslant \lambda\mathscr{R}(\theta\varphi)(x_0) = \lambda|\varphi|(x_0).$$

Remark. It is clear that if we take a real non-negative function φ in $C_c^\infty(G)$ such that $L\varphi(x_0) = 0$, where x_0 is a point at which φ assumes the maximum value, then

$$\|(\lambda - L)\varphi\|_\infty = \lambda\|\varphi\|_\infty.$$

Proof of proposition 3.1. It is an immediate consequence of lemmas 3.2 and 3.3 and the remark above that there is a unique operator R_λ on $C_0(G)$ such that

$$(\lambda - L)R_\lambda\varphi = R_\lambda(\lambda - L)\varphi = \varphi \quad \text{for all } \varphi \text{ in } C_c^\infty(G)$$

and

$$\|R_\lambda\|_\infty = \lambda^{-1}.$$

Of course, R_λ commutes with the right translations, i. e., R_λ is a multiplier on $C_0(G)$ and, consequently, there exists a bounded measure μ_λ on G such that

$$R_\lambda f = \mu_\lambda * f \quad \text{for all } f \text{ in } C_0(G).$$

Clearly enough,

$$\|\mu_\lambda\| = \lambda^{-1} \quad \text{and} \quad \mu_\lambda * \bar{f} = \overline{\mu_\lambda * f}.$$

It is not difficult to show that μ_λ is a non-negative measure. To do so, it is sufficient to show that R_λ maps non-negative C_c^∞-functions into non-negative functions.

Suppose then that $R_\lambda\varphi = \psi$, where $\varphi, \psi \in C_0^\infty(G)$ and $\varphi(x) \geqslant 0$. Then ψ is a real function. Assume that it takes the minimum at a point x_0 and $\psi(x_0) < 0$. Then

$$\varphi(x_0) = \lambda\psi(x_0) - (L\psi)(x_0) < 0,$$

since $\lambda > 0$ and $(L\psi)(x_0) \geqslant 0$, which is a contradiction. The proof of proposition 3.1 is thus completed.

Proposition 3.1 shows that the resolvent R_λ of L is a bounded operator on all $L_p(G)$, $1 \leqslant p < \infty$, for $\lambda > 0$. This enables us to apply the theory of one-parameter semigroups to extent the operators T^t, $t > 0$, as defined by (2.19) to all $L_p(G)$, $1 \leqslant p < \infty$. But before doing it, let us recall a few notions and a theorem concerning semigroups of operators on Banach spaces.

A one-parameter family $\{T_t\}_{t\geqslant 0}$ of bounded operators in a Banach space E is called a *semigroup* of operators in E if the following conditions are satisfied:

(a) for every x in E the mapping $[0, \infty) \ni t \to T_t x \in E$ is continuous;

(b) $T_0 = I$;

(c) $T_{s+t} = T_s T_t$ for all $s, t \geqslant 0$.

An operator A in E is called the *infinitesimal generator* of a semigroup $\{T_t\}_{t\geqslant 0}$, if

$$Ax = \lim_{t\to 0} t^{-1}(T_t - I)x \tag{3.2}$$

and if the domain $D(A)$ of A consists of all elements x in E for which limit (3.2) exists. If $\varphi \in C_c^\infty[0, \infty)$, we put

$$T_\varphi x = \int_0^\infty \varphi(t) T_t x dt.$$

It is easy to verify that

$$\{T_\varphi x\colon \varphi \in C_c^\infty[0, \infty), x \in E\}$$

is contained in $D(A)$ and is dense in E; moreover, T_t commutes with A.

The following theorem plays an important role in what follows:

THEOREM OF HILLE AND YOSHIDA. *Let A be a densely defined operator in a Banach space E. If for every $\lambda > 0$ the operator $(\lambda - A)^{-1} = R_\lambda$ exists and $\|R_\lambda\| \leqslant \lambda^{-1}$, then there exists a unique semigroup $\{T_t\}_{t\geqslant 0}$ of operators in E such that the infinitesimal generator of $\{T_t\}_{t\geqslant 0}$ contains A and*

$$T_t x = \lim_{n\to 0} \left(\frac{n}{t} R_{n/t}\right)^n x \quad \text{for all } x \text{ in } E. \tag{3.3}$$

Moreover, for $\lambda > 0$ we have

$$R_\lambda x = \int_0^\infty e^{-\lambda t} T_t x dt, \qquad x \in E. \tag{3.4}$$

Now we are ready to prove the following theorem:

THEOREM 3.4. *Let G be a Lie group, $X_1, \ldots, X_n$ a basis of the Lie algebra LG of G and $L = X_1^2 + \ldots + X_n^2$. Then there exists a unique semigroup of operators*

$$T^t\colon C_c(G) \to C(G)$$

such that

(a) *For each p, $1 \leqslant p \leqslant \infty$, and each $t \geqslant 0$, T^t is a bounded operator of the norm 1 on $L_p(G)$ and $\{T^t\}_{t\geqslant 0}$ is a semigroup of operators on $L_p(G)$.*

(b) *T^t is a self-adjoint operator on $L_2(G)$.*

(c) *T^t is positive: $T^t f \geqslant 0$ for $f \geqslant 0$.*

(d) *For each p, $1 \leqslant p < \infty$, the infinitesimal generator of the semigroup $\{T^t\}_{t\geqslant 0}$ on $L_p(G)$ contains the operator L, i. e.,*

$$\frac{d}{dt}T^t\varphi = LT^t\varphi \quad \text{for all } \varphi \in C^\infty(G)\cap L_p(G),\ 1 \leqslant p < \infty,$$

and

$$\frac{d}{dt}T^t f = \Delta T^t f \quad \text{for all } f \text{ in } \boldsymbol{D}(\Delta) \subset L_2(G).$$

(e) *For every $t > 0$ there is a non-negative function p_t such that $p_t \in L_1(G)\cap L_2(G)$,*

$$T^t f = p_t * f \quad \text{for all } f \text{ in } L_p(G),\ 1 \leqslant p \leqslant \infty,$$

and

$$(0, \infty)\times G \ni t, x \to p_t(x) \in \boldsymbol{R}$$

is a C^∞-function.

(f) *For every $t > 0$, $\|p_t\|_1 = 1$.*

(g) *For every p, $1 \leqslant p < \infty$, f in $L_p(G)$ and left-uniformly continuous bounded function f,*

$$u(t, x) = p_t * f(x)$$

is a C^∞-function on $(0, \infty)\times G$, satisfies the heat equation

$$\frac{\partial}{\partial t}u(t, x) = Lu(t, x),$$

and $u(t, \cdot)$ tends to f in L_p-norm if f is in $L_p(G)$ for $1 \leqslant p < \infty$ and also in L_∞-norm if f is a left-uniformly continuous bounded function.

Proof. To prove the existence of T^t, $t \geqslant 0$, we consider L as an operator with the domain $C_c^\infty(G)$ acting on $C_0(G)$. In view of proposition 3.1 (b) we may apply the Hille-Yoshida theorem to obtain the semigroup $\{T^t\}_{t\geqslant 0}$ of operators acting on the Banach space $C_0(G)$. Since R_λ commutes with the right translations, by (3.3), so do all T^t, whence

$$T^t f = \nu_t * f, \tag{3.5}$$

where ν_t is a bounded measure. By proposition 3.1 (a) and again by (3.3) we see that ν_t is a non-negative measure and by proposition 3.1 (b) and (3.3) we get $\|\nu_t\| \leqslant 1$. Suppose (e) is proved. Then 0 belongs to the spectrum of T^t considered as an operator on $L_2(G)$, whence by the equality in (2.20), $\|T^t\| = 1$ and, consequently,

$$\|\nu_t\| = 1 \quad \text{for all } t > 0. \tag{3.6}$$

(d) is a simple consequence of the fact that the infinitesimal generator of T^t contains L (and Δ in case of $L_2(G)$) and commutes with T^t, $t > 0$.

(f) follows from (3.5) and (e), and (g) is a consequence of the fact that p_t, $t > 0$, is an approximate unit in $L_1(G)$ and of (e), which in turn follows from the theorem of Hille and Yoshida. Thus all reduce to the proof of (e).

Let us first recall a version of Sobolev's lemma.

SOBOLEV LEMMA. *Let M be a riemannian manifold and let L be the Laplace-Beltrami operator on M* (cf. [5]). *Let $\|\cdot\|_p$ denote the L_p-norm with respect to any measure equivalent to the riemannian measure on M. For every compact set Ω there exist C and l such that, for all f in the common domain of $L, \ldots, L^l$ in $L_2(M)$, we have*

$$|f(x)| \leqslant C \sum_{k=0}^{l} \|L^k f\|_2 \quad \textit{for all } x \textit{ in } \Omega.$$

Now to prove (e) we take an f in $L_2(G)$, arbitrary non-negative integers k and N, and a $t_0 > 0$. Then, if $u(t, x) = T^t f$, we have, by the spectral theorem,

$$\left\| \frac{\partial^k}{\partial t^k} \Delta_x^N u(t, x) \right\|_2^2 = \int_{-\infty}^{0} 2^k \lambda^{2N+k} e^{2\lambda t} \langle dE(\lambda) f, f \rangle \tag{3.7}$$

$$\leqslant \max\{2^k |\lambda|^{2N+k} e^{2\lambda t_0} \colon \lambda \leqslant 0\} \|f\|_2^2 \quad \text{for all } 0 < t < t_0,$$

which, by Sobolev's lemma, proves that $u(t, x)$ is a C^∞-function. Applying (3.7) with $k = 0$, in view of (3.5), we see that there exists a constant C such that, by Sobolev's lemma,

$$\|\nu_t * f\|_\infty = \|T^t f\|_\infty \leqslant C \|f\|_2 \quad \text{for all } f \text{ in } L_2(G),$$

which shows that ν_t is absolutely continuous with respect to the Haar measure whose Radon-Nikodym derivative p_t belongs to $L_2(G)$. But for a fixed t and arbitrary m an application of inequality (3.7) yields

$$\|\Delta^m p_t\|_2 = \|\Delta^m (p_{t/2} * p_{t/2})\|_2 \leqslant C_m \|p_{t/2}\|_2,$$

which, via Sobolev's lemma, shows, as before, that $p_t \in C^\infty(G)$.

COROLLARY 3.5. *For $\lambda > 0$ the measure μ_λ, as defined in proposition* 3.1, *is absolutely continuous with respect to the Haar measure; its Radon-Nikodym derivative we denote by k_λ.*

Proof. By (3.4) we have

$$\mu_\lambda * f = \int_0^\infty e^{-\lambda t} p_t * f \, dt \tag{3.8}$$

for all f in $L_2(G)$. Since $\|p_t\|_1 = 1$, the function

$$(0, \infty) \ni t \to e^{-\lambda t} p_t \in L_1(G)$$

is Bochner integrable and so, if

$$k_\lambda = \int_0^\infty e^{-\lambda t} p_t \, dt, \tag{3.9}$$

k_λ is an L_1-function and, by (3.8), the Radon-Nikodym derivative of μ_λ is k_λ.

4. Behavior of the p_t's at infinity. For a Lie group G we fix a basis $X_1, \ldots, X_n$ in the Lie algebra LG. Consider the laplacian defined by it and the semi-group of L_1-functions p_t, $t > 0$, as defined in section 3. Our aim now is to show that the p_t's decrease very rapidly at infinity. We prove the following

PROPOSITION 4.1. *For every submultiplicative function φ on G and a $t_0 > 0$ there is a constant C such that*

$$\langle p_t, \varphi \rangle = \int p_t(x)\varphi(x)\,dx \leqslant C \quad \textit{for all } t \leqslant t_0. \tag{4.1}$$

The selection of the basis induces the norm $\|X\|$ in LG, the riemannian structure in G, and the subadditive function τ as defined in (1.9).

For every natural number m let

$$\tau_m(x) = \begin{cases} \tau(x) & \text{if } \tau(x) \leqslant m, \\ m & \text{if } \tau(x) > m. \end{cases}$$

It will be convenient to use the notation

$$\tau_\infty(x) = \tau(x).$$

Clearly enough, τ_m is a subadditive function vanishing at e, whence, by propositions 1.1 and 1.6, we have for all X in LG

$$|X\tau_m|(x) \leqslant |X\tau_m|(e) \leqslant \|X\|. \tag{4.2}$$

Let

$$U = \{x \in G\colon \tau(x) < 1\}.$$

Then U is an open set with compact closure. Let $f \in C_c^\infty(U)$ be a non-negative function such that $\int f(x)\,dx = 1$. We then have

LEMMA 4.2. *For every $m = \infty, 1, 2, \ldots$ the inequalities*

$$\tau_m(x) - 1 \leqslant f * \tau_m(x) \leqslant \tau_m(x) + 1 \tag{4.3}$$

hold for all x in G.

In fact,

$$f * \tau_m(x) = \int f(y)\tau_m(y^{-1}x)\,dy \leqslant \int_U f(y)\tau_m(y)\,dy + \tau_m(x) \leqslant \tau_m(x) + 1$$

and

$$f*\tau_m(x) = \int f(y)\tau_m(y^{-1}x)\,dy \geqslant -\int_U f(y)\,\tau_m(y)\,dy+\tau_m(x) \geqslant \tau_m(x)-1.$$

LEMMA 4.3. *For every X in LG there exist constants A_X and B_X such that for all $m = \infty, 1, 2, \ldots$ we have*

$$|X(f*\tau_m)(x)| \leqslant A_X, \qquad |X^2(f*\tau_m)(x)| \leqslant B_X$$

for all x in G.

Proof. Since $f*\tau_m$ is a C^∞-function, $f\in C_c^\infty(G)$ and X commutes with the right translations, we have

$$X^2(f*\tau_m) = X\big((Xf)*\tau_m\big). \tag{4.4}$$

For a y in G we write

$$y^{-1}\exp tX\cdot y = \exp tAd_yX,$$

whence

$$y^{-1}\exp tX = \exp tAd_yX\cdot y^{-1}.$$

Now, using this and (4.2), we get

$$\begin{aligned}(4.5)\quad & |X(f*\tau_m)(x)| \\ &= \Big|\lim_{t\to 0} t^{-1}\int f(y)\tau_m(y^{-1}\exp tX\cdot x)-f(y)\tau_m(y^{-1}x)\,dy\Big| \\ &= \Big|\lim_{t\to 0} t^{-1}\int f(y)\big(\tau_m(\exp tAd_yX\cdot y^{-1}x)-\tau_m(y^{-1}x)\big)\,dx\Big| \\ &\leqslant \int |f(y)|\limsup_{t\to 0}|t|^{-1}|\tau_m(\exp tAd_y\cdot y^{-1}x)-\tau_m(y^{-1}x)|\,dy \\ &\leqslant \int |f(y)|\,|Ad_yX\tau_m|\,(y^{-1}x)\,dy \leqslant \int |f(y)|\,\|Ad_yX\|\,dy = A_X.\end{aligned}$$

Putting Xf in place of f in (4.5) and applying (4.4) we obtain

$$B_X = \int |Xf(y)|\,|Ad_yX|\,dy.$$

LEMMA 4.4. *Let C be an arbitrary positive constant and let for $m = \infty$, $1, 2, \ldots$*

$$\varphi_m(x) = e^{Cf*\tau_m(x)+C}. \tag{4.6}$$

Then there is a constant K independent of $m = \infty, 1, 2, \ldots$ such that

$$|L\varphi_m(x)| \leqslant K\varphi_m(x) \quad \textit{for all } x \textit{ in } G.$$

Proof. For each $j = 1, \ldots, n$ we have

$$|X_j^2\varphi_m| = \big|CX_j^2(f*\tau_m)+\big(CX_j(f*\tau_m)\big)^2\big|\varphi_m \leqslant (CB_{X_j}+C^2A_{X_j}^2)\varphi_m,$$

whence, summing over $j = 1, \ldots, n$, we get the result.

Now we are ready to prove proposition 4.1.

Proof of proposition 4.1. In virtue of proposition 1.2 and (4.3) it is sufficient to prove that for an arbitrary positive constant C there is a constant K such that

$$\langle p_t, \varphi_\infty\rangle < e^{tK+2C}. \tag{4.7}$$

Let e_k be the sequence of functions with properties (2.12) and let

$$u_{k,t}(x) = p_t * \check{e}_k(x), \qquad u_{k,0} = e_k.$$

By theorem 3.4 (e), $u_{k,t}$ is a non-negative function in $L_1(G)$. Since, for $m = 1, 2, \ldots$, the function φ_m is bounded,

$$\langle u_{k,t}, \varphi_m\rangle = \int u_{k,t}(x)\varphi_m(x)\,dx$$

is finite. By lemma 4.4, there exists a constant K, independent of m, such that

$$\langle u_{k,t}, |L\varphi_m|\rangle \leqslant K\langle u_{k,t}, \varphi_m\rangle. \tag{4.8}$$

Hence, since $u_{k,t}$ is non-negative,

$$|\langle u_{k,t}, L\varphi_m\rangle| \leqslant K\langle u_{k,t}, \varphi_m\rangle. \tag{4.9}$$

But, if ξ_n is the sequence of functions defined in section 2 which satisfies (2.14), we have

$$\begin{aligned}\langle u_{k,t}, X^2\varphi_m\rangle &= \lim_{n\to\infty}\langle \xi_n u_{k,t}, X^2\varphi_m\rangle \\ &= -\lim_{n\to\infty}\langle (X\xi_n)u_{k,t}, X\varphi_m\rangle - \lim_{n\to\infty}\langle \xi_n Xu_{k,t}, X\varphi_m\rangle \\ &= -\langle Xu_{k,t}, X\varphi_m\rangle.\end{aligned}$$

Applying this procedure once again, we see that

$$\langle u_{k,t}, L\varphi_m\rangle = \langle Lu_{k,t}, \varphi_m\rangle = \frac{d}{dt}\langle u_{k,t}, \varphi_m\rangle. \tag{4.10}$$

Now, by (4.9) and (4.10), we get

$$\left|\frac{d}{dt}\langle u_{k,t}, \varphi_m\rangle\right| \leqslant K\langle u_{k,t}, \varphi_m\rangle.$$

Consequently,

$$\langle u_{k,t}, \varphi_m\rangle \leqslant e^{tK}\langle u_{k,0}, \varphi_m\rangle = e^{tK}\langle e_k, \varphi_m\rangle.$$

Passing with k to infinity we obtain

$$\langle p_t, \varphi_m\rangle \leqslant e^{tK}\varphi_m(e) \leqslant e^{tK+2C},$$

which, as m tends to infinity, gives (4.7) and completes the proof of proposition 4.1.

5. The subalgebra of $L_1(G)$ generated by the p_t's. Let $\mathscr{A}$ denote the linear span (over C) of the functions $p_t, t > 0$. Then, since $p_s * p_t = p_{s+t}$ $(s, t > 0)$ and $p_t^* = p_t$ $(t > 0)$, $\mathscr{A}$ is a $*$-subalgebra of $L_1(G)$. It follows immediately from proposition 4.1 that if $f \in \mathscr{A}$ and φ is a submultiplicative function on G, then

$$\|f\|_\varphi = \int |f(x)|\varphi(x)\,dx < \infty. \tag{5.1}$$

Of course, since φ is submultiplicative,

$$\|f*g\|_\varphi \leqslant \|f\|_\varphi \|g\|_\varphi \quad \text{for all } f, g \text{ in } \mathscr{A}. \tag{5.2}$$

Let A be the closure in the norm $\|\cdot\|_1$ of $\mathscr{A}$.

PROPOSITION 5.1. *Let, for a $\lambda > 0$, k_λ be the function defined in corollary* 3.5, *and let for a $t > 0$*

$$P^t(x) = (\pi)^{-1/2} \int_0^\infty \lambda^{-1/2} e^{-\lambda} p_{t^2/4\lambda}\,d\lambda. \tag{5.3}$$

Then P^t and k_λ belong to A.

Proof. By (3.9) we have

$$k_\lambda = \int_0^\infty e^{-t\lambda} p_t\,dt, \tag{5.4}$$

so it suffices to note that both integrals (5.3) and (5.4) are convergent in the L_1-norm.

Remark 5.2. The function P^t is called the *Poisson kernel* and for every function f in $L_p(G)$, $1 \leqslant p < \infty$, the function

$$u(t, x) = P^t * f(x)$$

satisfies the Laplace equation

$$\frac{\partial^2}{\partial t^2} u(t, x) + L_x u(t, x) = 0.$$

Moreover,

$$P^{t*} = P^t, \quad t > 0,$$

$$P^s * P^t = P^{s+t}, \quad s, t > 0$$

and for every $1 \leqslant p < \infty$ and f in $L_p(G)$ we have

$$\lim_{t \to 0} \|P^t * f - f\|_p = 0.$$

For these and many other properties of the Poisson kernel see, e.g., Stein [18].

For a commutative Banach algebra $\boldsymbol{B}$ we denote by $\mathfrak{M}_{\boldsymbol{B}}$ the space of regular maximal ideals of $\boldsymbol{B}$, and for each x in $\boldsymbol{B}$ we write $\hat{x}$ for its Gelfand transform.

A Banach *-algebra $\boldsymbol{B}$ is called *symmetric* if the spectrum of each element x^*x is real non-negative.

Let us list here several simple and well-known properties of symmetric Banach *-algebras (cf., e. g., [17]).

A Banach *-algebra is symmetric if and only if there is a *-representation T of $\boldsymbol{B}$ into the algebra of bounded operators on a Hilbert space such that

$$\nu(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \|T_x\| \quad \text{for all } x = x^* \text{ in } \boldsymbol{B}. \tag{5.5}$$

We note that the inequality

$$\nu(x) \geqslant \|T_x\| \tag{5.6}$$

is always satisfied for hermitian elements x in $\boldsymbol{B}$ and any *-representation T.

We say that a locally compact group G is *amenable* if

$$\|T_x\| \leqslant \|R_x\| \quad \text{for all } x = x^* \text{ in } L_1(G), \tag{5.7}$$

where T is any *-representation of $L_1(G)$ and R is the regular representation of $L_1(G)$ on $L_2(G)$.

It is not difficult to prove that (5.5) implies that for every hermitian element x in $\boldsymbol{B}$ we have

$$\mathrm{Sp}_{\boldsymbol{B}} x = \mathrm{Sp}\, T_x \tag{5.8}$$

(cf., e. g., [9]).

If $\boldsymbol{B}$ is a commutative Banach *-algebra, then $\boldsymbol{B}$ is symmetric if and only if, for every element x in $\boldsymbol{B}$, we have $\widehat{x^*} = \bar{\hat{x}}$.

If a commutative Banach *-algebra is symmetric, then the set of functions $\{\hat{x}\colon x \in \boldsymbol{B}\}$ is dense in the space of continuous functions vanishing at infinity on $\mathfrak{M}$.

A Banach *-algebra is symmetric if and only if all the commutative Banach *-subalgebras of $\boldsymbol{B}$ are symmetric.

PROPOSITION 5.3. *If A is symmetric and G is amenable, then for all $1 \leqslant p < \infty$ we have*

$$\mathrm{Sp}_2 L = \mathrm{Sp}_p L,$$

where by $\mathrm{Sp}_p L$ we mean the complement of the set of complex numbers λ such that $(\lambda - L)^{-1}$ is a bounded operator on $L_p(G)$.

First we prove

LEMMA 5.4. *Let M be a measure space and let X be a dense subspace of all $L_p(M)$, $1 \leqslant p < \infty$. Suppose A is an operator defined on X such that A maps X into itself and is essentially self-adjoint on $L_2(M)$. If* $\mathrm{Sp}_1 A = \mathrm{Sp}_2 A$, *then* $\mathrm{Sp}_p A = \mathrm{Sp}_2 A$ *for all* p, $1 \leqslant p < \infty$.

Proof. Suppose $\lambda \notin \mathrm{Sp}_1 A = \mathrm{Sp}_2 A$. Then $R_\lambda = (\lambda - A)^{-1}$ is bounded on both $L_1(M)$ and $L_2(M)$ and so, by Riesz-Thorin interpolation theorem, R_λ is bounded on $L_p(M)$, $1 \leqslant p \leqslant 2$. Hence

$$\mathrm{Sp}_p A \subset \mathrm{Sp}_2 A \quad \text{for } 1 \leqslant p \leqslant 2.$$

This shows that $\mathrm{Sp}_p A$ is real for $1 \leqslant p \leqslant 2$. Since A is symmetric on $L_2(M)$, we have

$$Af = A'f \quad \text{for } f \text{ in } X, \tag{5.9}$$

whence, by (2.1) (b), A has a closed extension on $L_p(M)$ for every p, $1 \leqslant p < \infty$. Consequently, by a theorem in [19], p. 225,

$$\mathrm{Sp}_p A = \mathrm{Sp}_q A', \quad \text{where } q = p(p-1)^{-1}.$$

Hence, in virtue of (5.9), we have

$$\mathrm{Sp}_p A = \mathrm{Sp}_q A, \quad 1 < p \leqslant 2. \tag{5.10}$$

But, since $1 < p \leqslant 2$ implies $2 \leqslant q < \infty$, (5.10) together with the Riesz-Thorin interpolation theorem give

$$\mathrm{Sp}_2 A \subset \mathrm{Sp}_p A = \mathrm{Sp}_q A,$$

which completes the proof of the lemma.

Proof of proposition 5.2. In virtue of (5.5), (5.7) and (5.8), if G is amenable and A is symmetric, then for every element $f = f^*$ of A we have

$$\mathrm{Sp}_A f = \mathrm{Sp}\, R_f, \tag{5.11}$$

where $R_f g = f * g$, $g \in L_2(G)$. Since $\mathrm{Sp}_A f$ is then real non-negative, i. e. does not separate the complex plane, (5.11) yields

$$\mathrm{Sp}_{L_1(G)} f = \mathrm{Sp}_A f = \mathrm{Sp}\, R_f. \tag{5.12}$$

But (5.12) applied to the function p_t, $t > 0$, gives

$$\mathrm{Sp}_1 T^t = \mathrm{Sp}_2 T^t \quad \text{for all } t > 0,$$

since the spectrum of p_t in the Banach algebra $L_1(G)$ is equal to the spectrum of T^t considered as an operator on $L_1(G)$. Hence, by lemma 5.4, for all p, $1 \leqslant p < \infty$, we have

$$\mathrm{Sp}_p T^t = \mathrm{Sp}_2 T^t \quad \text{for all } t > 0. \tag{5.13}$$

Now, by [7], p. 457, Corollary 2,

$$\exp[t\mathrm{Sp}_p L] \subset \mathrm{Sp}_p T^t = \mathrm{Sp}_2 e^{t\Delta} \quad \text{for all } t > 0.$$

In virtue of the spectral theorem and (2.5), since Δ is non-positive,

$$\mathrm{Sp}_2 e^{t\Delta} = \exp[t\mathrm{Sp}_2 \Delta]^- = \exp[t\mathrm{Sp}_2\Delta]\cup\{0\}.$$

Thus

$$\exp[t\mathrm{Sp}_p L] \subset \exp[t\mathrm{Sp}_2 \Delta] \quad \text{for all } t > 0,$$

whence, since $t\mathrm{Sp}_2 \Delta$ is real for all $t > 0$,

$$\mathrm{Sp}_p L \subset \mathrm{Sp}_2 L, \quad 1 \leqslant p < \infty. \tag{5.14}$$

To prove the converse inclusion, suppose that $\lambda \notin \mathrm{Sp}_1 L$. Then $R_\lambda = (\lambda - L)^{-1}$ is a bounded operator on $L_1(G)$ which commutes with the right translations. Therefore R_λ is a multiplier on $L_1(G)$ and so $R_\lambda f = \mu * f$ for all f in $L_1(G)$, where μ is a bounded measure on G. Consequently, R_λ is bounded on all $L_p(G)$, $1 \leqslant p < \infty$, and so $\lambda \notin \mathrm{Sp}_p L$. This, together with (5.14) implies

$$\mathrm{Sp}_1 L = \mathrm{Sp}_2 L$$

and, by lemma 5.4, completes the proof of proposition 5.3.

A locally compact group G is called of *polynomial growth* if for every compact set U there exists a natural number r such that

$$|U^n| = O(n^r) \quad \text{as } n \text{ tends to infinity}. \tag{5.15}$$

If G is connected, then the minimal natural number r for which (5.15) is satisfied is the same for all compact sets with non-void interior; the number r is then called the *degree of growth* in G_0. For a characterization of connected Lie groups with polynomial growth see [11]. Compact extensions of nilpotent groups are of polynomial growth. A group of polynomial growth is amenable (cf. [8]).

PROPOSITION 5.5. *If G is of polynomial growth, then A is symmetric.*

Proof. First we are going to show that for every $f = f^*$ in $\mathscr{A}$ we have

$$\nu(f) \leqslant \|R_f\|, \tag{5.16}$$

where $R_f g = f * g$, $g \in L_2(G)$. To do this we let U to be a compact symmetric neighbourhood of e in G and we let τ_U be as in proposition 1.3. We put

$$\varphi(x) = \exp \tau_U(x) \quad \text{for } x \in G.$$

Then for an arbitrary m we have

$$\|f^{*n}\|_1 = \int_{U^m} |f^{*n}(x)|\,dx + \int_{G\setminus U^m} |f^{*n}(x)|\varphi(x)\varphi(x)^{-1}dx$$
$$\leqslant |U^m|^{1/2}\|f^{*n}\|_2 + \|f\|_\varphi^n$$
$$\leqslant Cm^{r/2}\|R_f\|^{n-1}\|f\|_2 + \|f\|_\varphi^n.$$

Putting $m = n^2$, for n large enough we get $\|f^{*n}\|_1 \leqslant C'n^r\|R_f\|^n$, whence

$$\nu(f) = \lim_{n\to\infty} \|f^{*n}\|^{1/n} \leqslant \|R_f\|.$$

Now since hermitian elements of $\mathscr{A}$ form a dense subset in the set of hermitian elements of A and both $\nu\colon A \ni f \to \nu(f) \in \boldsymbol{R}$ and $A \ni f \to \|R_f\| \in \boldsymbol{R}$ are continuous functions (because A is commutative), inequality (5.16) holds for all hermitian elements in A and so, by (5.6), equality (5.5) follows, whence A is symmetric.

From now on we shall consider only Lie groups G of polynomial growth. For such G the algebra A is symmetric and, as we are going to see soon, it is regular and has the Tauber-Wiener property. To show this let us recall first few notions.

We say that a function $F\colon \boldsymbol{R} \to \boldsymbol{C}$ *operates* on an element f in a semi-simple commutative Banach algebra $\boldsymbol{B}$, if $\hat{f}$ is a real-valued function on $\mathfrak{M}_{\boldsymbol{B}}$ and there exists (necessarily unique) element g in $\boldsymbol{B}$ such that

$$F(\hat{f}(M)) = \hat{g}(M) \quad \text{for all } M \text{ in } \mathfrak{M}_{\boldsymbol{B}}.$$

We then write $g = F\circ f$.

If $\boldsymbol{B}$ does not contain the unit, then only functions F for which $F(0) = 0$ can operate on $\boldsymbol{B}$.

Let

$$e(t) = e^{it} - 1 = \sum_{1}^{\infty} \frac{(it)^k}{k!}.$$

For an element f in a Banach algebra $\boldsymbol{B}$ we write

$$e(f) = \sum_{1}^{\infty} \frac{(if)^k}{k!}.$$

The following proposition and its proof are well-known (cf., e. g., Kahane [13]). We include it here for completeness sake.

PROPOSITION 5.6. *If for a Banach algebra $\boldsymbol{B}$ and an element f in $\boldsymbol{B}$ such that f is a real-valued function we have*

(5.17) $$e(nf) = O(n^k) \quad \textit{as } n \textit{ tends to infinity},$$

then the functions F in $C_c^{k+2}(\boldsymbol{R})$ such that $F(0) = 0$ operate on f.

Proof. Let

$$F \in C_c^{k+2}(\boldsymbol{R}),\ F(0) = 0 \text{ and } F(s) = 0 \text{ for } |s| > a.$$

Then, of course, if

$$\hat{F}(n) = (2a)^{-1} \int_{-a}^{a} f(s) \exp\left(-i\frac{\pi}{a} ns\right) ds,$$

then

$$\hat{F}(n) = O(n^{-(k+2)}) \quad \text{as } n \text{ tends to infinity}$$

and

$$\sum_{-\infty}^{+\infty} \hat{F}(n) = F(0) = 0. \tag{5.18}$$

In virtue of (5.18) we have

$$F(s) = \sum_{-\infty}^{+\infty} \hat{F}(n) e\left(n\frac{\pi}{a} s\right).$$

Let

$$g = \sum_{-\infty}^{+\infty} \hat{F}(n) e\left(n\frac{\pi}{a} f\right). \tag{5.19}$$

By assumption,

$$\hat{F}(n) \left\| e\left(n\frac{\pi}{a} f\right) \right\| = O(n^{-2}),$$

and so the series (5.19) is absolutely convergent. Moreover, for every M in $\mathfrak{M}_B$ we have

$$\hat{g}(M) = \sum_{-\infty}^{+\infty} \hat{F}(n) e\left(n\frac{\pi}{a} \hat{f}(M)\right) = F(\hat{f}(M)).$$

If G is of polynomial growth and a function $f^* = f$ in $L_1(G)$ has compact support, then there is an r such that $C_c^r(\boldsymbol{R})$ functions operate on f. This was proved by Dixmier [2]. Here we need a stronger version of this theorem.

PROPOSITION 5.7. *If G is of polynomial growth of degree r, then the unc tions F such that*

$$F \in C_c^{(r/2)+3}(\boldsymbol{R}) \quad \textit{and} \quad F(0) = 0$$

operate on functions $f = f^*$ of $\mathscr{A}$ into $\boldsymbol{A}$.

Proof. Since A is symmetric, it suffices to show that for a function $f = f^*$ in $\mathscr{A}$ we have

(5.20) $$\|e(nf)\|_1 = O(n^{(r/2)+1}) \quad \text{as } n \to \infty.$$

Let U be a compact symmetric neighbourhood of e in G and let τ_U be defined as in proposition 1.3. Let $\varphi(x) = \exp \tau_U(x)$. We then have

$$\|e(nf)\|_\varphi = \left\| \sum_1^\infty \frac{(inf)^{*k}}{k!} \right\|_\varphi \leqslant \sum_1^\infty \frac{\|nf\|_\varphi^k}{k!} \leqslant \exp[n\|f\|_\varphi].$$

Hence for each m

(5.21) $$\begin{aligned} \|e(nf)\|_1 &= \int_{U^m} |e(nf)(x)|\,dx + \int_{G\setminus U^m} |e(nf)(x)|\varphi(x)\varphi(x)^{-1}\,dx \\ &\leqslant |U^m|^{1/2}\|e(nf)\|_2 + \sup\{\varphi(x)^{-1}: x \in U^m\}\exp[n\|f\|_\varphi] \\ &\leqslant Cm^{r/2}\|e(nf)\|_2 + \exp[n\|f\|_\varphi - m], \end{aligned}$$

where C depends on U and G only. But

$$\|e(nf)\|_2 \leqslant \|T\|\,\|f\|_2,$$

where T is the operator on $L_2(G)$ defined by

$$T = inI + \sum_{k=1}^\infty [(k+1)!]^{-1}(inR_f)^k$$

and

$$R_f g = f * g, \qquad g \in L_2(G).$$

Since $f = f^*$, T is normal and, by the spectral theorem,

$$\|T\| \leqslant \sup\{|e^{int}-1|\,|t|^{-1}: t \in \boldsymbol{R}\} = n.$$

Thus putting $m = n\|f\|_\varphi$ in (5.21) we obtain (5.20).

A commutative semi-simple Banach algebra $\boldsymbol{B}$ is called *regular* if for every closed subset C in $\mathfrak{M}_{\boldsymbol{B}}$ and a point $M_0 \notin C$ there is an element g in $\boldsymbol{B}$ such that

$$\hat{g}(M_0) = 1 \text{ and } \hat{g}(M) = 0 \quad \text{for all } M \text{ in } C.$$

PROPOSITION 5.8. *If G is of polynomial growth, then A is regular.*

Proof. Since $\mathscr{A}$ is dense in A and A is symmetric, for every compact set C in $\mathfrak{M}_A$ and a point $M_0 \notin C$ there is a function f in $\mathscr{A}$ such that $\hat{f}(M_0) = 1$ and $|\hat{f}(M)| < 1/2$ for all M in C. Let

(5.22) $$F \in C_c^\infty(-\pi, \pi), \quad F(1) = 1, \quad F(s) = 0 \quad \text{for } |s| < 1/2.$$

Then, by proposition 5.7, there is a g in A such that

$$F(\hat{f}(M)) = \hat{g}(M) \quad \text{for all } M \text{ in } \mathfrak{M}_A.$$

We see that $\hat{g}(M_0) = 1$ and $\hat{g}(M) = 0$ for M in C, as required.

PROPOSITION 5.9. *Let G be of polynomial growth. Let further a function F satisfies* (5.22) *and let*

$$q_t = F \circ p_t, \quad t > 0.$$

Then $\operatorname{supp} \hat{q}_t$ *is compact and*

$$\lim_{t \to 0} \|q_t * h - h\|_1 = 0 \quad \textit{for all } h \textit{ in } \mathscr{A}. \tag{5.23}$$

Proof. Since $F(0) = 0$ and $F(1) = 1$, we have

$$1 = \sum_{-\infty}^{\infty} \hat{F}(n) e^{in} = \sum_{-\infty}^{\infty} \hat{F}(n) e(n).$$

Consequently,

$$q_t * h - h = \sum_{-\infty}^{\infty} \hat{F}(n) e(np_t) * h - h$$
$$= \sum_{-\infty}^{\infty} \hat{F}(n) [e(np_t) * h - e(n) h],$$

whence

$$\|q_t * h - h\|_1 \leqslant \sum_{-\infty}^{\infty} \|\hat{F}(n) e(np_t) * h - e(n) h\|_1. \tag{5.24}$$

Now repeating the argument of the proof of proposition 5.7 with $e(np_t) * h$ in place of $e(nf)$ we see, by (5.21), that for all natural numbers m we have

$$\begin{aligned} \|e(np_t) * h\|_1 &\leqslant Cm^{r/2} \|e(np_t) * h\|_2 + \exp[n \|p_t\|_\varphi - m] \cdot \|h\|_\varphi \\ &\leqslant Cm^{r/2} \|S\| \|h\|_2 + \exp[n \|p_t\|_\varphi - m] \cdot \|h\|_\varphi, \end{aligned} \tag{5.25}$$

where S is the operator on $L_2(G)$ defined by

$$Sf = e(np_t) * f, \quad f \in L_2(G).$$

Consequently, since $p_t = p_t^*$, we get, by the spectral theorem,

$$\|S\| \leqslant \sup\{|e^{int} - 1|: t \in \boldsymbol{R}\} = 2.$$

In virtue of proposition 4.1, for $t \leqslant t_0$ there is a constant d (independent of $t < t_0$) such that $\|p_t\|_\varphi \leqslant d$ for $t \leqslant t_0$. Putting $m = (d+1)n$ in (5.25), we get

$$\|e(np_t) * h\|_1 \leqslant Kn^{r/2},$$

where K does not depend on $t \leqslant t_0$ and n. This shows that since $F \in C_c^{\infty}(-\pi, \pi)$, the series (5.24) is convergent absolutely and uniformly with respect to $t \in (0, t_0]$. Thus to prove proposition 5.9 it suffices to show that

$$\lim_{t\to 0} \|e(np_t) * h - e(n)h\|_1 = 0 \quad \text{for each } n.$$

But

$$\|e(np_t) * h - e(n) * h\|_1 \leqslant \sum_{k=1}^{\infty} n^k (k!)^{-1} \|p_{tk} * h - h\|_1$$

and this series again is convergent uniformly with respect to $t \in \boldsymbol{R}^+$, since $\|p_{tk} * h - h\|_1 \leqslant 2\|h\|_1$. Consequently,

$$\lim_{t\to 0} \|e(np_t) * h - e(n)h\|_1 \leqslant \sum_{k=1}^{\infty} n^k (k!)^{-1} \lim_{t\to 0} \|p_{tk} * h - h\|_1 = 0,$$

which completes the proof of proposition 5.9.

COROLLARY 5.10. *The set of elements f in A such that* supp $\hat{f}$ *is compact, is dense in A.*

Proof. We have just shown that the set $\{q_t * h : h \in \mathscr{A}, t > 0\}$ is dense in $\mathscr{A}$, so it is dense in A. On the other hand, since $(q_t * h)^{\wedge} = \hat{q}_t \hat{h}$, the support of $(q_t * h)^{\wedge}$ is compact.

A commutative Banach algebra $\boldsymbol{B}$ has the Tauber-Wiener property if every ideal I of $\boldsymbol{B}$ is contained in a maximal regular ideal.

The following theorem is well-known (cf., e. g., [17]):

If a commutative semi-simple symmetric Banach algebra is regular and the set of elements x in $\boldsymbol{B}$ such that supp $\hat{x}$ *is compact is dense in $\boldsymbol{B}$, then $\boldsymbol{B}$ has the Tauber-Wiener property.*

Thus proposition 5.8 and corollary 5.10 yield the following

THEOREM 5.11. *If G has polynomial growth, then the algebra A has the Tauber-Wiener property.*

From this we deduce

THEOREM 5.12. *If G is a Lie group of polynomial growth, then for every $t > 0$ each of the functions p_t and P^t is cyclic in $L_1(G)$.*

Proof. By proposition 5.1 and remark 5.2, theorem 5.12 follows immediately from the following

LEMMA 5.13. *Let $\{Q_t\}$, $t > 0$, be a family of functions in A such that*

$$Q_t^* = Q_t \quad \text{for all } t > 0,$$
$$Q_s * Q_t = Q_{s+t} \quad \text{for all } s, t > 0,$$
$$\lim_{t\to 0} \|f * Q_t - f\|_1 = 0 \quad \text{for all } f \text{ in } L_1(G).$$

Then for each $t_0 > 0$ the function Q_{t_0} is cyclic in $L_1(G)$.

Proof. Let J be the closure of $L_1(G)*Q_{t_0}$. Then J is a closed left ideal in $L_1(G)$. Suppose $J \neq L_1(G)$. Then, since for every sequence $t_j \to 0$, Q_{t_j} form an approximate identity in $L_1(G)$, there is an $s_0 > 0$ such that, for $s < s_0$, the functions Q_s do not belong to J. Consequently, $I = J \cap A$ is a proper closed ideal in A. By theorem 5.11, there is a homomorphism $\chi \neq 0$ of A into complex numbers such that $\chi(I) = 0$. Consequently,

$$\chi(Q_{t_0}) = 0.$$

But then for all natural numbers n

$$\chi(Q_{t_0/n})^n = \chi(Q_{t_0}) = 0,$$

which shows that all functions $Q_{t_0/n}$ $(n = 1, 2, \ldots)$ belong to $M = \{f\colon \chi(f) = 0\}$, which is impossible because M is a proper ideal in A and $Q_{t_0/n}$ $(n = 1, 2, \ldots)$ form an approximate identity.

From theorem 5.12 we deduce a theorem of tauberian type for solutions of the heat equation (or the Laplace equation) on a Lie group of polynomial growth in the same manner as the classical Wiener's tauberian theorem is deduced from the fact that $L_1(\boldsymbol{R})$ has the Tauber-Wiener property (cf., e. g., [14], p. 228-229).

THEOREM 5.14. *Suppose for an f in $L_\infty(G)$ the function $u(x, t)$ satisfies the heat equation*

$$L_x u(x, t) = \frac{\partial}{\partial t} u(x, t)$$

with the boundary condition

$$u(x, 0) = f(x), \quad x \in G, \text{ and } \|u(\cdot, t)\|_\infty \leqslant \|f\|_\infty.$$

If G is of polynomial growth, then the following implication holds:

(5.26) *if, for a t_0, $\lim\limits_{x\to\infty} u(x, t_0)$ exists and equals a, then for every g in $L_1(G)$*

$$\lim_{x\to\infty} g*f(x) = a\int g(x)\,dx.$$

Similarly, we have

THEOREM 5.14'. *Suppose that for an f in $L_\infty(G)$ the function $u(x, t)$ satisfies the Laplace equation*

$$L_x u(x, t) = -\frac{\partial^2}{\partial t^2} u(x, t)$$

with the boundary condition

$$u(x, 0) = f(x), \quad x \in G, \text{ and } \|u(\cdot, x)\|_\infty \leqslant \|f\|_\infty.$$

If G is of polynomial growth, then the following implication holds

(5.27) *if, for a t_0, $\lim_{x\to\infty} u(x, t_0)$ exists and equals a, then for every g in $L_1(G)$*

$$\lim_{x\to\infty} g * f(x) = a \int g(x)\,dx.$$

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