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# COMPLETE RINGS OF FUNCTIONS AND WALLMAN–FRINK COMPACTIFICATIONS

BY

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**1. Introduction.** In [4] Frink generalized Wallman's method of compactification. Frink used a normal base of closed sets instead of the family of all closed sets as employed by Wallman. Frink posed the question whether all compactifications are obtainable by using suitable normal bases. He also asked which compactifications can be obtained from normal bases consisting of zero-sets. Frink observed that each compactification $cX$ of $X$ determines a normal base for $X$, consisting of the zero-sets of those continuous real-valued functions defined on $X$ which may be extended to continuous functions over $cX$. This theorem was proved by Biles in [1]. In [4] Frink also cited an example showing that Wallman's method applied to the zero-sets of functions continuously extendable to $cX$ does not necessarily yield $cX$ (see also [2], Major Problem 8.12).

In this paper, we shall be primarily concerned with those compactifications $cX$ of a space $X$ for which the Wallman space determined by the family of all zero-sets of real-valued functions continuously extendable to $cX$ is a compactification of $X$ equivalent to $cX$. Next, we consider the spaces $X$ for which every compactification $cX$ can be obtained as the Wallman space determined by the zero-sets of continuous real-valued functions defined on $X$ and continuously extendable to $cX$, and the spaces $X$ for which only $\beta X$ is obtainable in this manner. Most of our theorems are formulated in the language of rings of functions.

**2. Definitions and basic properties.** Let $X$ be a Tychonoff space. The ring of all continuous functions from the space $X$ into the space $\boldsymbol{R}$ of real numbers is denoted by $C(X)$ and its subring of all bounded functions is denoted by $C^*(X)$.

A subring $P$ of $C^*(X)$ is called a *complete ring* of functions on the space $X$ if $P$ contains all constant functions, separates points from closed sets (i.e., for any closed set $F$ and for any $x \in X \setminus F$ there is an $f \in P$ such that

$f(x) \notin \mathrm{cl}_R f(F)$), and is closed with respect to uniform convergence. The family of all complete rings of functions on the space $X$ will be denoted by $P(X)$.

Let $K(X)$ be the family of all classes consisting of equivalent compactifications of $X$. For compactifications $c_1 X$ and $c_2 X$, we say that $c_1 X \leqslant c_2 X$ if there is a continuous mapping $q: c_2 X \to c_1 X$ for which $q \circ c_2 = c_1$, where $c_1$ and $c_2$ are homeomorphic embeddings of $X$ in the corresponding compactifications. By assigning to any compactification $cX$ the ring $P \in P(X)$ consisting of all functions continuously extendable to $cX$, we establish an isomorphism of the partially ordered set $(K(X), \leqslant)$ onto the partially ordered set $(P(X), \subset)$.

For any $f \in C(X)$, the set $f^{-1}(0)$ is called the *zero-set* of $f$. If $P$ is a subring of $C(X)$, we put

$$Z[P] = \{f^{-1}(0): f \in P\};$$

$Z[C(X)] = Z[C^*(X)]$ is denoted, for simplicity, by $Z(X)$.

Let $P \in P(X)$. We shall denote by $wZ[P]$ the space of ultrafilters of $Z[P]$ with the Wallman–Frink topology (see [4] or [2], Section 8). From Theorems 2.6 and 3.9 of [1] (see also [5], Problem 7M, and [3], Problem 3.12.21(e)) we obtain

**2.1.** THEOREM. *For each $P \in P(X)$, $wZ[P] \in K(X)$. Moreover, for the compactification $cX$ corresponding to the ring $P$, we have $cX \leqslant wZ[P]$.*

If $P \in P(X)$, then the compactification $wZ[P]$, in general, is not equivalent to the compactification of $X$ corresponding to the ring $P$. R. M. Brooks observed that if $X = N$ is the space of positive integers, $\omega X$ is the one-point compactification of $X$ and $P \in P(X)$ is the ring corresponding to $\omega X$, then $wZ[P] = \beta X \neq \omega X$ (see [4] or [2], Major Problem 8.12). The next theorem is an immediate consequence of Theorems 2.6 and 3.8 of [1].

**2.2.** THEOREM. *Suppose that $P \in P[X]$ and $cX$ is the compactification corresponding to the ring $P$. Then $cX = wZ[P]$ if and only if*

$$\mathrm{cl}_{cX} Z_1 \cap \mathrm{cl}_{cX} Z_2 = \emptyset$$

*whenever $Z_1 \cap Z_2 = \emptyset$ and $Z_1, Z_2 \in Z[P]$.*

The proof of the following lemma can be obtained by a slight modification of the proof of Corollary 3.6.2 in [3].

**2.3.** LEMMA. *For each $P \in P(X)$ and any disjoint subsets $A$ and $B$ of $X$, the following conditions are equivalent:*

(i) $\mathrm{cl}_{cX} A \cap \mathrm{cl}_{cX} B = \emptyset$, *where $cX$ is the compactification of $X$ corresponding to the ring $P$;*

(ii) *there exists a function $f \in P$ such that*

$$A \subset f^{-1}(0) \quad \text{and} \quad B \subset f^{-1}(1).$$

Two subsets $A$ and $B$ of $X$ are said to be *separated* by the ring $P \in P(X)$ if there exists a function $f \in P$ such that $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$; we then say that $P$ *separates* $A$ and $B$. Subsets $A$ and $B$ of $X$ separated by the ring $C^*(X)$ are called *completely separated.*

Denote by $B(X)$ the subfamily of $P(X)$ consisting of all rings $P \in P(X)$ with the property that any disjoint sets $Z_1, Z_2 \in Z[P]$ are separated by the ring $P$. Let us observe that the family $B(X)$ is non-empty because $C^*(X) \in B(X)$.

Theorem 2.2 and Lemma 2.3 imply

2.4. COROLLARY. *Let $P \in P(X)$ and let $cX$ be the compactification of $X$ corresponding to the ring $P$. Then $cX = wZ[P]$ if and only if $P \in B(X)$.*

We shall now establish an algebraic characterization of the ring $P_w \in P(X)$ corresponding to the compactification $wZ[P]$. Before doing this, however, we shall prove two lemmas.

2.5. LEMMA. *Let $X$ be a dense subspace of a topological space $T$ and $f$ a continuous mapping of $X$ into a compact space $Y$. The mapping $f$ has a continuous extension over $T$ if and only if there exists a base $\mathscr{F}$ for closed sets in $Y$ satisfying the conditions:*

(i) *$\mathscr{F}$ is closed under finite intersections;*

(ii) $\mathrm{cl}_T f^{-1}(F_1) \cap \mathrm{cl}_T f^{-1}(F_2) = \emptyset$ *for each pair $F_1$, $F_2$ of disjoint sets from $\mathscr{F}$.*

Proof. The necessity is obvious.

Sufficiency. Let $B_1$ and $B_2$ be disjoint closed subsets of $Y$. By Theorem 3.2.1 of [3], it suffices to prove that

$$\mathrm{cl}_T f^{-1}(B_1) \cap \mathrm{cl}_T f^{-1}(B_2) = \emptyset.$$

We then have

$$B_1 = \bigcap_{s \in S_1} F_s \quad \text{and} \quad B_2 = \bigcap_{t \in S_2} F_t,$$

where $F_s$, $F_t \in \mathscr{F}$ for $s \in S_1$ and $t \in S_2$. Since

$$B_1 \subset Y \setminus B_2 = \bigcup_{t \in S_2} (Y \setminus F_t),$$

it follows from the compactness of $B_1$ that there is a finite set $\{t_1, t_2, \ldots, t_k\} \subset S_2$ such that

$$B_1 \subset \bigcup_{i=1}^{k} (Y \setminus F_{t_i}) = Y \setminus \bigcap_{i=1}^{k} F_{t_i}.$$

Moreover,

$$B_2 \subset \bigcap_{i=1}^{k} F_{t_i} \subset Y \setminus B_1 = \bigcup_{s \in S_1} (Y \setminus F_s).$$

The set $\bigcap_{i=1}^{k} F_{t_i}$ is compact; hence there exists a finite set $\{s_1, s_2, \ldots, s_l\} \subset S_1$ such that

$$\bigcap_{i=1}^{k} F_{t_i} \subset \bigcup_{j=1}^{l} (Y \setminus F_{s_j}) = Y \setminus \bigcap_{j=1}^{l} F_{s_j}.$$

We put

$$F_1 = \bigcap_{j=1}^{l} F_{s_j} \quad \text{and} \quad F_2 = \bigcap_{i=1}^{k} F_{t_i}.$$

By hypothesis, $F_1, F_2 \in \mathcal{F}$ and

$$\mathrm{cl}_T f^{-1}(F_1) \cap \mathrm{cl}_T f^{-1}(F_2) = \emptyset.$$

Since $f^{-1}(B_i) \subset f^{-1}(F_i)$ for $i = 1, 2$, we have

$$\mathrm{cl}_T f^{-1}(B_1) \cap \mathrm{cl}_T f^{-1}(B_2) = \emptyset.$$

Let us note that the assumption that the base $\mathcal{F}$ is closed under finite intersections is essential in the last lemma.

2.6. Example. Denote by $T$ the interval $[0, 1]$ with the natural topology; let $X = T \setminus \{\frac{1}{2}\}$, and $Y = \{1, 2, 3\}$ with the discrete topology. Let

$$f([0, \tfrac{1}{2})) = \{1\} \quad \text{and} \quad f((\tfrac{1}{2}, 1]) = \{2\}.$$

The family

$$\mathcal{F} = \{\{2, 3\}, \{1, 3\}, \{1, 2\}, \{1, 2, 3\}\}$$

is a base for closed sets in $Y$ (that is not closed under finite intersections) and the implication

$$(F_1, F_2 \in \mathcal{F} \wedge F_1 \cap F_2 = \emptyset) \Rightarrow \mathrm{cl}_T f^{-1}(F_1) \cap \mathrm{cl}_T f^{-1}(F_2) = \emptyset$$

is valid. However, the mapping $f$ is not continuously extendable over $T$.

We shall now show that the ring corresponding to the compactification $wZ[P]$ is the least element of $P(X)$ (with respect to the partial order on the family $P(X)$ given by inclusion) which separates any two disjoint members of $Z[P]$.

For each $P \in P(X)$, let $S(P)$ be the collection of all rings $S \in P(X)$ such that $S$ separates each pair of disjoint elements of $Z[P]$, and let $P_w \in P(X)$ be the ring corresponding to the compactification $wZ[P]$.

2.7. Lemma. *For each* $P \in P(X)$,

$$P_w \in S(P) \quad \textit{and} \quad P_w = \bigcap \{S : S \in S(P)\}.$$

Proof. Let $W = wZ[P]$. It is known that

$$\mathrm{cl}_W Z_1 \cap \mathrm{cl}_W Z_2 = \mathrm{cl}_W (Z_1 \cap Z_2) \quad \text{whenever } Z_1, Z_2 \in Z[P];$$

from Lemma 2.3 we infer that

$$P_w \in S(P) \quad \text{and} \quad \bigcap \{S\colon S \in S(P)\} \subset P_w.$$

Now, let $S \in S(P)$ and let $sX$ be the compactification corresponding to the ring $S$. Consider the embedding $i_X$ of the subspace $X$ in the space $W$. We shall check that $i_X$ has a continuous extension over $S$.

The family $\{\mathrm{cl}_W Z\colon Z \in Z[P]\}$ is a base for closed sets in $W$ and is closed under finite intersections. For any $Z_1, Z_2 \in Z[P]$, if $\mathrm{cl}_W Z_1 \cap \mathrm{cl}_W Z_2 = \emptyset$, then $Z_1 \cap Z_2 = \emptyset$; since $S \in S(P)$, by virtue of Lemma 2.3, we infer that

$$\mathrm{cl}_{sX} Z_1 \cap \mathrm{cl}_{sX} Z_2 = \emptyset.$$

From Lemma 2.5 it follows that the mapping $i_X$ is extendable to a continuous mapping of $sX$ onto $W$. Thus,

$$W \leqslant sX, \quad P_w \subset S \quad \text{and} \quad P_w \subset \bigcap \{S\colon S(P)\}.$$

For $E \subset C^*(X)$, let $[E]$ be the ring generated by $E$ (i.e., $[E]$ is the common part of all subrings of $C^*(X)$ containing $E$).

**2.8.** THEOREM. *For $P \in P(X)$ and*

$$E = \left\{\frac{|f|}{|f|+|g|}\colon f, g \in P \text{ and } f^{-1}(0) \cap g^{-1}(0) = \emptyset\right\},$$

*we have the equality $P_w = \overline{[E]}$, where $\overline{[E]}$ is the closure of $[E]$ in $C^*(X)$ with the topology of uniform convergence.*

Proof. By Lemma 2.7, it suffices to show that $\overline{[E]} \in S(P)$ and $E \subset S$ for each $S \in S(P)$.

Since $P$ contains all constant functions, every constant function belongs to $[E]$. The family $Z[P]$ is a base for closed sets in $X$; thus, if $H$ is a closed set in $X$ and $x \in X \setminus H$, then there is an $f \in P$ such that $H \subset f^{-1}(0)$ and $x \notin f^{-1}(0)$. The function $h = |f|/(|f|+1)$ is an element of $E$, and $h(x) \notin \mathrm{cl}_R h(H)$; hence $E$ separates points from closed sets. We conclude that $\overline{[E]} \in P(X)$, and this – together with the definition of $E$ – implies that $\overline{[E]} \in S(P)$.

Now, consider $S \in S(P)$ and let $sX$ be the compactification corresponding to the ring $S$. Let

$$k = \frac{|f|}{|f|+|g|},$$

where $f, g \in P$ and $f^{-1}(0) \cap g^{-1}(0) = \emptyset$, be an element of $E$. The family $\mathscr{F}$ of all finite unions of disjoint closed intervals contained in $[0, 1]$ is a base for closed sets in $[0, 1]$ and is closed under finite intersections. If $[a, b]$ is an interval, then the set

$$k^{-1}([a, b]) = \{x: |f(x)| - b|f(x)| - b|g(x)| \leqslant 0\}$$
$$\cap \{x: |f(x)| - a|f(x)| - a|g(x)| \geqslant 0\}$$

belongs to $Z[P]$ (see [5], 1.10 and 1.11). Thus, $k^{-1}(F) \in Z[P]$ for each $F \in \mathcal{F}$. Since $S \in S(P)$, from Lemmas 2.3 and 2.5 it follows that $k$ has a continuous extension over $sX$ and, consequently, $k \in S$ and $E \subset S$.

**2.9.** THEOREM. *For each $P \in P(X)$, we have the equality $Z[P] = Z[P_w]$.*

Proof. Let $\overline{[E]}$ be the ring defined in Theorem 2.8. We prove that $Z[\overline{[E]}] \subset Z[P]$.

If $h \in [E]$, then

$$h = \sum_{i=1}^{n} \prod_{j=1}^{m_i} \frac{|f_{i,j}|}{|f_{i,j}| + |g_{i,j}|} - \sum_{i=n+1}^{m} \prod_{j=1}^{m_i} \frac{|f_{i,j}|}{|f_{i,j}| + |g_{i,j}|}$$

for some $f_{i,j}$, $g_{i,j} \in P$ such that $f_{i,j}^{-1}(0) \cap g_{i,j}^{-1}(0) = \emptyset$ $(i = 1, 2, \ldots, m;$ $j = 1, 2, \ldots, m_i)$. Arguing similarly as in the proof of Theorem 2.8, one can easily see that for any real number $r$ the set $\{x: |h(x)| \leqslant r\}$ belongs to $Z[P]$.

If $h \in \overline{[E]}$, then there exists a sequence $(h_n)$ of functions from $[E]$ uniformly convergent to $h$. It can be assumed that

$$|h(x) - h_n(x)| \leqslant 1/n$$

for each $x \in X$ and $n = 1, 2, \ldots$ Let us observe that

$$\{x: h(x) = 0\} = \bigcap_{n=1}^{\infty} \{x: |h_n(x)| \leqslant 1/n\};$$

since $\{x: |h_n(x)| \leqslant 1/n\} \in Z[P]$ for $n = 1, 2, \ldots$, we have $h^{-1}(0) \in Z[P]$ (see [5], 1.14(a)). Therefore, $Z[\overline{[E]}] \subset Z[P]$.

From Theorems 2.1 and 2.8 it follows that $Z[P] \subset Z[\overline{[E]}]$. Thus, $Z[P] = Z[\overline{[E]}] = Z[P_w]$.

**2.10.** COROLLARY. *For each $P \in P(X)$,*

$$P_w \in B(X) \quad \text{and} \quad P_w = \bigcap \{S \in B(X): P \subset S\}.$$

Proof. The first part of the proposition follows from Theorem 2.9 and Lemma 2.7. But

$$\{S \in B(X): P \subset S\} \subset S(P),$$

and we complete the proof by applying Lemma 2.7.

**3. The family** $B(X)$. In this section, we are going to study the spaces $X$ for which $B(X) = P(X)$ and those for which $B(X) = \{C^*(X)\}$.

First, we present the following generalization of a well-known result concerning the Čech–Stone compactification:

**3.1.** THEOREM. *Let $P \in B(X)$ and let $cX$ be the compactification corresponding to the ring $P$. Each non-empty closed $G_\delta$-set in $cX$ disjoint from $X$ contains a copy of $\beta N$, and thus has at least cardinality $2^{\mathfrak{c}}$.*

Proof. Assume that $F \subset cX \setminus X$ is a non-empty closed $G_\delta$-set in $cX$. There exists a continuous function $\bar{f}\colon cX \to [0, 1]$ such that $F = \bar{f}^{-1}(0)$. Let $(a_n)$ be a sequence of points of $X$ such that

$$\bar{f}(a_n) > \bar{f}(a_{n+1}) \text{ for } n = 1, 2, \ldots, \quad \text{and} \quad \lim_{n \to \infty} \bar{f}(a_n) = 0$$

(see [3], the hint to Exercise 3.6.G(a)). We put $M = \{a_1, a_2, \ldots\}$ and $f = \bar{f}|_X$; clearly, $f \in P$. Observe that if $A \subset M$, then the set $f^{-1}[f(A)]$ is an element of $Z[P]$, and

$$f^{-1}[f(A)] \cap M = A.$$

Indeed, if $A \subset M$, then $f(A) \cup \{0\}$ is a closed subset of the interval $[0, 1]$, and thus $f(A) \cup \{0\} = h^{-1}(0)$ for some function $h \in C([0, 1])$. The function $g = h \circ f$ belongs to $P$, and since $\bar{f}^{-1}(0) \cap X = \emptyset$, we have $g^{-1}(0) = f^{-1}[f(A)]$. We conclude that each subset of $M$ is a zero-set in $M$, so $M$ is a copy of $N$. Moreover, if $A$, $B$ is a pair of disjoint subsets of $M$, then the sets $f^{-1}[f(A)]$ and $f^{-1}[f(B)]$ are disjoint members of $Z[P]$; by hypothesis and Lemma 2.3, we obtain

$$\mathrm{cl}_{cX} A \cap \mathrm{cl}_{cX} B = \emptyset.$$

Thus, $\mathrm{cl}_{cX} M$ is homeomorphic to $\beta N$ (see [3], Corollary 3.6.4). To complete the proof, it suffices to observe that $\mathrm{cl}_{cX} M \setminus M \subset F$ (see [5], 6.10(a) and 9.3).

**3.2.** COROLLARY. *If $P \in B(X)$, then no point of the remainder of the compactification $cX$ corresponding to the ring $P$ is a $G_\delta$-set in $cX$.*

**3.3.** THEOREM. *$B(X) = P(X)$ if and only if the remainder of $\beta X$ does not contain non-empty closed $G_\delta$-subsets of $\beta X$.*

Proof. Necessity. Assume that $F \subset \beta X \setminus X$ is a non-empty closed $G_\delta$-subset of $\beta X$. Let $cX$ be the quotient space obtained from $\beta X$ by identifying the set $F$ to a point. Clearly, $cX \in K(X)$ and the one-point set $\{F\} \subset cX \setminus X$ is a $G_\delta$-set in $cX$. From Corollary 3.2 it follows that the ring corresponding to the compactification $cX$ does not belong to $B(X)$, which is a contradiction.

Sufficiency. Now, assume that $P \in P(X) \setminus B(X)$. By Lemma 2.3, there exist functions $f_1, f_2 \in P$ for which

$$(1) \qquad f_1^{-1}(0) \cap f_2^{-1}(0) = \emptyset \quad \text{and} \quad \mathrm{cl}_{cX} f_1^{-1}(0) \cap \mathrm{cl}_{cX} f_2^{-1}(0) \neq \emptyset,$$

where $cX$ is the compactification corresponding to the ring $P$.

For $i = 1, 2$, let $\bar{f}_i \in C(cX)$ be an extension of $f_i$. Then, by (1), the set $Z = \bar{f}_1^{-1}(0) \cap \bar{f}_2^{-1}(0)$ is a non-empty zero-set in $cX$ contained in the remainder of $cX$.

Let $q\colon \beta X \to cX$ be a continuous mapping such that $q \circ \beta = c$. Hence the set $q^{-1}(Z)$ is a non-empty closed $G_\delta$-subset of $\beta X$. Since

$$q^{-1}(cX \setminus X) = \beta X \setminus X \quad \text{and} \quad Z \subset cX \setminus X,$$

we have $q^{-1}(Z) \subset \beta X \setminus X$, a contradiction.

Since a Tychonoff space $X$ is pseudocompact if and only if the remainder $\beta X \setminus X$ does not contain non-empty closed $G_\delta$-subsets of $\beta X$ (see [3], Exercise 3.10E), we have

3.4. Corollary. *$B(X) = P(X)$ if any only if $X$ is a pseudocompact space.*

The Wallman-type compactifications which arise from normal bases consisting of zero-sets are called *z-compactifications*. From Corollary 2.4 it follows that if $P \in B(X)$, then the compactification of $X$ corresponding to the ring $P$ is a $z$-compactification. In [8] A. K. Steiner and E. F. Steiner proved that the Alexandroff compactification is a $z$-compactification. In particular, the one-point compactification $\omega N$ of $N$ is a $z$-compactification; however, the complete ring of functions corresponding to $\omega N$ is not an element of $B(N)$.

From Corollaries 2.4 and 3.4 we obtain

3.5. Corollary. *If $X$ is a pseudocompact space, then each compactification of $X$ is a z-compactification.*

In the sequel, we shall use the following

3.6. Theorem. *A Tychonoff space $X$ has the Lindelöf property if and only if, for each compact subset $F \subset \beta X \setminus X$, in $\beta X$ there exists a compact $G_\delta$-set $H$ such that $F \subset H \subset \beta X \setminus X$; moreover, in the above characterization, $\beta X$ can be replaced by any compactification of $X$* (see [3], Problem 3.12.24(a)).

We say that the space $X$ is *almost compact* if it has a unique (up to equivalence) compactification (see [7]).

3.7. Lemma. *If $B(X) = \{C^*(X)\}$, then the space $X$ is either Lindelöf or almost compact.*

Proof. Assume that the space $X$ has more than one compactification and is not a Lindelöf space. By Theorem 3.6, there exists a closed set $F \subset \beta X \setminus X$ for which

(2) there is no $G_\delta$-subset $H$ of $\beta X$ satisfying $F \subset H \subset \beta X \setminus X$.

Since $\operatorname{card}(\beta X \setminus X) \geqslant 2$ (see [3], Problem 3.12.16(a)), one can assume that $\operatorname{card}(F) \geqslant 2$. Then the quotient space $cX = \beta X/\mathscr{R}$, where $\mathscr{R}$ is the equivalence relation determined by the decomposition of $\beta X$ into the set $F$ and one-point subsets of $\beta X \setminus F$, is a compactification of $X$ and is not equivalent to

$\beta X$. Let $P$ be the ring of functions corresponding to $cX$. Clearly, $P \neq C^*(X)$. We shall prove that $P \in B(X)$.

Assume that $P \notin B(X)$. From Lemma 2.3 it follows that there exist sets $Z_1, Z_2 \in Z[P]$ such that

$$Z_1 \cap Z_2 = \emptyset \quad \text{and} \quad \mathrm{cl}_{cX} Z_1 \cap \mathrm{cl}_{cX} Z_2 \neq \emptyset.$$

We know that $\mathrm{cl}_{\beta X} Z_1 \cap \mathrm{cl}_{\beta X} Z_2 = \emptyset$. Thus, from the definition of the space $cX$ we obtain

$$\mathrm{cl}_{cX} Z_1 \cap \mathrm{cl}_{cX} Z_2 = \{F\}.$$

Let $f_1, f_2 \in P$ and $Z_i = f_i^{-1}(0)$ for $i = 1, 2$. The functions $f_i$ are extendable to functions $\tilde{f}_i \in C(cX)$ $(i = 1, 2)$, and

$$\mathrm{cl}_{cX} Z_1 \cap \mathrm{cl}_{cX} Z_2 \subset \tilde{f}_1^{-1}(0) \cap \tilde{f}_2^{-1}(0).$$

The set $\tilde{H} = \tilde{f}_1^{-1}(0) \cap \tilde{f}_2^{-1}(0)$ is a closed $G_\delta$-subset of $cX$, and $F \in \tilde{H}$. The union $H$ of all equivalence classes belonging to $\tilde{H}$ is a closed $G_\delta$-subset of $\beta X$ such that $F \subset H \subset \beta X \setminus X$. This contradicts (2).

**3.8.** Lemma. *If $P \in B(X)$ and $Z[P] = Z(X)$, then $P = C^*(X)$.*

Proof. By hypothesis, $P \in S(C^*(X))$. Applying Theorem 2.1 and Lemma 2.7 we deduce that $C^*(X) \subset P$ and $P = C^*(X)$.

**3.9.** Lemma. *If the space $X$ is either Lindelöf or almost compact, then $B(X) = \{C^*(X)\}$.*

Proof. In [7] Henriksen and Johnson proved that if $X$ is a Lindelöf space contained in a compact space $Y$, then for each $h \in C(X)$ there is an $f \in C(Y)$ such that $h^{-1}(0) = f^{-1}(0) \cap X$. Thus, for any Lindelöf space $X$ and $P \in P(X)$, we have the equality $Z[P] = Z(X)$. The equality $B(X) = \{C^*(X)\}$ follows from Lemma 3.8.

Clearly, if $X$ is an almost compact space, then $B(X) = \{C^*(X)\}$.

**3.10.** Theorem. *For any Tychonoff space $X$, the following conditions are equivalent:*

(i) $B(X) = \{C^*(X)\}$;

(ii) *for each $P \in B(X)$, we have $Z[P] = Z(X)$;*

(iii) *for each $P \in P(X)$, we have $Z[P] = Z(X)$;*

(iv) *for each $P \in P(X)$, we have $wZ[P] = \beta X$;*

(v) *for each $P \in B(X)$, we have $wZ[P] = \beta X$;*

(vi) *the space $X$ is either Lindelöf or almost compact.*

Proof. Implications (i) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) are obvious. Since $wZ(X) = \beta X$, implication (iii) $\Rightarrow$ (iv) holds. The equivalence of (i) and (vi) follows from Lemmas 3.7 and 3.9. To complete the proof, it suffices to show that (v) implies (i).

If $P \in B(X)$ and $wZ[P] = \beta X$, then, by Corollary 2.4, the compactification of $X$ corresponding to the ring $P$ is equivalent to the Čech–Stone compactification; therefore, $P = C^*(X)$ and $B(X) = \{C^*(X)\}$.

**3.11.** Remark. In [6] Hager and Johnson considered the subrings $\mathscr{A}$ of $C(X)$ which contain all constant functions, separate points from closed sets, are closed with respect to uniform convergence and are closed under inversion (i.e., if $f \in \mathscr{A}$ and $f^{-1}(0) = \emptyset$, then $1/f \in \mathscr{A}$); those subrings are called *algebras* on $X$. One can easily check that the conditions in the above theorem are equivalent to the property "$C(X)$ is the only algebra on $X$" (see [6], Theorem 3).

Finally, let us observe that the converse of Theorem 3.1 does not hold. Indeed, if $X$ is a non-compact Lindelöf space, $z_1$ and $z_2$ are distinct points of $\beta X \setminus X$, $cX$ is the quotient space obtained from $\beta X$ by identifying the set $\{z_1, z_2\}$ to a point and $P \in P(X)$ is the ring corresponding to the compactification $cX$, then, by Theorem 3.10, $P \notin B(X)$; however, each non-empty closed $G_\delta$-set in $cX$, contained in the remainder $cX \setminus X$, contains a subset homeomorphic to $\beta N$.

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