COLLOQUIUM MATHEMATICUM

VOL. LVI 1988 FASC. 2

FUNCTIONS HAVING STATIONARY CONSTANT SETS

BY

GREG G. GIBBON (ST. LUCIA)

A function $f\colon \varkappa \to \varkappa$ is *regressive on* $\varkappa$ if $f(\alpha) < \alpha$ for all $\alpha \neq 0$. In this paper, we investigate a generalization of the classical result of Fodor which states that there is always a stationary set on which a regressive function is constant. We develop conditions that guarantee a stationary constant set for a function $f\colon \varkappa \to [\varkappa]^{\leqslant \lambda}$ for some cardinal λ.

Further, we investigate a similar generalization of the results of Jech concerning regressive functions on the set $[\varrho]^{<\varkappa}$.

1. Introduction and notation. A subset C of the cardinal $\varkappa$ is said to be *unbounded* in $\varkappa$ if for all $\alpha < \lambda$ there is γ in C with $\gamma \geqslant \alpha$. The subset C of $\varkappa$ is said to be *closed* in $\varkappa$ if whenever $B \subseteq C$, then $\bigcup B \in C \cup \{\varkappa\}$. If $\varkappa$ is regular, it is easily seen that the intersection of any family of fewer than $\varkappa$ closed unbounded subsets of $\varkappa$ is itself closed and unbounded in $\varkappa$. A subset S of $\varkappa$ is said to be *stationary* in $\varkappa$ if S intersects every closed unbounded subset of $\varkappa$.

The following theorem is a classical result of Fodor [1]:

THEOREM 1.1. *Let $\varkappa$ be a regular cardinal greater than $\aleph_0$, and let $f\colon \varkappa \to \varkappa$ have the property that $f(\alpha) < \alpha$ for all non-zero α in S, where S is some stationary subset of $\varkappa$. Then there exists a stationary subset T of S such that f is constant on T.*

We will call such a set T a *constant set* with respect to f, and in Section 2 we will investigate the following generalization:

For a given cardinal $\varkappa$, for what values of the cardinal λ will every function $f\colon S \to [\varkappa]^{\leqslant \lambda}$ with the property that $f(\alpha) \subseteq \alpha$ for all α in S, where S is a stationary subset of $\varkappa$, have a large constant set?

In Section 3 we investigate a similar generalization of a theorem of Jech.

Our notation is conventional. We use $\varkappa$, λ, η, σ, and θ for infinite cardinal numbers. Other lower case Greek letters will denote ordinal numbers. The cardinal numbers are identified with the initial ordinals in the usual way. We use α' to denote the cofinality of α for α being an ordinal or cardinal number. The cardinal $\varkappa$ is said to be *regular* if $\varkappa' = \varkappa$, otherwise singular. We denote the cardinal successor of $\varkappa$ by $\varkappa^+$, and the ordinal successor of α by $\alpha+1$. Let $[A]^{\leqslant\eta}$ denote the set $\{B \subseteq A;\ |B| \leqslant \eta\}$, and let $[A]^{<\eta}$ and $[A]^{\eta}$ have the obvious meaning. Let

$$f[X] = \{f(x);\ x \in X\},$$

and let $\mathrm{P}X$ denote the powerset of X. Finally, let

$$f^{-1}(A) = \{\alpha; f(\alpha) = A\},$$

and let $f|X$ be the restriction of the function f to the set X.

2. Functions on ordinals. We will need the following lemmas:

LEMMA 2.1. *Let λ be an infinite cardinal and let $\{\beta_\alpha;\ \alpha < \lambda\}$ be a strictly increasing sequence of ordinals. Put*

$$\delta = \bigcup \{\beta_\alpha;\ \alpha < \lambda\}.$$

Then $\delta' = \lambda'$.

Proof. The map $\lambda \to \delta$ defined by $\alpha \to \beta_\alpha$ is order-preserving and cofinal in δ, and so clearly there is a map $r\colon \lambda' \to \delta$ which is order-preserving and cofinal in δ. Hence $\delta' \leqslant \lambda'$.

For a contradiction, assume that $\delta' < \lambda'$. Let $f\colon \delta' \to \delta$ be cofinal in δ. Define $h\colon \delta' \to \lambda'$ by $h(\alpha) = \gamma$, where γ is the least ordinal such that $r(\gamma) \geqslant f(\alpha)$. Now, h is cofinal in λ', since given $\gamma < \lambda'$, $r(\gamma) < \delta$, and so there exists $\alpha < \delta'$ with $f(\alpha) \geqslant r(\gamma)$. Hence $h(\alpha) \geqslant \gamma$, since r is order-preserving. Since $\delta' < \lambda'$ and λ' is regular, we have the required contradiction.

DEFINITION. Let P_θ denote the subset of the infinite cardinal $\varkappa$ defined by

$$P_\theta = \{\alpha < \varkappa;\ \alpha' = \theta\}.$$

Clearly, P_θ is empty when θ is not a regular cardinal.

LEMMA 2.2. *If θ is regular and $\omega \leqslant \theta < \varkappa'$, then P_θ is stationary in $\varkappa$.*

Proof. Let C be a closed unbounded set in $\varkappa$. Define $\{\beta_\alpha;\ \alpha \leqslant \theta\}$ as follows. Choose any $\beta_0 \in C$. Choose $\beta_{\alpha+1} \in C$ with $\beta_{\alpha+1} > \beta_\alpha$. For γ being a limit ordinal, put $\beta_\gamma = \bigcup \{\beta_\alpha;\ \alpha < \gamma\}$. Then $\beta_\gamma \in C$ since C is closed. So $\beta_\theta \in C$ (since $\theta < \varkappa'$) and $\beta'_\theta = \theta$, since Lemma 2.1 applies. Hence $\beta_\theta \in C \cap P_\theta$.

We are ready to consider the case where $\varkappa$ is regular:

THEOREM 2.1. *Let $\varkappa$ be a regular cardinal greater than $\aleph_0$ and let λ be a cardinal such that $\sigma^\lambda < \varkappa$ whenever σ is a cardinal less than $\varkappa$.*

Let $f\colon S \to [\varkappa]^{\leqslant\lambda}$ have the property $f(\alpha) \subseteq \alpha$ for all α in S, where S satisfies the following property:

For some regular cardinal θ satisfying $\lambda^+ \leqslant \theta < \varkappa$, there is a set F closed *and unbounded in $\varkappa$ such that*

$$\left(X \subseteq F \wedge \left(\bigcup X\right)' = \theta\right) \Rightarrow \bigcup X \in S.$$

Then there is A in $[\varkappa]^{\leqslant\lambda}$ such that $f^{-1}(A)$ is stationary.

Proof. For a contradiction, suppose that $f^{-1}(X)$ is non-stationary for all X in $[\varkappa]^{\leqslant\lambda}$. Thus for each X in $[\varkappa]^{\leqslant\lambda}$ we infer that $f^{-1}(X)$ is not stationary, and hence there is a closed and unbounded set C_X satisfying $C_X \cap f^{-1}(X) = \emptyset$. Put

$$D = \left\{\alpha < \varkappa;\ \alpha \in \bigcap \{C_X;\ X \subseteq \alpha\}\right\}.$$

We will show that $D \cap S \neq \emptyset$, and this contradicts that $f(\alpha) \subseteq \alpha$ for all α in S.

Define $\{\beta_\alpha;\ \alpha \leqslant \theta\}$ inductively as follows:

Choose $\beta_0 \in F$, where F is the closed and unbounded set guaranteed by the hypothesis. If α is such that β_α is defined, choose $\beta_{\alpha+1}$ such that $\beta_{\alpha+1} > \beta_\alpha$ holds and

$$\beta_{\alpha+1} \in \bigcap \{C_X;\ X \subseteq \beta_\alpha\} \cap F.$$

This choice is possible since $|[\beta_\alpha]^{\leqslant\lambda}| < \varkappa$, and so there are less than $\varkappa$ sets X, and so we have an intersection of less than $\varkappa$ closed unbounded sets, which is itself closed and unbounded. For a limit ordinal γ such that β_α has been defined for all $\alpha < \gamma$, define β_γ by

$$\beta_\gamma = \bigcup \{\beta_\alpha;\ \alpha < \gamma\}.$$

Clearly, $\beta_\gamma \in F$.

Put $\delta = \beta_\theta = \bigcup \{\beta_\alpha;\ \alpha < \theta\}$. Then $\delta < \varkappa$, and $\delta' = \theta$ from Lemma 2.1, and so $\delta \in S$, since $\beta_\alpha \in F$ for all $\alpha \leqslant \theta$. It remains to show that $\delta \in D$; that is, given $X \subseteq \delta$, then $\delta \in C_X$.

If $X \subseteq \delta$, then X is not cofinal in δ since $\lambda < \theta$, and so there is $\alpha < \theta$ such that $\beta_\alpha < \delta$ and $X \subseteq \beta_\alpha$. Hence $\beta_{\alpha+1} \in C_X$, by construction. Now we will show that $\varepsilon > \alpha \Rightarrow \beta_\varepsilon \in C_X$. We will use induction. We have $\beta_{\alpha+1} \in C_X$. Assume true for ε, so $\beta_\varepsilon \in C_X$. Now,

$$\beta_{\varepsilon+1} \in \bigcap \{C_X;\ X \subseteq \beta_\varepsilon\}.$$

But $X \subseteq \beta_\alpha \subseteq \beta_\varepsilon$, and so $\beta_{\varepsilon+1} \in C_X$. If ε is a limit ordinal, then

$$\beta_\varepsilon = \bigcup \{\beta_\alpha;\ \alpha < \gamma < \varepsilon\},$$

and so $\beta_\varepsilon \in C_X$, since C_X is closed. Hence

$$\delta = \bigcup \{\beta_\varepsilon;\ \alpha < \varepsilon < \theta\} \in C_X,$$

and the result follows.

It is easily seen that for S to satisfy the hypothesis of Theorem 2.1, it is necessarily stationary. However, the following example shows that being stationary is not sufficient:

EXAMPLE 2.1. Let η be a regular infinite cardinal with $\eta \leqslant \lambda$. Then putting $S = P_\eta$ we infer that S is stationary from Lemma 2.2. Write, for $\alpha \in S$, $f(\alpha) = h_\alpha[\eta]$, where h_α: $\eta \to \alpha$ is a function cofinal in α. This is always possible since $\alpha' = \eta$ by choice. Clearly, f is 1-1 on S, and so the result of Theorem 2.1 is not possible.

The condition on the cardinal λ, i.e., $\sigma < \varkappa \Rightarrow \sigma^\lambda < \varkappa$ for all cardinals σ, is shown to be necessary by the following example:

EXAMPLE 2.2. Suppose there is $\sigma < \varkappa$ with $\sigma^\lambda \geqslant \varkappa$. Then

$$|[\sigma]^{\leqslant\lambda}| = \sigma^\lambda \geqslant \varkappa.$$

We will list $[\sigma]^{\leqslant\lambda}$ by $[\sigma]^{\leqslant\lambda} = \{A_\alpha;\ \alpha < \sigma^\lambda\}$. Define, for all $\alpha < \varkappa$,

$$f(\alpha) = \begin{cases} \emptyset & \text{for } \alpha \leqslant \sigma, \\ A_\alpha & \text{for } \sigma < \alpha < \varkappa. \end{cases}$$

Clearly, $f(\alpha) \subseteq \alpha$ for all $\alpha < \varkappa$, but f is 1-1 on $\varkappa - \sigma$, and so there is no stationary constant set.

To conclude the case for $\varkappa$ a regular cardinal, we will abandon the condition on the cardinal λ, putting $\lambda = \varkappa$, and impose conditions on the function f that ensure the existence of a stationary constant set.

We will refer to the following conditions:

(1) Putting $E_\alpha = \{X \in \mathrm{P}\varkappa;\ X \subseteq \alpha \wedge \exists \eta < \varkappa\ (f(\eta) = X)\}$, we have $|E_\alpha| < \varkappa$ for all α in S.

Note. Since $E_\alpha \subseteq E_\beta$ when $\alpha < \beta$, condition (1) implies that $|E_\alpha| < \varkappa$ for all $\alpha < \varkappa$.

(2) $\bigcup f(\alpha) < \alpha$ for all α in S.

Condition (1) is not satisfied by Example 2.2, since here

$$\begin{aligned} E_\sigma &= \{X \in \mathrm{P}\varkappa;\ X \subseteq \sigma \wedge \exists \eta < \varkappa\ (f(\eta) = X)\} \\ &= \{A_\alpha;\ \alpha < \varkappa\} \cup \{\emptyset\}, \end{aligned}$$

and so $|E_\sigma| = \varkappa$.

Condition (2) is clearly not satisfied by Example 2.1, since for each α in S we have $f(\alpha) = h_\alpha[\eta]$, and so $\bigcup f(\alpha) = \alpha$, by construction.

Theorem 2.2 shows that the imposition of conditions (1) and (2) suffices to ensure the result when $\lambda = \varkappa$, also allowing a slight relaxation of the condition on S.

THEOREM 2.2. *Let $\varkappa$ be a regular cardinal and let f: $S \to \mathrm{P}\varkappa$ have the property $f(\alpha) \subseteq \alpha$ for all α in S, where $S \subseteq \varkappa$ satisfies:*

For some regular cardinal θ satisfying $\omega \leqslant \theta < \varkappa$, there is a set F closed *and unbounded in* $\varkappa$ *such that*

$$\left(X \subseteq F \wedge \left(\bigcup X\right)' = \theta\right) \Rightarrow \bigcup X \in S.$$

Then if f *satisfies conditions* (1) *and* (2) *on* S, *there is* A *in* $\mathrm{P}\varkappa$ *such that* $f^{-1}(A)$ *is stationary.*

Proof. As in the proof of Theorem 2.1, assume that for each X in $\mathrm{P}\varkappa$ the set $f^{-1}(X)$ is not stationary, and so there is a closed and unbounded set C_X satisfying $C_X \cap f^{-1}(X) = \emptyset$.

Define $\{\beta_\alpha;\ \alpha \leqslant \theta\}$ as follows:

Choose $\beta_0 \in F$, where F is the closed and unbounded set guaranteed by the hypothesis. If α is such that β_α is defined, choose $\beta_{\alpha+1}$ such that $\beta_{\alpha+1} > \beta_\alpha$ and

$$\beta_{\alpha+1} \in \bigcap \{X \in \mathrm{P}\varkappa;\ X \subseteq \beta_\alpha \wedge \exists \eta < \varkappa\ (f(\eta) = X)\} \cap F.$$

By condition (1), there are less than $\varkappa$ such sets X, and so the choice is possible. For a limit ordinal γ such that β_α has been defined for all $\alpha < \gamma$, define β_γ by $\beta_\gamma = \bigcup \{\beta_\alpha;\ \alpha < \gamma\}$. Clearly, $\beta_\gamma \in F$. Putting $\delta = \beta_\theta$, we have $\delta < \varkappa$ and $\delta' = \theta$ from Lemma 2.1, and so $\delta \in S$. It follows from condition (2) that $\bigcup f(\delta) < \delta$. We will show that $X \subseteq \delta \Rightarrow f(\delta) \neq X$, and this contradicts $f(\delta) \subseteq \delta$.

Now, if $X \subseteq \delta$, we can assume that X is not cofinal in δ (otherwise, by condition (2), $f(\delta) \neq X$), and so there is $\alpha < \theta$ such that β_α satisfies $X \subseteq \beta_\alpha$. Clearly, either $f(\eta) \neq X$ for all $\eta < \varkappa$ or $f(\eta) = X$ for some $\eta < \varkappa$. The former case implies that $f(\delta) \neq X$ and we are done. The latter case implies that $\beta_{\alpha+1} \in C_X$. We claim that

$$\varepsilon > \alpha \Rightarrow \beta_\varepsilon \in C_X.$$

Proceeding inductively, assume that $\beta_\varepsilon \in C_X$. Now,

$$\beta_{\varepsilon+1} \in \bigcap \{C_X;\ X \subseteq \beta_\varepsilon \wedge \exists \eta < \varkappa\ (f(\eta) = X)\} \cap F.$$

But $X \subseteq \beta_\alpha \subseteq \beta_\varepsilon$ and $f(\eta) = X$, so $\beta_{\alpha+1} \in C_X$. For ε a limit ordinal,

$$\beta_\varepsilon = \bigcup \{\beta_\gamma;\ \alpha < \gamma < \varepsilon\},$$

and so $\beta_\varepsilon \in C_X$ since C_X is closed. Hence

$$\delta = \bigcup \{\beta_\varepsilon;\ \alpha < \varepsilon < \theta\} \in C_X,$$

and so $f(\delta) \neq X$. The proof is complete.

We have now covered all cases for $\varkappa$ being a regular cardinal. For singular $\varkappa$, we have

THEOREM 2.3. *Let* $\varkappa$ *and* λ *be cardinals with* $\varkappa$ *singular and* $\lambda < \varkappa$. *Let* $\eta < \varkappa$ *be a regular cardinal greater than* $\aleph_0$ *such that* $\sigma^\lambda < \eta$ *whenever* $\sigma < \eta$ *for all cardinals* σ. *Let* $S \subseteq \varkappa$ *satisfy*:

For some regular cardinal θ satisfying $\lambda^+ \leqslant \theta < \eta$, there is a set F closed and unbounded in η such that

$$\left(X \subseteq F \wedge \left(\bigcup X\right)' = \theta\right) \Rightarrow \bigcup X \in S.$$

Then, if $f\colon S \to [\varkappa]^{\leqslant\lambda}$ has the property $f(\alpha) \subseteq \alpha$ for all α in S, there is A in $[\varkappa]^{\leqslant\lambda}$ such that $f^{-1}(A) \cap \eta$ is stationary in η.

Note. If we assume GCH and if $S = \varkappa$, then the statement of the theorem simplifies to:

Let $\varkappa$ and λ be cardinals with $\varkappa$ singular and $\lambda < \varkappa$. Then, given any cardinal less than $\varkappa$, there exists a larger cardinal η, with $\eta < \varkappa$, such that $f^{-1}(A) \cap \eta$ is stationary in η for some A in $[\varkappa]^{\leqslant\lambda}$.

Proof of Theorem 2.3. $f|\eta$ satisfies the conditions of Theorem 2.1, since $f|\eta$ is a map $S \cap \eta \to [\eta]^{\leqslant\lambda}$.

We will conclude the case for $\varkappa$ singular by showing that the result of Theorem 2.3 is the best possible by constructing a function $f\colon \varkappa \to [\varkappa]^{\leqslant 1}$ with $f(\alpha) \subseteq \alpha$ for all $\alpha < \varkappa$, but with no constant set of size $\varkappa$, and hence with no stationary constant set.

Let $\{\varkappa_\sigma;\ \sigma < \varkappa'\}$ be a strictly increasing sequence of cardinals with $\bigcup\{\varkappa_\sigma;\ \sigma < \varkappa'\} = \varkappa$ and with $\varkappa_0 = 0$ and $\varkappa_1 = \varkappa'$. Define $F\colon \varkappa \to \varkappa$ by

$$F(\alpha) = \begin{cases} \varkappa_\varepsilon & \text{if } \varkappa_\varepsilon < \alpha < \varkappa_{\varepsilon+1}, \\ \varepsilon & \text{if } \varkappa_\varepsilon = \alpha. \end{cases}$$

So $F(\alpha) < \alpha$ for all $\alpha \neq 0$. Put $f(\alpha) = \{F(\alpha)\}$. Clearly, we have constant sets of every power $\varkappa_\varepsilon$, but no constant set of size $\varkappa$.

3. Functions on sets of ordinals. Following Jech [2], we give definitions of closed and unbounded subsets of $[\varrho]^{<\varkappa}$, where $[\varrho]^{<\varkappa} = \{A \subseteq \varrho;\ |A| < \varkappa\}$.

Let $\varkappa$ be a regular uncountable cardinal and let ϱ be an ordinal such that $\varrho \geqslant \varkappa$.

Definitions. A set D satisfying $D \subseteq [\varrho]^{<\varkappa}$ is a *chain* if

$$D = \{P_\alpha;\ \alpha < \gamma\}$$

so that $P_0 \subseteq P_1 \subseteq \ldots \subseteq P_\alpha \subseteq \ldots$ for all $\alpha < \gamma$.

A set C satisfying $C \subseteq [\varrho]^{<\varkappa}$ is *closed* in $[\varrho]^{<\varkappa}$ if, for every non-empty chain D such that $D \subseteq C$ and $|D| < \varkappa$, the set $\bigcup\{P;\ P \in D\}$ is in C.

C is *unbounded* in $[\varrho]^{<\varkappa}$ if for all P in $[\varrho]^{<\varkappa}$ there exists Q in C such that $P \subseteq Q$. A set S satisfying $S \subseteq [\varrho]^{<\varkappa}$ is *stationary* in $[\varrho]^{<\varkappa}$ if $S \cap C \neq \emptyset$ for all closed and unbounded sets C such that $C \subseteq [\varrho]^{<\varkappa}$.

Let $\hat{R} = \{P \in [\varrho]^{<\varkappa};\ R \subseteq P\}$. Then clearly $\hat{R}$ is closed and unbounded in $[\varrho]^{<\varkappa}$ for all R in $[\varrho]^{<\varkappa}$.

The following is Theorem 3.2 (d) of [1]:

THEOREM 3.1. *If S is a stationary subset of $[\varrho]^{<\varkappa}$ and $f\colon S \to \varrho$ has the property that $f(P) \in P$, then f is constant on some stationary subset of S.*

We will investigate the following generalization of Theorem 3.1:

For a given regular uncountable cardinal $\varkappa$ and an ordinal ϱ such that $\varrho \geqslant \varkappa$, for what values of the cardinal η will a function $G\colon [\varrho]^{<\varkappa} \to [\varrho]^{\leqslant\eta}$ with the property that $G(P) \subseteq P$ for all P belonging to a stationary set, always have a large constant set?

We will need the following lemmas:

LEMMA 3.1. *The intersection of less than $\varkappa$ closed and unbounded sets is itself closed and unbounded.*

For the proof see Theorem 3.2 (b) of [1].

LEMMA 3.2. *Let $Q = \bigcup \{P_\alpha;\ \alpha < \theta\}$, where $P_\alpha \subseteq P_\beta$ whenever $\alpha < \beta$. Let $A \subseteq Q$ and $|A| < \theta'$ hold. Then there is an ordinal δ such that $\delta < \theta$ and $A \subseteq P_\delta$.*

Proof. Define a function $h\colon A \to \theta$ by $h(a)$, being the least ordinal γ such that $a \in P_\gamma$ for all a in A. Since $|A| < \theta'$, the range of h is not cofinal in θ, and so there is an ordinal δ such that $\delta < \theta$ and $h(a) \leqslant \delta$ for all a in A, and hence $A \subseteq P_\delta$ holds.

LEMMA 3.3. *Let λ be a regular cardinal with $\omega \leqslant \lambda < \varkappa$ and let $\varrho' \geqslant \varkappa$. Put*

$$S_\lambda = \left\{P \in [\varrho]^{<\varkappa};\ \left(\bigcup P\right)' = \lambda\right\}.$$

Then S_λ is a stationary set in $[\varrho]^{<\varkappa}$.

Proof. Choose any closed and unbounded set C. It will suffice to show that $C \cap S_\lambda \neq \emptyset$. Define inductively the sequence $\{P_\alpha;\ \alpha \leqslant \lambda\}$ as follows. Choose P_0 from C. For an ordinal α for which P_α is defined, choose $P_{\alpha+1}$ from C such that

$$P_\alpha \cup \left\{\left(\bigcup P_\alpha\right) + 1\right\} \subseteq P_{\alpha+1}.$$

This is possible by the unboundedness of C and the fact that $\varrho' \geqslant \varkappa$ holds. For a limit ordinal γ, define

$$P_\gamma = \bigcup \{P_\alpha;\ \alpha < \gamma\},$$

and so $P_\gamma \in C$ by closure. Note that

$$\bigcup P_\lambda = \bigcup \left\{\bigcup P_\alpha;\ \alpha < \lambda\right\},$$

and since the ordinals $\bigcup P_\alpha$ are strictly increasing, by Lemma 2.1 we have

$$\left(\bigcup P_\lambda\right)' = \lambda' = \lambda.$$

Hence $P_\lambda \in S_\lambda$.

We are now prepared to present sufficient conditions for a constant set.

THEOREM 3.2. *Let $\varrho' \geqslant \varkappa$, and let $G\colon [\varrho]^{<\varkappa} \to [\varrho]^{\leqslant\eta}$ be a function with the property that $G(P) \subseteq P$ for all P in S, where S is a subset of $[\varrho]^{<\varkappa}$ such that, for some R in $[\varrho]^{<\varkappa}$ and some regular cardinal λ with $\omega \leqslant \lambda < \varkappa$, we have $S_\lambda \cap \hat{R} \subseteq S$. Further, let η be a cardinal such that $\sigma^\eta < \varkappa$ for all cardinals σ with $\sigma < \varkappa$. Then for some Y in $[\varrho]^{\leqslant\eta}$ the set $G^{-1}(Y) \cap S$ is stationary in $[\varrho]^{<\varkappa}$.*

Note. If GCH is assumed and if we let $S = [\varrho]^{<\varkappa}$, then the statement of the theorem simplifies to:

Let $\varrho' \geqslant \varkappa$, and let η, λ be cardinals with λ regular, $\omega \leqslant \lambda < \varkappa$ and $\eta < \lambda$. Then if $G\colon [\varrho]^{<\varkappa} \to [\varrho]^{\leqslant\eta}$ has the property $G(P) \subseteq P$, there is some Y such that $G^{-1}(Y)$ is stationary in $[\varrho]^{<\varkappa}$.

Proof. Suppose $G^{-1}(X) \cap S$ is non-stationary for each X in $[\varrho]^{\leqslant\eta}$. Hence, for each X in $[\varrho]^{\leqslant\eta}$ there is some closed and unbounded set C_X such that

$$C_X \cap G^{-1}(X) \cap S = \emptyset$$

holds. Inductively define a sequence $\{P_\alpha;\ \alpha < \lambda\}$ as follows. Choose $P_0 \in \hat{R}$. For an ordinal α for which P_α is defined, choose $P_{\alpha+1}$ from $[\varrho]^{<\varkappa}$ such that

$$P_\alpha \cup \left\{\left(\bigcup P_\alpha\right)+1\right\} \subseteq P_{\alpha+1} \quad \text{and} \quad P_{\alpha+1} \in \bigcap \{C_X;\ X \subseteq P_\alpha\}.$$

This choice is possible since

$$|[P_\alpha]^{\leqslant\eta}| \leqslant |P_\alpha|^\eta < \varkappa,$$

and so the above intersection is unbounded by Lemma 3.1. Also, $\bigcup P_\alpha < \varrho$ since $\varrho' \geqslant \varkappa$ holds. For a limit ordinal γ with $\gamma < \varkappa$, put $P_\gamma = \bigcup \{P_\alpha;\ \alpha < \gamma\}$, noting that P_γ is in $[\varrho]^{<\varkappa}$. Consider P_λ, noting as in the proof of Lemma 3.3 that $P_\lambda \in S_\lambda$, and hence $P_\lambda \in S_\lambda \cap \hat{R}$, and so $G(P_\lambda) \subseteq P_\lambda$. Let X be a set such that $G(P_\lambda) = X$. We will show that $P_\lambda \in C_X$. This contradicts that $P_\lambda \in G^{-1}(X) \cap S$ since

$$C_X \cap G^{-1}(X) \cap S = \emptyset.$$

Now, $|X| \leqslant \eta < \lambda$, and so by Lemma 3.2 applied to P_λ we infer that $X \subseteq P_\delta$ for some $\delta < \lambda$, and so $P_{\delta+1} \in C_X$ by construction. We verify by induction that $P_\varepsilon \in C_X$ for all ε satisfying $\delta < \varepsilon \leqslant \lambda$. This is true, as above, if $\varepsilon = \delta + 1$. If $P_\varepsilon \in C_X$, then since

$$P_{\varepsilon+1} \in \bigcap \{C_X;\ X \subseteq P_\varepsilon\} \quad \text{and} \quad X \subseteq P_\delta \subseteq P_\varepsilon,$$

we have $P_{\varepsilon+1} \in C_X$. The limit case follows from the closedness of C_X. Hence $P_\lambda \in C_X$ and the required contradiction results.

The following example shows that the conditions on S are necessary:

Let G be as in Theorem 3.2 except that $S = S_\lambda$ with $\lambda \leqslant \eta$. Define G by

$$G(X) = \begin{cases} \eta & \text{if } X \notin S_\lambda, \\ X' \subseteq X, \text{ where } |X'| \leqslant \eta \text{ and } \bigcup X' = \bigcup X & \text{if } X \in S_\lambda. \end{cases}$$

Clearly, $G(X) \subseteq X$ on S_λ, but for each set X in $[\varrho]^{\leqslant\eta}$ the set $G^{-1}(X) \cap S_\lambda$ is not stationary, since choosing any $P \in [\varrho]^{<\varkappa}$ with $(\bigcup A_\alpha)+1 \in P$, it follows that $\hat{P}$ is closed and unbounded but

$$\hat{P} \cap G^{-1}(X) \cap S_\lambda = \emptyset.$$

This is true since if $Z \in G^{-1}(X) \cap S_\lambda$, then $G(Z) = X$, and so $\bigcup X = \bigcup Z$, and hence $Z \notin \hat{P}$.

Acknowledgement. The author would like to thank Dr. Neil H. Williams for the invaluable help and encouragement he gave while this work was being carried out.

REFERENCES

[1] G. Fodor, *Eine Bemerkung zur Theorie der Regressiven Funktionen*, Acta Sci. Math. (Szeged) 17 (1956), pp. 139–202.

[2] T. J. Jech, *Some combinatorial problems concerning uncountable cardinals*, Ann. Math. Logic 5 (1973), pp. 165–198.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF QUEENSLAND
ST. LUCIA 4067, AUSTRALIA

Reçu par la Rédaction le 22.2.1984