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GENERALIZATION OF FORMULAE OF FREDHOLM TYPE

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1. Introduction. Sikorski [5] has given formulae of Fredholm type for solutions of a Fredholm linear equation $(I+T)x = x_0$ in a Banach space X, and for the adjoint equation $\xi(I+T) = \xi_0$ in the case where T is a quasi-nuclear operator. Later, the author showed how to obtain the same formulae in a more general case using an algebraic argument (see [2]-[4]).

The purpose of this paper is to give a further generalization of Sikorski's formulae to a larger class of linear equations $(S+T)x = x_0$ in a vector space X, and for the adjoint equation $\xi(S+T) = \xi_0$ in a conjugate space Ξ, where S is an arbitrary fixed generalized Fredholm operator such that $S+T$ is also a generalized Fredholm operator.

2. Terminology and notation. In what follows Ξ and X denote two fixed vector spaces over the same real or complex field F. The letters σ, ξ, η, ζ and s, x, y, z (with indices, if necessary) always denote elements of Ξ and X, respectively. Every mapping into F will be called a *functional*.

To make the paper self-contained, we recall here (see [1]) that Ξ and X are conjugate in the sense that there is a bilinear functional $\xi x\colon \Xi \times X \to F$ such that

(a) $\xi x = 0$ for every $x \in X$ implies $\xi = 0$;

(a') $\xi x = 0$ for every $\xi \in \Xi$ implies $x = 0$.

Let $\mathfrak{A}$ denote the class of all bilinear functionals A defined in $\Xi \times X$, $\xi A x$ being the value of A at (ξ, x) such that each $A \in \mathfrak{A}$ can simultaneously be interpreted as an endomorphism $\eta = \xi A$ in Ξ and as an endomorphism $y = Ax$ in X defined by the relationship

$$\xi A x = (\xi A)x = \xi(Ax).$$

Clearly, a bilinear functional $A \in \mathfrak{A}$, interpreted as an endomorphism $\eta = \xi A$ in Ξ, is the adjoint of the endomorphism $y = Ax$ in X, and the elements of $\mathfrak{A}$ will be called *operators*. The bilinear functional K defined by the formula

$$\xi K x = \xi x_0 \cdot \xi_0 x,$$

where ξ_0 and x_0 are fixed non-zero elements, will be called a *one-dimensional operator* denoted by $x_0 \cdot \xi_0$. Every finite sum of one-dimensional operators will be called a *finite-dimensional operator*.

Let $S \in \mathfrak{A}$ be a fixed generalized Fredholm operator [1], and let $U \in \mathfrak{A}$ be a quasi-inverse of S, i.e., $SUS = S$ and $USU = U$.

We denote by $\sigma_1, \ldots, \sigma_{m'}$ and $s_1, \ldots, s_{n'}$ complete systems of solutions of the equations $\xi S = 0$ and $Sx = 0$, respectively. There exist points $\omega_1, \ldots, \omega_{n'}$ and $w_1, \ldots, w_{m'}$ such that

$$US = I - \sum_{i=1}^{n'} s_i \cdot \omega_i, \qquad SU = I - \sum_{i=1}^{m'} w_i \cdot \sigma_i, \tag{1}$$

where $\omega_i s_j = \delta_{ij}$ $(i, j = 1, \ldots, n')$, $\sigma_i w_j = \delta_{ij}$ $(i, j = 1, \ldots, m')$, and $I \in \mathfrak{A}$ is the identity operator. It is obvious that $\omega_1, \ldots, \omega_{n'}$ and $w_1, \ldots, w_{m'}$ are complete systems of solutions of the equations $\xi U = 0$ and $Ux = 0$, respectively.

Next we define the finite-dimensional operator

$$R = \sum_{i=1}^{\varrho} s_i \cdot \sigma_i, \tag{2}$$

where $\varrho = \min(m', n')$ is the order of S. It is obvious that $SR = RS = 0$. Suppose now that $T \in \mathfrak{A}$ is any operator such that $S+T$ is a generalized Fredholm operator of order $r = \min(\overline{m}, \overline{n})$ and index $d = \overline{n} - \overline{m}$. The operator $S+T$ has a determinant system $\{D_m^n\}$ also of order r and index d [1], i.e., $D_0^d, D_1^d, \ldots$ if $d \geqslant 0$, and $D_{-d}^0, D_{-d+1}^1, \ldots$ if $d < 0$, D_m^n being a multilinear functional on $\Xi^n \times X^m$, whose value at a point $(\xi_1, \ldots, \xi_n, x_1, \ldots, x_m)$ is

$$D_m^n \begin{pmatrix} \xi_1, \ldots, \xi_n \\ x_1, \ldots, x_m \end{pmatrix}.$$

Since r is the order of the determinant system, $D_{\overline{m}}^{\overline{n}} \neq 0$ and $D_m^n = 0$ for $n = \max(0, d), \max(0, d)+1, \ldots, \overline{n}-1$; $n - m = d$. We recall that the determinant system $\{D_m^n\}$ satisfies [1] the following identities:

$$D_{m+1}^{n+1} \begin{pmatrix} \xi_0, \ldots, \xi_n \\ (S+T)x_0, x_1, \ldots, x_m \end{pmatrix} = \sum_{i=0}^{n} (-1)^i \xi_i x_0 D_m^n \begin{pmatrix} \xi_0, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_n \\ x_1, \ldots, x_m \end{pmatrix}, \tag{3}$$

$$D_{m+1}^{n+1} \begin{pmatrix} \xi_0(S+T), \xi_1, \ldots, \xi_n \\ x_0, \ldots, x_m \end{pmatrix} = \sum_{i=0}^{m} (-1)^i \xi_0 x_i D_m^n \begin{pmatrix} \xi_1, \ldots, \xi_n \\ x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m \end{pmatrix}. \tag{4}$$

Let $\eta_1, \ldots, \eta_{\bar{n}}$ and $y_1, \ldots, y_{\bar{m}}$ be points such that

$$\delta = D_{\bar{m}}^{\bar{n}}\begin{pmatrix}\eta_1, \ldots, \eta_{\bar{n}} \\ y_1, \ldots, y_{\bar{m}}\end{pmatrix} \neq 0.$$

Then a complete system $\zeta_1, \ldots, \zeta_{\bar{m}}$ of solutions of $\xi(S+T) = 0$ is given [1] by the formulae

$$(5) \qquad \zeta_i x = \frac{1}{\delta} D_{\bar{m}}^{\bar{n}}\begin{pmatrix} \eta_1, \ldots, \eta_{\bar{n}} \\ y_1, \ldots, y_{i-1}, x, y_{i+1}, \ldots, y_{\bar{m}}\end{pmatrix} \quad \text{for every } x \in X,$$

where $\zeta_i y_j = \delta_{ij}$ $(i, j = 1, \ldots, \bar{m})$. Similarly, a complete system $z_1, \ldots, z_{\bar{n}}$ of solutions of $(S+T)x = 0$ is given by

$$(6) \qquad \xi z_j = \frac{1}{\delta} D_{\bar{m}}^{\bar{n}}\begin{pmatrix}\eta_1, \ldots, \eta_{j-1}, \xi, \eta_{j+1}, \ldots, \eta_{\bar{n}} \\ y_1, \ldots, y_{\bar{m}}\end{pmatrix} \quad \text{for every } \xi \in \Xi,$$

where $\eta_i z_j = \delta_{ij}$ $(i, j = 1, \ldots, \bar{n})$. The operator B defined by the formula

$$\xi B x = \frac{1}{\delta} D_{\bar{m}+1}^{\bar{n}+1}\begin{pmatrix}\xi, \eta_1, \ldots, \eta_{\bar{n}} \\ x, y_1, \ldots, y_{\bar{m}}\end{pmatrix}$$

is a quasi-inverse of $S+T$.

Next, using (3), (4) and properties of the determinant system $\{D_m^n\}$ for $S+T$, it can be shown [1] that

$$(7) \qquad (S+T)B = I - \sum_{i=1}^{\bar{m}} y_i \cdot \zeta_i, \qquad B(S+T) = I - \sum_{i=1}^{\bar{n}} z_i \cdot \eta_i.$$

Let

$$(8) \qquad M = (U+R)(S+T) - I, \qquad N = (S+T)(U+R) - I.$$

Multiplying M (N) on the right (left) by B and using (7), we obtain

$$(9) \qquad MB - \sum_{i=1}^{\bar{n}} z_i \cdot \eta_i (U+R) = BN - \sum_{i=1}^{\bar{m}} (U+R) y_i \cdot \zeta_i.$$

Since any determinant system for $S+T$ differs from $\{D_m^n\}$ by a non--vanishing coefficient, we can assume that the system $\{D_m^n\}$ for $S+T$ is of the form (cf. [1])

$$(10) \quad D_m^n\begin{pmatrix}\xi_1, \ldots, \xi_n \\ x_1, \ldots, x_m\end{pmatrix} = 0 \quad \text{for } n = \max(0, d), \ldots, \bar{n}-1;\ n - m = d,$$

$$(11) \qquad D_{\bar{m}}^{\bar{n}}\begin{pmatrix}\xi_1, \ldots, \xi_{\bar{n}} \\ x_1, \ldots, x_{\bar{m}}\end{pmatrix} = \begin{vmatrix}\xi_1 z_1 & \ldots & \xi_1 z_{\bar{n}} \\ \cdot & \cdots & \cdot \\ \xi_{\bar{n}} z_1 & \ldots & \xi_{\bar{n}} z_{\bar{n}}\end{vmatrix} \times \begin{vmatrix}\zeta_1 x_1 & \ldots & \zeta_1 x_{\bar{m}} \\ \cdot & \cdots & \cdot \\ \zeta_{\bar{m}} z_1 & \ldots & \zeta_{\bar{m}} z_{\bar{m}}\end{vmatrix},$$

and

$$(12)\quad D^{\bar{n}+k}_{\bar{m}+k}\begin{pmatrix}\xi_1,\dots,\xi_{\bar{n}+k}\\ x_1,\dots,x_{\bar{m}+k}\end{pmatrix}$$

$$=\sum_{\mathfrak{p},\mathfrak{q}}\operatorname{sgn}\mathfrak{p}\cdot\operatorname{sgn}\mathfrak{q}\begin{vmatrix}\xi_{p_1}Bx_{q_1} & \dots & \xi_{p_1}Bx_{q_k}\\ \cdot & \cdots & \cdot\\ \xi_{p_k}Bx_{q_1} & \dots & \xi_{p_k}Bx_{q_k}\end{vmatrix}\times D^{\bar{n}}_{\bar{m}}\begin{pmatrix}\xi_{p_{k+1}},\dots,\xi_{p_{k+\bar{n}}}\\ x_{q_{k+1}},\dots,x_{q_{k+\bar{m}}}\end{pmatrix}$$

for $k=1,2,\dots,$

where $\sum_{\mathfrak{p},\mathfrak{q}}$ is extended over all permutations $\mathfrak{p}=(p_1,\dots,p_{k+\bar{n}})$ and $\mathfrak{q}=(q_1,\dots,q_{k+\bar{m}})$ of the integers $1,\dots,k+\bar{n}$ and $1,\dots,k+\bar{m}$, respectively, such that

$$(13)\qquad \begin{aligned} &p_1<p_2<\dots<p_k, \quad p_{k+1}<p_{k+2}<\dots<p_{k+\bar{n}},\\ &q_1<q_2<\dots<q_k, \quad q_{k+1}<q_{k+2}<\dots<q_{k+\bar{m}}.\end{aligned}$$

3. Formulae of Fredholm type. To prove these formulae we shall need the following theorem:

THEOREM 1. *If $\{D^n_m\}$ is a determinant system for $S+T$ of order $r=\min(\bar{m},\bar{n})$ and index $d=\bar{n}-\bar{m}$, then*

$$(14)\qquad D^n_m\begin{pmatrix}\xi_1M,\dots,\xi_nM\\ x_1,\dots,x_m\end{pmatrix}=(-1)^d D^n_m\begin{pmatrix}\xi_1,\dots,\xi_n\\ Nx_1,\dots,Nx_m\end{pmatrix}$$

($n=\max(0,d),\ \max(0,d)+1,\dots;\ n-m=d$) and

$$(15)\qquad D^{\bar{n}}_{\bar{m}}\begin{pmatrix}\xi_1M,\dots,\xi_{\bar{n}}M\\ x_1,\dots,x_{\bar{m}}\end{pmatrix}=(-1)^{\bar{n}}D^{\bar{n}}_{\bar{m}}\begin{pmatrix}\xi_1,\dots,\xi_{\bar{n}}\\ x_1,\dots,x_{\bar{m}}\end{pmatrix},$$

where the operators M and N are defined by (8).

Proof. Since $Mz_i=-z_i$ $(i=1,\dots,\bar{n})$ and $\zeta_iN=-\zeta_i$ $(i=1,\dots,\bar{m})$, by virtue of (8), formulae (15) and (14) (for $m=\bar{m}$, $n=\bar{n}$) follow from (11).

The proof of (14) is based on the well-known formula

$$(16)\quad \begin{vmatrix}a_{1,1} & \dots & a_{1,k+\bar{n}}\\ \cdot & \cdots & \cdot\\ a_{k+\bar{n},1} & \dots & a_{k+\bar{n},k+\bar{n}}\end{vmatrix}$$

$$=\sum_{\mathfrak{p}}\operatorname{sgn}\mathfrak{p}\begin{vmatrix}a_{p_1,1} & \dots & a_{p_1,k}\\ \cdot & \cdots & \cdot\\ a_{p_k,1} & \dots & a_{p_k,k}\end{vmatrix}\times\begin{vmatrix}a_{p_{k+1},1} & \dots & a_{p_{k+1},\bar{n}}\\ \cdot & \cdots & \cdot\\ a_{p_{k+\bar{n}},1} & \dots & a_{p_{k+\bar{n}},\bar{n}}\end{vmatrix},$$

where the permutation $\mathfrak{p}$ is the same as in (13). Therefore, using (11), (12), (9), (15), (16), well-known properties of classical determinants, and introducing the notation

$$L = MB - \sum_{i=1}^{\bar{n}} z_i \cdot \eta_i (U+R), \qquad K = BN - \sum_{i=1}^{\bar{m}} (U+R) y_i \cdot \zeta_i,$$

we obtain

$$D_{\bar{m}+k}^{\bar{n}+k}\begin{pmatrix} \xi_1 M, \ldots, \xi_{\bar{n}+k} M \\ x_1, \ldots, x_{\bar{m}+k}\end{pmatrix}$$

$$= (-1)^{\bar{n}} \sum_{p,q} \operatorname{sgn} p \cdot \operatorname{sgn} q \begin{vmatrix} \xi_{p_1} MBx_{q_1} & \ldots & \xi_{p_1} MBx_{q_k} \\ \cdot & \cdots & \cdot \\ \xi_{p_k} MBx_{q_1} & \ldots & \xi_{p_k} MBx_{q_k} \end{vmatrix} \times D_{\bar{m}}^{\bar{n}} \begin{vmatrix} \xi_{p_{k+1}} & \ldots & \xi_{p_{k+\bar{n}}} \\ x_{q_{k+1}} & \ldots & x_{q_{k+\bar{m}}} \end{vmatrix}$$

$$= (-1)^{\bar{n}} \sum_{q} \operatorname{sgn} q \begin{vmatrix} \xi_1 Lx_{q_1} & \ldots & \xi_1 Lx_{q_k} & \xi_1 z_1 & \ldots & \xi_1 z_{\bar{n}} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \xi_{k+\bar{n}} Lx_{q_1} & \ldots & \xi_{k+\bar{n}} Lx_{q_k} & \xi_{k+\bar{n}} z_1 & \ldots & \xi_{k+\bar{n}} z_{\bar{n}} \end{vmatrix} \times$$

$$\times \begin{vmatrix} \zeta_1 x_{q_{k+1}} & \ldots & \zeta_1 x_{q_{k+\bar{m}}} \\ \cdot & \cdots & \cdot \\ \zeta_{\bar{m}} x_{q_{k+1}} & \ldots & \zeta_1 x_{q_{k+\bar{m}}} \end{vmatrix}$$

$$= (-1)^{\bar{n}} \sum_{p\cdot q} \operatorname{sgn} p \cdot \operatorname{sgn} q \begin{vmatrix} \xi_{p_1} Kx_{q_1} & \ldots & \xi_{p_1} Kx_{q_k} \\ \cdot & \cdots & \cdot \\ \xi_{p_k} Kx_{q_1} & \ldots & \xi_{p_k} Kx_{q_k} \end{vmatrix} \times D_{\bar{m}}^{\bar{n}} \begin{pmatrix} \xi_{p_{k+1}}, \ldots, \xi_{p_{k+\bar{n}}} \\ x_{q_{k+1}}, \ldots, x_{q_{k+\bar{m}}} \end{pmatrix}$$

$$= (-1)^{\bar{n}} \sum_{p} \operatorname{sgn} p \begin{vmatrix} \xi_{p_1} Kx_1 & \ldots & \xi_{p_1} Kx_{k+\bar{m}} \\ \cdot & \cdots & \cdot \\ \xi_{p_k} Kx_1 & \ldots & \xi_{p_k} Kx_{k+\bar{m}} \\ \zeta_1 x_1 & \ldots & \zeta_1 x_{k+\bar{m}} \\ \cdot & \cdots & \cdot \\ \zeta_{\bar{m}} x_1 & \ldots & \zeta_{\bar{m}} x_{k+\bar{m}} \end{vmatrix} \times \begin{vmatrix} \xi_{p_{k+1}} z_1 & \ldots & \xi_{p_{k+1}} z_{\bar{n}} \\ \cdot & \cdots & \cdot \\ \xi_{p_{k+\bar{n}}} z_1 & \ldots & \xi_{p_{k+\bar{n}}} z_{\bar{n}} \end{vmatrix}$$

$$= (-1)^{d} \sum_{p\cdot q} \operatorname{sgn} p \cdot \operatorname{sgn} q \begin{vmatrix} \xi_{p_1} BNx_{q_1} & \ldots & \xi_{p_1} BNx_{q_k} \\ \cdot & \cdots & \cdot \\ \xi_{p_k} BNx_{q_1} & \ldots & \xi_{p_k} BNx_{q_k} \end{vmatrix} \times D_{\bar{m}}^{\bar{n}} \begin{pmatrix} \xi_1, \ldots, \xi_n \\ Nx_1, \ldots, Nx_{\bar{m}} \end{pmatrix}$$

$$= (-1)^{d} D_{\bar{m}+k}^{\bar{n}+k} \begin{pmatrix} \xi_1, \ldots, \xi_{\bar{n}+k} \\ Nx_1, \ldots, Nx_{\bar{m}+k} \end{pmatrix}.$$

This completes the proof.

With the above-given assumptions we have the following theorem:

THEOREM 2. *Let*

$$\bar{D}^n_m\begin{pmatrix}\xi_1, \ldots, \xi_n \\ x_1, \ldots, x_m\end{pmatrix} = D^n_m\begin{pmatrix}\xi_1 M, \ldots, \xi_n M \\ x_1, \ldots, x_m\end{pmatrix} \tag{17}$$

($n = \max(0, d), \max(0, d)+1, \ldots; n-m = d$) *and let us fix the points* $\eta_1, \ldots, \eta_{\bar{n}}$ *and* $y_1, \ldots, y_{\bar{m}}$ *such that*

$$\delta = \bar{D}^{\bar{m}}_{\bar{n}}\begin{pmatrix}\eta_1, \ldots, \eta_{\bar{n}} \\ y_1, \ldots, y_{\bar{m}}\end{pmatrix} \neq 0.$$

Let ζ_j ($j = 1, \ldots, \bar{m}$), z_i ($i = 1, \ldots, \bar{n}$) *and let the operator* $\bar{B}$ *be defined as follows*:

$$\zeta_j x = \frac{1}{\delta}\bar{D}^{\bar{n}}_{\bar{m}}\begin{pmatrix}\eta_1, \ldots, \eta_{\bar{n}} \\ y_1, \ldots, y_{j-1}, x, y_{j+1}, \ldots, y_{\bar{m}}\end{pmatrix} \quad \text{for every } x \in X, \tag{18}$$

$$\xi z_i = \frac{1}{\delta}\bar{D}^{\bar{n}}_{\bar{m}}\begin{pmatrix}\eta_1, \ldots, \eta_{i-1}, \xi, \eta_{i+1}, \ldots, \eta_{\bar{n}} \\ y_1, \ldots, y_{\bar{m}}\end{pmatrix} \quad \text{for every } \xi \in \Xi, \tag{19}$$

$$\xi \bar{B} x = \frac{1}{\delta}\bar{D}^{\bar{n}+1}_{\bar{m}+1}\begin{pmatrix}\xi, \eta_1, \ldots, \eta_{\bar{n}} \\ x, y_1, \ldots, y_{\bar{m}}\end{pmatrix}. \tag{20}$$

Then the equation

$$(S+T)x = x_0 \tag{*}$$

has a solution x *iff* $\zeta_j x_0 = 0$ *for* $j = 1, \ldots, \bar{n}$, *and the equation*

$$\xi(S+T) = \xi_0 \tag{**}$$

has a solution ξ *iff* $\xi_0 z_i = 0$ *for* $i = 1, \ldots, \bar{n}$. *The general form of the solution of* (*) *is*

$$x = (U+R-\bar{B})x_0 + a_1 z_1 + \ldots + a_{\bar{n}} z_{\bar{n}},$$

and the general form of the solution of (**) *is*

$$\xi = \xi_0(U+R-\bar{B}) + b_1\zeta_1 + \ldots + b_{\bar{m}}\zeta_{\bar{m}},$$

where $a_1, \ldots, a_{\bar{n}}$ *and* $b_1, \ldots, b_{\bar{m}}$ *are arbitrary constants.*

Proof. The formulae for $\zeta_1, \ldots, \zeta_{\bar{m}}$ and $z_1, \ldots, z_{\bar{n}}$ can be obtained immediately, by virtue of (15), from (5) and (6), constituting complete systems of solutions of $\xi(S+T) = 0$ and $(S+T)x = 0$, respectively. Substituting $(S+T)x$ for x and $\xi(S+T)$ for ξ, and then using identities (3) and (4), by virtue of the skew symmetry of $\bar{D}^{\bar{n}}_{\bar{m}}$ and Theorem 1, we obtain

$$\xi\bar{B}(S+T)x = \xi Mx - \sum_{i=1}^{\bar{n}} \xi z_i \cdot \eta_i Mx, \qquad \xi(S+T)\bar{B}x = \xi Nx - \sum_{i=1}^{\bar{m}} \xi N y_i \cdot \zeta_i x$$

or, equivalently, in view of (8),

$$(U+R-\bar{B})(S+T) = I+\sum_{i=1}^{\bar{n}} z_i\cdot\eta_i M,$$

$$(S+T)(U+R-\bar{B}) = I+\sum_{i=1}^{\bar{m}} Ny_i\cdot\zeta_i.$$

Multiplying the first equation by ξ_0 on the left, and the second equation by x_0 on the right and assuming that $\xi_0 z_i = 0$ $(i = 1, \ldots, \bar{n})$ and $\zeta_i x_0 = 0$ $(i = 1, \ldots, \bar{m})$, we obtain

$$\xi_0(U+R-\bar{B})(S+T) = \xi_0, \quad (S+T)(U+R-\bar{B})x_0 = x_0.$$

This completes the proof.

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