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DISJOINTING INFINITE SUMS IN INCOMPLETE BOOLEAN ALGEBRAS

BY

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Let $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1\rangle$ be a Boolean algebra. The main result in this paper* is a solution (negative) of Problem 1 of [1], which asks whether every indexed set $\{a_t \colon t \in T\} \subseteq A$ with a supremum in $\mathfrak{A}$ can be disjointed (see Definition 1). A weak form of disjointing is considered and we give sufficient conditions under which an indexed set $\{a_t \colon t \in T\}$ can be disjointed or weakly disjointed.

We use $\mathfrak{m}$, $\mathfrak{n}$ to denote infinite cardinal numbers, α, β to denote ordinal numbers. $\overline{\overline{S}}$ is the power of the set S. If h is a function, $\operatorname{dmn} h$ and $\operatorname{rng} h$ denote the domain of h and the range of h, respectively. Boolean algebras are denoted by $\mathfrak{A}$, $\mathfrak{B}$, it being understood that A is the universe of $\mathfrak{A}$ etc. We use $\sum$ to denote sum (supremum). A Boolean algebra $\mathfrak{A}$ is $\mathfrak{n}$-complete if $\sum D$ exists in $\mathfrak{A}$ whenever $D \subseteq A$ and $\overline{\overline{D}} \leqslant \mathfrak{n}$.

Definition 1. Let $\mathfrak{A}$ be a Boolean algebra, $\{a_t \colon t \in T\} \subseteq A$ and $a = \sum_{t \in T} a_t$. We say that $\{a_t \colon t \in T\}$ *can be disjointed* provided there is an indexed set $\{b_t \colon t \in T\} \subseteq A$ satisfying:

(i) $b_t \leqslant a_t$ for each $t \in T$,

(ii) $\sum_{t \in T} b_t = a$,

(iii) $\{b_t \colon t \in T\}$ is disjoint, namely if $t, s \in T$ and $t \neq s$ then $b_t \cdot b_s = 0$.

It is well known [1] that if $a = \sum_{t \in T} a_t$ holds in $\mathfrak{A}$, and if $\mathfrak{A}$ is $\mathfrak{n}$-complete for every cardinal $\mathfrak{n} < \overline{\overline{T}}$, then $\{a_t \colon t \in T\}$ can be disjointed. In particular, $\{a_t \colon t \in T\}$ can be disjointed whenever T is countable. Recall that for an infinite cardinal $\mathfrak{m}$, $\mathfrak{A}$ satisfies the $\mathfrak{m}$-chain condition if every disjoint subset of $\mathfrak{A}$ has power $\leqslant \mathfrak{m}$. Theorems 2 and 4 are known (folklore) and are included for motivation.

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(1) For a proof of this fact see Sikorski [2], p. 69.

THEOREM 2. *Every infinite Boolean algebra* $\mathfrak{A}$ *contains an infinite disjoint subset.*

Proof. For each $a \in A$, let $c(a)$ be the power of the set $\{x: x \in A, x \leqslant a\}$. We define two sequences $\langle a_n: n < \omega \rangle$ and $\langle b_n: n < \omega \rangle$ of elements of A as follows:

a_0 is any element of A except $0, 1$, then either $c(a_0)$ or $c(-a_0)$ is infinite because for $a \in A$, $a = a \cdot a_0 + a \cdot - a_0$. If $c(a_0)$ is infinite let $b_0 = -a_0$; then $b_0 \neq 0$. Inductively assume we have defined $\langle a_i: i \leqslant n \rangle$ and $\langle b_i: : i < n \rangle$ such that $0 \neq b_i \leqslant -a_n$ for each $i < n$, and $b_i \cdot b_j = 0$ for $i < j < n$, and also $c(a_n)$ is infinite. Now let $a \leqslant a_n$, $a \neq 0$, a_n, and as before either $c(a)$ or $c(a_n \cdot - a)$ is infinite. Say $c(a)$ is infinite; then let $b_n = a_n \cdot - a$, $a_{n+1} = a$. Now for $i < n+1$, $0 \neq b_i \leqslant a_{n+1}$, $b_n \neq 0$, and $b_i \cdot b_j = 0$ for $i < j < n+1$. Hence $\{b_i: i < \omega\}$ is an infinite disjoint subset of A.

LEMMA 3. *Let* $\mathfrak{A}$ *be a Boolean algebra and let* $a = \sum_{t \in T} a_t \neq 0$ *in* $\mathfrak{A}$. *There is a disjoint set* D *of nonzero elements of* A *satisfying*:

(i) *for each* $d \in D$ *there is a* $t \in T$ *such that* $d \leqslant a_t$,

(ii) $a = \sum_{d \in D} d$.

Proof. Let $M = \{x \in A : x \neq 0$ and $x \leqslant a_t$ for some $t \in T\}$. By Zorn's Lemma let D be a maximal disjoint subset of M. To prove that $a = \sum_{d \in D} d$, suppose $b \in A$, $b \neq 0$ and $b \leqslant a$. Since $a = \sum_{t \in T} a_t$, there is a $t \in T$ such that $b \cdot a_t \neq 0$. Now $b \cdot a_t \in M$, so by the definition of D, there is a $d \in D$ such that $(b \cdot a_t) \cdot d \neq 0$, thus $b \cdot d \neq 0$. This, together with the fact that for each $d \in D$ there is a $t \in T$ such that $d \leqslant a_t \leqslant a$, proves that $a = \sum_{d \in D} d$.

THEOREM 4. *A Boolean algebra* $\mathfrak{A}$ *satisfies the* $\mathfrak{m}$*-chain condition if and only if whenever* $a = \sum_{t \in T} a_t$ *holds there is a set* $S \subseteq T$ *with* $\overline{\overline{S}} \leqslant \mathfrak{m}$ *and* $a = \sum_{t \in S} a_t$.

Proof. Assume $\mathfrak{A}$ satisfies the $\mathfrak{m}$-chain condition, and that $a = \sum_{t \in T} a_t \neq 0$. Choose a disjoint set D satisfying the conclusion of Lemma 3; then $\overline{\overline{D}} \leqslant \mathfrak{m}$. For each $d \in D$, let $f(d) \in T$ be chosen so that $d \leqslant a_{f(d)}$; then let $S = \operatorname{rng} f$. Now $\overline{\overline{S}} \leqslant \mathfrak{m}$ and $a = \sum_{t \in S} a_t$ follows from $a = \sum_{d \in D} d$.

Conversely, if $\mathfrak{A}$ does not satisfy the $\mathfrak{m}$-chain condition, let F be a disjoint subset of $\mathfrak{A}$ having power $> \mathfrak{m}$. F can be extended to a maximal disjoint set T by Zorn's Lemma. Then $1 = \sum_{a \in T} a$, and clearly there does not exist a set $S \subseteq T$ with the required properties.

THEOREM 5. *Let* $\mathfrak{A}$ *be a Boolean algebra, let* a, $a_t \in A$ *for* $t \in T$, *and* $a = \sum_{t \in T} a_t$. *If there is a cardinal* $\mathfrak{m}$ *such that*

(i) *for every* $s \in T$, $\{t \in T\colon a_s \leqslant a_t\}$ *has power at least* $\mathfrak{m}$,

(ii) $\mathfrak{A}$ *satisfies the* $\mathfrak{m}$*-chain condition,*

then $\{a_t : t \in T\}$ *can be disjointed.*

Proof. By Lemma 3 we choose disjoint set $D \subseteq A$ satisfying $\sum_{d\in D} d = a$ and for each $d \in D$ there is a $t \in T$ such that $d \leqslant a_t$. By hypothesis $\overline{\overline{D}} \leqslant \mathfrak{m}$, so we write $D = \{d_a : a < \eta \leqslant \mathfrak{m}\}$. Define a one-to-one function $h \in T^D$ as follows: $h(d_0)$ is any $t \in T$ such that $d_0 \leqslant a_t$, and if $h(d_a)$ is defined for $a < \beta < \eta$ so that $d_a \leqslant a_{f(d_a)}$ for each $a < \beta$ and $h|\{d_a : a < \beta\}$ is one-to-one; then let $h(d_\beta)$ be any $t \in T$ such that $d_\beta \leqslant a_t$ and $t \notin \{h(d_a)\colon a < \beta\}$. $h(d_\beta)$ exists because, by hypothesis $\{t : d_\beta \leqslant a_t\}$ has power $\geqslant \mathfrak{m}$ whereas $\{h(d_a)\colon a < \beta\}$ has power $< \mathfrak{m}$. Finally we define for each $t \in T$,

$$b_t = \begin{cases} d & \text{in case } h(d) = t, \\ 0 & \text{if } t \notin \operatorname{rng} h. \end{cases}$$

b_t is uniquely determined because h is one-to-one. Now for $t \in \operatorname{rng} h$, $b_t = d \leqslant a_{h(d)} = a_t$, and for $t \notin \operatorname{rng} h$, $b_t = 0 \leqslant a_t$. Moreover

$$\sum_{t\in T} b_t = \sum_{d\in D} d = a.$$

This proves that $\{a_t : t \in T\}$ can be disjointed.

Corollary 6. *Let* $a = \sum_{t\in T} a_t$ *in* $\mathfrak{A}$, *and let* $\overline{\overline{T}} = \mathfrak{m}$. *Assume*

(i) $\{a_t : t \in T\}$ *is linearly ordered by the Boolean inclusion,*

(ii) *no set* $S \subseteq T$ *satisfies simultaneously* $\overline{\overline{S}} < \mathfrak{m}$ *and* $a = \sum_{t\in S} a_t$,

(iii) $\mathfrak{A}$ *satisfies the* $\mathfrak{m}$*-chain condition.*

Then $\{a_t : t \in T\}$ *can be disjointed.*

Proof. To show that the hypotheses of Theorem 5 are satisfied, note that for any $s \in T$, $a = \sum\{a_t\colon a_s \leqslant a_t\}$ because $\{a_t\colon t \in T\}$ is linearly ordered. Then $\{t : a_s \leqslant a_t\}$ has power $\mathfrak{m}$ by (ii).

Lemma 7. *Let* $a = \sum_{t\in T} a_t$ *in* $\mathfrak{A}$, *let* F *be the set of non-void finite subsets of* T, *and for each* $f \in F$ *let* $b_f = \sum_{t\in f} a_t$. *Then* $a = \sum_{f\in F} b_f$.

Proof. Clearly $b_f \leqslant a$ for each $f \in F$, so $a = \sum_{f\in F} b_f$ follows from $a_t = b_{\{t\}}$ for each $t \in T$.

Definition 8. Let a, $\{a_t : t \in T\}$, $\{b_f\colon f \in F\}$ be as in Lemma 7. We say that $\{a_t\colon t \in T\}$ *can be weakly disjointed* if $\{b_f\colon f \in F\}$ can be disjointed.

Theorem 9. *Let* $a = \sum_{t\in T} a_t$ *in* $\mathfrak{A}$; *if* $\overline{\overline{T}} = \mathfrak{m}$ *and* $\mathfrak{A}$ *satisfies the* $\mathfrak{m}$*-chain condition, then* $\{a_t : t \in T\}$ *can be weakly disjointed.*

Proof. For $f, g \in F$, $a_g \leqslant a_f$ if $g \subseteq f$, thus since $\overline{\overline{T}} = \mathfrak{m}$, $\{f \in F\colon a_g \leqslant a_f\}$ has power $\mathfrak{m}$ for every $g \in F$. Hence Theorem 9 follows from Theorem 5.

The hypothesis that $\mathfrak{A}$ satisfy the $\mathfrak{m}$-chain condition in 5, 6 and 9 cannot be omitted. The Boolean algebra $\mathfrak{B}$ and the indexed set $\{-a_{h_\alpha}: : \alpha < \omega_1\}$ constructed below satisfy all hypotheses of 5, 6 and 9 except the $\aleph_1$-chain condition, however $\{-a_{h_\alpha} : \alpha < \omega_1\}$ can be neither disjointed nor weakly disjointed.

Definition 10. Let X be the set of all functions f satisfying

(i) dmn f is a successor ordinal α and $\alpha < \omega_1$,

(ii) rng $f \subseteq \omega_2$.

For each $g \in X$, $a_g = \{f \in X: f \supseteq g\}$. Let $\mathfrak{B}$ be the Boolean set algebra of subsets of X generated by $\{a_g: g \in X\}$.

The requirement that dmn f be a successor ordinal will insure incompleteness. Note that for $g, h \in X$, $a_g \cap a_h = a_{g \cup h}$ in case $g \cup h$ is a function and $a_g \cap a_h = 0$ otherwise. By (i) $g \cup h$ is a function if and only if either $g \subseteq h$ or $h \subseteq g$. Thus if $b \in B$ and $b \neq 0, 1$ (here $1 = X$), then b can be written as a finite union of sets each having one of the following forms: a_g, $\bigcap_{k \in K}(-a_{g_k})$, $a_g \cap \bigcap_{k \in K}(-a_{g_k})$, where K is finite, $K \neq 0$ and $g, g_k \in X$. Now $a_g \cap \bigcap_{k \in K}(-a_{g_k}) \neq 0$ if and only if there is an $f \in X$ such that $f \supseteq g$ and $f \not\supseteq g_k$ for each $k \in K$, and this holds if and only if $g \not\supseteq g_k$ for all $k \in K$ (i.e. if and only if $g \in a_g \cap \bigcap_{k \in K}(-a_{g_k})$). It will frequently be assumed that an expression $a_g \cap \bigcap_{k \in K}(-a_{g_k})$ is reduced meaning that none of the sets a_g, $-a_{g_k}$ contains another. We now define the set $\{-a_{h_\alpha} : \alpha < \omega_1\}$ mentioned above.

Definition 11. For each $\alpha < \omega_1$, let $h_\alpha \in X$ be defined by: (i) dmn $h_\alpha = \alpha + 1$ and (ii) rng $h_\alpha = \{0\}$.

LEMMA 12. *$\sum_{\alpha < \omega_1}(-a_{h_\alpha}) = 1$ in $\mathfrak{B}$.*

Proof. If $f \in X$, there is an ordinal $\beta < \omega_1$ such that dmn $f < \beta$; then $f \not\supseteq h_\beta$, so $f \in -a_{h_\beta}$. Thus $X = \bigcup_{\alpha < \omega_1}(-a_{h_\alpha})$.

In what follows we use the notation $\sum_{t \in T} b_t \neq b$ to mean that either $\sum_{t \in T} b_t$ does not exists or $\sum_{t \in T} b_t$ exists but is different from b.

LEMMA 13. *If $I \subseteq \omega_1$ and I is countable, then $\sum_{\alpha \in I}(-a_{h_\alpha}) \neq 1$.*

Proof. Choose $\beta < \omega_1$ such that $I \subseteq \beta$, and consider h_β, $h_\beta \supseteq h_\alpha$ for all $\alpha \in I$. If $f \in a_{h_\beta}$, then $f \supseteq h_\beta \supseteq h_\alpha$, so $f \in a_{h_\alpha}$, thus $f \notin -a_{h_\alpha}$. Thus $a_{h_\beta} \cap -a_{h_\alpha} = 0$ for all $\alpha \in I$ and this proves that $\sum_{\alpha \in I} a_{h_\alpha} \neq 1$.

It follows from Lemma 13 that if $\{-a_{h_\alpha}: \alpha < \omega_1\}$ could be either disjointed or weakly disjointed, then $\mathfrak{B}$ would contain a maximal disjoint set of power $\aleph_1$. It will be shown below that $\mathfrak{B}$ has no maximal disjoint subset of power $\aleph_1$. As a special case of our proof of this fact, if $\{g_\alpha: \alpha < \omega_1\} \subseteq X$, then $\sum_{\alpha<\omega_1} a_{g_\alpha} \neq 1$ whether $\{a_{g_\alpha}: \alpha < \omega_1\}$ is disjoint or not. This follows because $0 \in \operatorname{dmn} g_\alpha$ for each $\alpha < \omega_1$ and there is a $\beta < \omega_2$ such that $\beta \notin \bigcup_{\alpha<\omega_1} \operatorname{rng} g_\alpha$. Let f be any element of X such that $f(0) = \beta$; then if $h \in a_f$, we have $h(0) = \beta$, so $h \nsupseteq g_\alpha$, so $h \notin a_{g_\alpha}$ for all $\alpha < \omega_1$. Thus $a_f \cap a_{g_\alpha} = 0$ for $\alpha < \omega_1$. Roughly this idea underlies the proof of the next theorem. For $g \in X$, let $\mu(g)$ denote the greatest ordinal in dmn g. $\mu(g)$ exists because dmn g is a successor ordinal.

THEOREM 14. *The Boolean algebra $\mathfrak{B}$ of Definition 10 has no maximal disjoint subset of power $\aleph_1$.*

Proof. The proof is by contradiction and is divided into lemmas. We assume that $\mathfrak{B}$ does contain a maximal disjoint subset of power $\aleph_1$. It follows that there is an indexed set $\{b_i: i \in I\}$ of distinct non-zero elements such that $\overline{\overline{I}} = \aleph_1$, $b_i \cap b_j = 0$ for $i, j \in I$, $i \neq j$, and each b_i is written in one of the forms: a_{g_i}, $a_{g_i} \cap \bigcap_{k \in K_i} (-a_{g_{ik}})$, $\bigcap_{k \in K_i} (-a_{g_{ik}})$. It is further assumed that these expressions are reduced. In particular, we note that $-a_g \subseteq -a_f$ if and only if $a_f \subseteq a_g$ and this holds if and only if $g \subseteq f$, also $a_f \nsubseteq -a_g$ if and only if $a_f \cap a_g \neq 0$ if and only if $f \cup g$ is a function.

LEMMA 15. *There is exactly one $j \in I$ such that $b_j = \bigcap_{k \in K_j} (-a_{g_{jk}})$.*

Proof. If no such j exists, then $b_i \subseteq a_{g_i}$ for each $i \in I$ and by the special case considered above, $\sum_{i \in I} a_{g_i} \neq 1$, thus $\sum_{i \in I} b_i \neq 1$ contradicting the assumption.

Conversely if $i, j \in I$, $i \neq j$ and $b_i = \bigcap_{k \in K_i} (-a_{g_{ik}})$, $b_j = \bigcap_{k \in K_j} (-a_{g_{jk}})$, then choose $\beta \in \omega_2$ such that $\beta \notin \operatorname{rng} g_{ik}$ for $k \in K_i$ and $\beta \notin \operatorname{rng} g_{jk}$ for $k \in K_j$. Now choose $f \in X$ such that $f(0) = \beta$. $f(0) \neq g_{ik}(0)$ for $k \in K_i$, so that if $h \in a_f$, then $h \supseteq f$, so $h \nsupseteq g_{ik}$, hence $h \in -a_{g_{ik}}$. Thus $a_f \subseteq \bigcap_{k \in K_i} (-a_{g_{ik}}) = b_i$. Likewise $a_f \subseteq b_j$ so $b_i \cap b_j \neq 0$ contradicting the disjointness of $\{b_i: i \in I\}$.

LEMMA 16. *Suppose b_i is either a_{g_i} or $a_{g_i} \cap \bigcap_{k \in K_i} (-a_{g_{ik}})$ and b_j is either a_{g_j} or $a_{g_j} \cap \bigcap_{k \in K_j} (-a_{g_{jk}})$, where $i, j \in I$. If $i \neq j$, then $g_i \neq g_j$.*

Proof. Assume $g_i = g_j$. Since $b_i \neq 0$, $b_j \neq 0$, and $b_i \cap b_j = 0$, the only non-trivial case occurs when $b_i = a_{g_i} \cap \bigcap_{k \in K_i} (-a_{g_{ik}})$ and $b_j = a_{g_j} \cap \bigcap_{k \in K_j} (-a_{g_{jk}})$. But by the remarks following Definition 10 we have $g_i \in b_i$ and $g_j \in b_j$, a contradiction.

LEMMA 17. *For $b = a_g \cap \bigcap_{k \in K} (-a_{g_k})$ in reduced form, $b \neq 0$, $K \neq 0$, we have $g \subseteq g_k$ properly for each $k \in K$.*

Proof. Since $b \neq 0$, we have $g \in b$ and $g \nsupseteq g_k$ for $k \in K$. Moreover, $\mu(g) < \mu(g_k)$, for if $\mu(g_k) \leqslant \mu(g)$, then dmn $g_k \subseteq$ dmn g. By assumption $a_g \nsubseteq -a_{g_k}$, so, by the remark preceeding Lemma 15, $g \cup g_k$ is a function, thus $g_k \subseteq g$. Contradiction, hence $\mu(g) < \mu(g_k)$ for each $k \in K$. Using again the fact that $a_g \nsubseteq -a_{g_k}$, we have that $g \cup g_k$ is a function, so $g \subseteq g_k$ properly.

Definition 18. Let $b \in B$, $\{b_j : j \in J\} \subseteq B$; we say that $\{b_j : j \in J\}$ *covers* b *exactly* provided:

(i) If $a \leqslant b$, $a \neq 0$, there is a $j \in J$ such that $a \cap b_j \neq 0$,

(ii) $b \cap b_j \neq 0$ for all $j \in J$.

$\{b_j : j \in J\}$ *covers* b if (i) holds.

LEMMA 19. *Suppose that a_h is covered exactly by $\{b_i : i \in J\}$, where $J \subseteq I$, $\bar{J} = \aleph_1$. Assume that each b_i has as its reduced form either a_{g_i} or $a_{g_i} \cap \bigcap_{k \in K_i} (-a_{g_{ik}})$, and also that no countable subset of $\{b_i : i \in J\}$ covers a_h. Then there is a unique $j \in J$ such that $a_h \subseteq a_{g_j}$, and for this j we have $b_j = a_{g_j} \cap \bigcap_{k \in K_j} (-a_{g_{jk}})$ with $K_j \neq 0$.*

Proof. For $i \in J$, $b_i \subseteq a_{g_i}$, so $\{a_{g_i} : i \in J\}$ covers a_h exactly; we have $a_h \cap a_{g_i} \neq 0$, thus $a_h \nsubseteq -a_{g_i}$.

Assume $a_h \nsubseteq a_{g_i}$ for all $i \in J$; then $a_h \cap -a_{g_i} \neq 0$. Since $-a_{g_i} \nsubseteq a_h$ holds in general, it follows that $a_h \cap -a_{g_i}$ is in reduced form. By Lemma 17, $h \subseteq g_i$ properly, thus for all $i \in J$, $(\mu(h)+1) \in$ dmn g_i. Choose $\beta \notin \bigcup_{i \in J}$ rng g_i and let $f = h \cup \{\langle \mu(h)+1, \beta \rangle\}$. Then $f \in X$, and $f(\mu(h)+1) \neq g_i(\mu(h)+1)$ for every $i \in J$, so if $q \in a_f$, then $q \supseteq f$, so $q \nsupseteq g_i$ so that $q \in -a_{g_i}$. Hence $a_f \subseteq -a_{g_i}$ for every $i \in J$, and it follows that $a_f \cap a_{g_i} = 0$. However $h \subseteq f$, so that $a_f \subseteq a_h$, $a_f \neq 0$ and this contradicts the fact that $\{a_{g_i} : i \in J\}$ covers a_h.

Hence for some $j \in J$, $a_h \subseteq a_{g_j}$. Also $b_j \nsupseteq a_h$ by hypothesis, so $b_j = a_{g_j} \cap \bigcap_{k \in K_j} (-a_{g_{jk}})$ with $K_j \neq 0$.

Now assume $i, j \in J$, $i \neq j$ and $a_h \subseteq a_{g_i} \cap a_{g_j}$. Then $0 \neq a_h \cap b_i = a_h \cap a_{g_i} \cap \bigcap_{k \in K_i} (-a_{g_{ik}}) = a_h \cap \bigcap_{k \in K_i} (-a_{g_{ik}})$, so that by the remarks following Definition 10 $h \in a_h \cap b_i$. Likewise $h \in a_h \cap b_j$, so $b_i \cap b_j \neq 0$, contradiction. This proves the lemma.

We now define a sequence $\langle p_n : n < \omega \rangle$ of elements of X such that $p_n \subseteq p_{n+1}$ properly for each $n < \omega$, and a_{p_n} is not covered by any countable subset of $\{b_i : i \in I\}$. Let i' be the unique element of I such that $b_{i'} = \bigcap_{k \in K_{i'}} (-a_{g_{i'k}})$ by Lemma 15. $X = b_{i'} \cup \bigcup_{k \in K_{i'}} a_{g_{i'k}}$, and X is not covered by any countable subset of $\{b_i : i \in I\}$, because the b_i are disjoint and $\bar{I} = \aleph_1$. It follows that for some $k \in K_{i'}$, $a_{g_{i'k}}$ is not covered by any countable subset of $\{b_i : i \in I\}$. For any such k, let $p_0 = g_{i'k}$.

Inductively assume that $p_0, p_1, \ldots, p_n$ are defined such that $p_m \subseteq p_{m+1}$ properly for $0 \leqslant m < n$ and that a_{p_n} is not covered by any countable subset of $\{b_i: i \in I\}$. Let $J_n = \{i \in I: a_{p_n} \frown b_i \neq 0\}$; then $\{b_i: i \in J_n\}$ covers a_{p_n} exactly. The hypotheses of Lemma 19 are fulfilled, so choose $j \in J_n$, $b_j = a_{g_j} \frown \bigcap_{k \in K_j} (-a_{g_{jk}})$ and $a_{p_n} \subseteq a_{g_j}$. Also $j \in J_n$ is uniquely determined by $a_{p_n} \subseteq a_{g_j}$. By a straightforward set-theoretic calculation we obtain:

$$a_{p_n} = a_{p_n} \frown (b_j \smile -b_j) = (a_{p_n} \frown b_j) \smile \Big[\bigcup_{k \in K_j} (a_{p_n} \frown a_{g_{jk}})\Big].$$

It follows that for some $k \in K_j$, $a_{p_n} \frown a_{g_{jk}}$ is not covered by any countable subset of $\{b_i: i \in J_n\}$. Also $a_{p_n} \frown a_{g_{jk}} \neq 0$ so that $p_n \smile g_{jk}$ is a function. Thus by the remarks following Definition 10, either $p_n \subseteq g_{jk}$ or $g_{jk} \subseteq p_n$. Now if $g_{jk} \subseteq p_n$, then $a_{p_n} \subseteq a_{g_{jk}}$ so that $a_{p_n} \frown -a_{g_{jk}} = 0$, and it follows that $a_{p_n} \frown b_j = 0$, contradicting the definition of J_n. Thus $g_{jk} \nsubseteq p_n$, so $p_n \subseteq g_{jk}$ properly. Choose $p_{n+1} = g_{jk}$. It remains to show that $a_{g_{jk}}$ is not covered by any countable subset of $\{b_i: i \in I\}$. Assume that $R \subseteq I$, $\overline{\overline{R}} \leqslant \aleph_0$ and $\{b_i: i \in R\}$ covers $a_{g_{jk}}$. Then $\{b_i: i \in R\}$ covers $a_{p_n} \frown a_{g_{jk}}$. Let $R' = \{i \in R: b_i \frown (a_{p_n} \frown a_{g_{jk}}) \neq 0\}$; then $R' \subseteq J_n$ and $\overline{\overline{R'}} \leqslant \aleph_0$ and $\{b_i: i \in R'\}$ covers $a_{p_n} \frown a_{g_{jk}}$, contradiction. Hence $a_{g_{jk}}$ is not covered by any countable subset of $\{b_i: i \in I\}$. Thus a sequence $\langle p_n: n < \omega\rangle$ of elements of X is defined such that, for each $n < \omega$, $p_n \subseteq p_{n+1}$ properly and a_{p_n} is not covered by any countable subset of $\{b_i: i \in I\}$.

Now define $S = \bigcap_{n<\omega} a_{p_n}$, and let $p = \bigcup_{n<\omega} p_n$. We note that $\operatorname{dmn} p$ is a limit ordinal so that $p \notin X$, and $S = \{f \in X: f \supseteq p\}$. Let $J = \{i \in I: b_i \frown S \neq 0\}$. Since $S \subseteq a_{p_0}$, $i' \notin J$, so for each $i \in J$, b_i is either a_{g_i} or $a_{g_i} \frown \bigcap_{k \in K_i} (-a_{g_{ik}})$, thus $b_i \subseteq a_{g_i}$ each $i \in J$.

LEMMA 20. *For each* $j \in J$, $S \nsubseteq a_{g_j}$.

Proof. Assume that for some $j \in J$, $S \subseteq a_{g_j}$; then $g_j \subseteq p$ and in fact $\mu(g_j) \in \operatorname{dmn} p = \bigcup_{n<\omega} \operatorname{dmn} p_n$. Let m be any natural number such that $\mu(g_j) \in \operatorname{dmn} p_m$. Then $g_j \subseteq p_m$ so that $a_{p_m} \subseteq a_{g_j}$. However $J \subseteq J_m$, so $j \in J_m$, and following the construction of p_{m+1} from p_m we see, by using Lemma 19, that $b_j = a_{g_j} \frown \bigcap_{k \in K_j} (-a_{g_{jk}})$ and $p_{m+1} = g_{jk}$ for some $k \in K_j$. However, this implies that $b_j \frown a_{p_{m+1}} = 0$, so that $b_j \frown S = 0$, contradicting $j \in J$. This proves the lemma.

To complete the proof of Theorem 14 consider $\{b_i : i \in J\}$. For each $i \in J$, $b_i \subseteq a_{g_i}$ and by Lemma 20 $S \nsubseteq a_{g_i}$, so that $g_i \nsubseteq p$. However, $b_i \frown S \neq 0$ so that $a_{g_i} \frown S \neq 0$, thus as before $g_i \smile p$ is a function. Since $g_i \nsubseteq p$, we have $p \subseteq g_i$ properly for every $i \in J$. Let α be the least ordinal not in $\operatorname{dmn} p$, in fact $\alpha = \operatorname{dmn} p$. Then $\alpha \in \operatorname{dmn} g_i$ for all $i \in J$.

Choose $\beta \notin \bigcup_{i\in J} \operatorname{rng} g_i$ and $t = p \cup \{\langle \alpha, \beta\rangle\}$; then $t \in X$, so that $a_t \in B$ and $a_t \neq 0$. However, for each $i \in J$, $a_{g_i} \cap a_t = 0$, so that $b_i \cap a_t = 0$. Also $a_t \subseteq S$, so that for $i \in I - J$, $b_i \cap a_t = 0$, thus $b_i \cap a_t = 0$ for all $i \in I$. This proves that $\{b_i : i \in I\}$ is not a maximal disjoint subset of B, contradicting the original assumption. This completes the proof of Theorem 14.

We remark that the main result of this section can be generalized from $\aleph_1$ to any regular cardinal. There are incomplete Boolean algebras in which every sum can be disjointed, an example being the field $\mathfrak{F}$ of finite and cofinite subsets of an infinite set Y. Suppose $a = \sum_{t\in T} a_t$ holds in $\mathfrak{F}$. In case a_t is cofinite for some $t \in T$, there is a finite set $S \subseteq T$ such that $a = \sum_{t\in S} a_t$ and as remarked after Definition 1, $\{a_t \colon t \in S\}$ can be disjointed. In case a_t is finite for all $t \in T$, choose a well-ordering $\langle t_\alpha \colon \alpha < \overline{T}\rangle$ of T, and let $b_{t_0} = a_{t_0}$ and $b_{t_\beta} = a_{t_\beta} - \bigcup_{\alpha<\beta} a_{t_\alpha}$ for $0 < \beta < \overline{T}$. Then $\{b_t \colon t \in T\}$ is a disjointing of $\{a_t \colon t \in T\}$ even though $\bigcup_{\alpha<\beta} a_{t_\alpha}$ is not, in general, an element of F. Also if $\mathfrak{A}$ satisfies the $\aleph_0$-chain condition, every sum in $\mathfrak{A}$ can be disjointed by Theorem 4.

Some problems are suggested.

PROBLEM 1. Find conditions, not involving completeness or the $\aleph_0$-chain condition, on a Boolean algebra $\mathfrak{A}$ under which every sum in $\mathfrak{A}$ can be disjointed (**P 612**).

PROBLEM 2. Is there a Boolean algebra $\mathfrak{A}$, and a set $\{a_t : t \in T\} \subseteq A$ where $\sum_{t\in T} a_t$ exists such that $\{a_t \colon t \in T\}$ can be weakly disjointed but not disjointed? (**P 613**)

REFERENCES

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