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THE SEPARABILITY OF CARTESIAN PRODUCTS

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1. Introduction. A topological space is *separable* if it contains a countable dense subset. It is *reducible* if it can be expressed as a union of two proper closed subsets. It is *irreducible* if it is not reducible. Equivalently, a space is irreducible if and only if the family of its non-empty open sets has the finite intersection property. This paper deals with the following problem:

Under which conditions is the cartesian product X of a family $\{X_i, i \in I\}$ of topological spaces separable?

Hewitt [2], Marczewski [3] and Pondiczery [5] have proved that the following condition is sufficient: Each X_i is separable, and $\operatorname{card} I \leqslant \mathfrak{c} =$ the power of continuum. Moreover, if each X_i is reducible, this is also a necessary condition. For arbitrary spaces, the separability of X implies the separability of each X_i, but card I may be larger than $\mathfrak{c}$. As an example, consider any separable irreducible space X. It is easy to see that *every power X^I of X is separable.* In fact, if D is dense in X, then the set of all constant functions $I \to D$ is dense in X^I. In view of this example, one might be tempted to think that any product of separable irreducible spaces would be separable. However, this is not true, as is seen from the following counter-example. Let N be the set of all positive integers, and let F be the collection of all ultrafilters of N. Then each $u \in F$ defines an irreducible topology $u \cup \{\emptyset\}$ of N. Let X_u be the corresponding topological space, and let X be the cartesian product of the spaces X_u, $u \in F$. Then each X_u is separable but X is not. To prove this, assume that there exists a mapping $p\colon N \to X$ such that pN is dense in X. Let $P_u : X \to X_u$ be the projection mapping. Since $\operatorname{card} F = 2^{\mathfrak{c}}$ ([1], p. 130), there exist ultrafilters $u \neq v$ such that $P_u p = P_v p$. We may assume that u contains an element U which does not belong to v. Since v is an ultrafilter, there is a $V \in v$ such that $U \cap V = \emptyset$. The set $P_u^{-1} U \cap P_v^{-1} V$ is open in X. Hence it contains $p(n)$ for some $n \in N$. But this implies that the element $P_u(p(n)) = P_v(p(n))$ belongs to $U \cap V$, and we have obtained a contradiction.

These examples show that the solution of our problem cannot be so simple as in the case of reducible spaces. We need a condition which guarantees that there are not "too many too different spaces" among the spaces X_i. We will give a precise meaning for this in the next section.

2. The main theorem. In what follows, N will always denote the set of positive integers. A *separation* of a topological space X is a mapping $p : N \to X$ such that pN is dense in X. Since we will not deal with connectedness, no confusion will arise from the fact that in the literature this word is sometimes used to mean a representation of X as the union of two disjoint open sets. A separation of an indexed family $\mathbf{X} = \{X_i, i \in I\}$ of spaces is a family $\mathbf{p} = \{p_i, i \in I\}$ where each p_i is a separation of X_i. We say that $\mathbf{p}$ is *strong* if the following condition is satisfied: For each finite subset J of I and for each family $\{U_j, j \in J\}$ of non-empty open sets $U_j \subset X_j$, the intersection $\bigcap_{j \in J} p_j^{-1} U_j$ is not empty.

THEOREM 1. *Let X be the cartesian product of a family $\mathbf{X} = \{X_i, i \in I\}$ of non-empty topological spaces. Then the following conditions are equivalent:*

(a) *X is separable.*

(b) *Each X_i contains a countable dense set D_i such that for every family $\mathbf{p}$ of surjections $p_i : N \to D_i$, $i \in I$, we can express I as a disjoint union of sets I_a, $a \in A$, where $\operatorname{card} A \leqslant \mathfrak{c}$ and $\mathbf{p}$ induces a strong separation $\mathbf{p}_a$ for each family $\mathbf{X}_a = \{X_i, i \in I_a\}$.*

(c) *There exists a separation $\mathbf{p}$ of $\mathbf{X}$ such that I can be expressed as in condition* (b).

(d) *There exists a strong separation of $\mathbf{X}$.*

Proof. We will denote by P_i the projection mapping $X \to X_i$. Obviously, (b) implies (c). Furthermore, (a) and (d) are easily seen to be equivalent. In fact, if $p : N \to X$ is any mapping, then p is a separation of X if and only if the family $\{P_i p, i \in I\}$ is a strong separation of $\mathbf{X}$.

To prove that (c) implies (a), set $Y_a = \prod_{i \in I_a} X_i$. Since (d) $\Rightarrow$ (a), Y_a is separable. Since $\operatorname{card} A \leqslant \mathfrak{c}$, X is separable by the theorem of Hewitt, Marczewski and Pondiczery.

It remains to prove that (a) $\Rightarrow$ (b). Let $p : N \to X$ be a separation of X, and set $D_i = P_i p N$. For each $i \in I$ choose a mapping $q_i : D_i \to N$ such that $p_i q_i$ is the identity mapping. Let A be the set of all mappings $N \to N$, and let $I_a = \{i \in I : q_i P_i p = a\}$, $a \in A$. Then $\operatorname{card} A = \mathfrak{c}$, and it suffices to prove that each $\mathbf{p}_a$ is a strong separation of $\mathbf{X}_a$. Let $J \subset I_a$ be finite, and let $U_j \neq \emptyset$ be open in X_j, $j \in J$. Then there exists an $n \in N$ such that $p(n) \in \bigcap_{j \in J} P_j^{-1} U_j$. Since $p_j a = P_j p$, this implies that $a(n) \in p_j^{-1} U_j$ for each $j \in J$. Hence $\mathbf{p}_a$ is a strong separation of $\mathbf{X}_a$.

Remark. It is natural to ask whether (b) can be replaced by the following stronger condition: "Each X_i is separable, and for each separation $\mathbf{p} = \{p_i, i \in I\}$ of $\mathbf{X}$, we can express I as in (b)." The following counter-example shows that the answer is in the negative. Let F again be the collection of all ultrafilters of N. For each $u \in F$ we define a topological space X_u as follows: The points of X_u are the non-negative integers, and the open sets in X_u are $\emptyset$ and all sets $U \cup \{0\}$, $U \in u$. Then the cartesian product X of the spaces X_u, $u \in F$, is separable, because the constant function $F \to \{0\}$ alone is dense in X. Moreover, N is dense in each X_u. Let $p_u : N \to X_u$ be the inclusion mapping. Then $\mathbf{p} = \{p_u, u \in F\}$ is a separation of $\mathbf{X}$ which does not satisfy the condition. For, if $F = \bigcup_{a \in A} F_a$ with $\operatorname{card} A \leqslant \mathfrak{c}$, at least one F_a contains two distinct ultrafilters u, v. We can then find disjoint sets $U \in u$, $V \in v$. Thus $p_u^{-1} U' \cap p_v^{-1} V' = \emptyset$, where $U' = U \cup \{0\}$, $V' = V \cup \{0\}$. Hence $\mathbf{p}_a$ is not strong.

3. Special cases. We first remark that the theorem of Hewitt, Marczewski and Pondiczery is contained in Theorem 1. In fact, if card $I \leqslant \mathfrak{c}$ and if each X_i is separable, then (c) is satisfied with each I_a containing only one element. Conversely, assume that X is separable. In each reducible X_i choose disjoint non-empty open sets U_i, V_i. The mappings $p_i \colon N \to D_i$ in (b) can be chosen so that n is even or odd if $p_i(n)$ belongs to U_i or V_i, respectively. Hence $p_i^{-1} U_i \cap p_j^{-1} V_j = \emptyset$, which implies that each I_a may contain only one index i for which X_i is reducible. Thus, at most $\mathfrak{c}$ of the spaces X_i can be reducible.

There is another special case in which our problem has a simple solution. A *pseudo-base* [4] of a topological space X is a family B of non-empty open sets such that every non-empty open set contains a member of B. If X has a countable base or, more generally, if X is separable and satisfies the first axiom of countability, then X has a countable pseudo-base. On the other hand, every space which has a countable pseudo-base is separable.

THEOREM 2. *Let X be the cartesian product of a family $\{X_i, i \in I\}$ of non-empty topological spaces, each of which has a countable pseudo-base. Then X is separable if and only if there exists a subset I_0 of I such that* card $(I - I_0) \leqslant \mathfrak{c}$ *and X_i is irreducible for all $i \in I_0$.*

Proof. The necessity of the condition is contained in the theorem of Hewitt, Marczewski and Pondiczery, and was reproved above. To prove the sufficiency, let $\{U_i(n), n \in N\}$ be a countable pseudo-base for X_i, $i \in I_0$. Since X_i is irreducible, we can choose a point $p_i(n)$ in $U_i(1) \cap \cap \ldots \cap U_i(n)$. We obtain mappings $p_i \colon N \to X_i$, which clearly give a separation $\mathbf{p}_0$ of the family $\{X_i, i \in I_0\}$. We show that $\mathbf{p}_0$ is strong. Let $J \subset I_0$ be finite, and let $V_j \neq \emptyset$ be open in X_j, $j \in J$. Each V_j contains a set $U_j(n_j)$. Thus $p_j(n) \in V_j$ for $n \geqslant n_j$. Since J is finite, this implies that $\bigcap_{j \in J} p_j^{-1} V_j \neq \emptyset$. Hence $\mathbf{p}_0$ is strong, and the condition (c) of Theorem 1 is satisfied.

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