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A THEOREM OF FIXED-POINT TYPE FOR NON-COMPACT LOCALLY CONNECTED SPACES

BY

A. J. WARD (CAMBRIDGE)

In a recent paper [5], L. E. Ward, following upon earlier work of Wallace [3] and himself, has obtained a generalization of Borsuk's fixed-point theorem [1] which applies to any semi-locally connected continuum equipped with a partial order which has certain properties in common with the cut-point order. We shall show that close analogues of Ward's preliminary lemmas apply in any connected and locally connected Hausdorff space; in fact, owing to the simplicity of the structure of such a space in relation to its cut-points, we can slightly relax the conditions imposed on the partial order. It is of course impossible to prove a fixed-point theorem, properly so called, for such spaces, as is shown by the simplest examples; e.g. we may take the space $\{x; x \geqslant 0\}$ with the natural order and topology, and consider the mappings defined by $f_1(x) = \{x+1\}$ and $f_2(x) = \{y; y \geqslant x+1\}$. We shall however prove a general theorem which may be considered as saying that, under certain conditions, the two functions just mentioned exemplify the *only* ways in which a mapping can fail to have a fixed point.

I should like to thank Prof. L. E. Ward for his kindness in supplying me with a copy of his paper [5] in advance of publication.

1. Properties of the partial order. Throughout this section X is a connected and locally connected Hausdorff space, with a partial order (asymmetric) satisfying:

(a) $M(x) = \{y; x \leqslant y\}$ is closed (all $x \in X$);

(b) $M^0(x) = \{y; x < y\}$ is open (all $x \in X$);

(c) writing $L(x) = \{z; z \leqslant x\}$, $L(x) \cap L(y)$ is a compact chain (all $x, y \in X$, $x = y$ allowed).

Condition (b) is formally weaker than condition (ii) of [5], but we shall show that our conditions in fact imply condition (ii).

We begin by listing some known or trivial consequences of our assumptions. The natural order-topology of any compact chain in X

coincides with its subspace topology, and any subset of the chain has a supremum and infimum in the chain (which are also the supremum and infimum in X of the same set, though we shall not need this fact). X has a least element, x_0 say. For any x, $M^0(x)$ is open and closed in $X\setminus\{x\}$ and hence either x is maximal or $M^0(x)$ is a union of components of $X\setminus\{x\}$; if $x \neq x_0$ and $M^0(x) \neq \emptyset$, then x is a cut-point. Thus if $y > x$ then x separates x_0 and y, so that the given order is contained in the cut-point order [1] with respect to x_0. The point x is in the closure of every component of $X\setminus\{x\}$; it follows that $M(x)$ and $X\setminus M^0(x)$ are connected.

As in [5], we define $C(x) = \{y;\ M(x) \cap L(y) = \{x, y\}\}$. We index those components, if any, of $X\setminus\{x\}$ which meet (and hence are contained in) $M^0(x)$ as $\{A_i(x);\ i \in I(x)\}$, and write $C_i(x) = \{x\} \cup (A_i(x) \cap C(x))$. (If $M^0(x) = \emptyset$, we put conventionally $I(x) = \{1\}$ and write $C_1(x) = C(x) = \{x\}$). We write for brevity $C^0(x) = C(x)\setminus\{x\}$, $C_i^0(x) = C_i(x)\setminus\{x\}$. We write further $L(E) = \bigcup(L(x);\ x \in E)$ and $M(E) = \bigcup(M(x);\ x \in E)$.

The following lemmas (1-4) correspond closely to Lemmas 1-3 of [5]. Some of the Corollaries are not actually necessary for the proof of our main theorem, but are included for their own interest and to clarify the consequences of our assumptions.

LEMMA 1. *The sets $C(x)$, $C_i(x)$ are closed.*

The proof of lemma 1 of [5] applies to $C(x)$. Since $\overline{A_i(x)} = A_i(x) \cup \{x\}$, we have $C_i(x) = C(x) \cap \overline{A_i(x)}$, hence closed.

LEMMA 2. *For all $E \subset X$, $\overline{M(E)} = M(E) \cup \overline{E}$; hence if F is closed, then $M(F)$ is closed.*

If $y \notin \overline{E}$, y has a connected neighbourhood U not meeting E. If U meets $M(x)$, $x \in E$, then for some $i \in I(x)$ we have $U \subset A_i(x) \subset M^0(x)$, so that $y \in M^0(x) \subset M(E)$. It follows that if $y \notin M(E) \cup \overline{E}$, then $U \cap M(E) = \emptyset$ so that $y \notin \overline{M(E)}$; the converse is trivial.

LEMMA 3. *If $y' \in M^*(x) = M^0(x)\setminus M(C^0(x))$, then $H(y') = \bigcup(M(y);\ x < y \leqslant y')$ is a subset of $M^*(x)$, open and closed in $M^0(x)$, and hence is of the form $\bigcup(A_i(x);\ i \in I^*(x, y') \subset I(x))$.*

Let $t' \in M(y)$, $x < y \leqslant y'$. If $\exists\, t \leqslant t'$ such that $t \in C^0(x)$, then since $L(t')$ is a chain, we have $t \leqslant y \leqslant y'$, a contradiction. Hence $t' \in M^*(x)$; that is, $H(y') \subset M^*(x)$.

Again, if $x < y_1 \leqslant y'$, $\exists\, y_2$, $x < y_2 < y_1$; as $M(y_1) \subset M^0(y_2)$, we have $H(y') = \bigcup(M^0(y);\ x < y < y')$, hence open. Now suppose $t \in M^0(x)\setminus H(y') \subset X\setminus[L(y') \cap M(x)]$. Let $E = L(y') \cap M^0(x)$, so that $M(E) = H(y')$. As $L(y') \cap M(x)$ is closed, $t \notin M(E) \cup \overline{E} = \overline{M(E)} = \overline{H(y')}$. Since t is arbitrary, $H(y')$ is closed in $M^0(x)$.

[1] For the definition and properties of the cut-point order see, e.g., [4] or [6], chapter III.

COROLLARY 1. *$M^*(x)$ and $M(C^0(x))$ are open and closed in $M^0(x)$, hence open in X.*

For $M^*(x) \subset \bigcup (H(y'); y' \in M^*(x)) \subset M^*(x)$, so that $M^*(x)$ is a union of (open) components of $M^0(x)$ and $M(C^0(x))$ is the union of the remaining components.

COROLLARY 2. *If y', z' are in the same component $A_i = A_i(x)$ of $M^0(x)$ and if $L(y') \cap L(z') \cap M^0(x) = \emptyset$, then there exist (distinct) y_1, z_1 in $C^0(x)$ such that $y_1 \leqslant y'$, $z_1 \leqslant z'$.*

For if $y' \in M^*(x)$ we have $i \in I^*(x, y')$, $z' \in A_i \subset H(y')$, so that $\exists y \in L(y') \cap L(z') \cap M^0(x)$; similarly if $z' \in M^*(x)$. Thus y', z' are both in $M(C^0(x))$ so that y_1, z_1 exist and are clearly distinct.

We now define the function $p_x : X \to C(x)$, just as in [5], by $p_x(y) = z$ iff $z \in C^0(x)$ and $y \in M(z)$, $p_x(y) = x$ otherwise; we define also the function $p_{i,x} : X \to C_i(x)$ (for $i \in I(x)$) by

$$p_{i,x}(y) = \begin{cases} p_x(y) & \text{if } p_x(y) \in C_i(x), \\ x & \text{otherwise.} \end{cases}$$

LEMMA 4. *The functions p_x and $p_{i,x}$ are order-preserving and continuous.*

As in lemma 3 of [5], the order-preserving properties are easily verified. If F is a closed subset of $C(x)$ (resp. $C_i(x)$) and if $x \notin F$, then $p_x^{-1}(F)$ (resp. $p_{i,x}^{-1}(F)) = M(F)$, closed, by lemma 2. If $x \in F$, then $G = C(x) \setminus F$ (resp. $C_i(x) \setminus F$) is a relatively open subset of $C^0(x)$. (In the case when $G = C_i(x) \setminus F$ this is because $A_i(x)$ is open, so that a set open in $A_i(x) \cap C(x)$ is open in $C(x)$.) By lemma 3, Corollary 1, it is sufficient to show that $M(G)$ is open in $M(C^0(x))$, or, taking complements, that $M(F \setminus \{x\})$ is closed in $M(C^0(x))$. By lemma 2, $[M(F \setminus \{x\})]^- = [F \setminus \{x\}]^- \cup M(F \setminus \{x\}) \subset F \cup M(F \setminus \{x\}) = M(F \setminus \{x\}) \cup \{x\}$. This establishes the required result and proves that p_x and $p_{i,x}$ are continuous.

LEMMA 5. *If E is connected, then $p_x(E)$ and $p_{i,x}(E)$ are either single points or equal to $E \cap C(x)$, $E \cap C_i(x)$ respectively: hence if E is closed and connected so also are $p_x(E)$ and $p_{i,x}(E)$.*

COROLLARY 1. *If E is a connected subset of $M^0(x)$, then either $p_x(E) = \{x\}$ or, for some $i \in I(x)$, $p_x(E) = p_{i,x}(E) \subset A_i(x) \cap C^0(x) = C_i^0(x)$.*

COROLLARY 2. *$A_i(x) \cap C(x) = C_i^0(x)$ is either empty or a component of $C^0(x)$.*

We need consider only the case when $p_x(E)$ contains at least two points; let $y \in p_x(E)$, $y \neq x$. Then $E \cap M(y) \neq \emptyset$, but $E \not\subset M(y)$ as $p_x(E) \neq \{y\}$. Since E is connected and $M^0(y)$ is open and closed in $X \setminus \{y\}$, this implies that $y \in E$. In the same way, since $p_x^{-1}(x) \setminus \{x\} = M^*(x) \cup [X \setminus M(x)]$ is open and closed in $X \setminus \{x\}$, we see that if $x \in p_x(E)$, then either $p_x(E) = \{x\}$ or $x \in E$. Thus in all cases we have either $p_x(E)$ a single point or $p_x(E) \subset E$, in which case $p_x(E) = E \cap C(x)$. We see similarly that $p_{i,x}(E)$ is either a single point or equal to $E \cap C_i(x)$, since if E meets two components of $X \setminus \{x\}$ it must contain x. This last remark proves Corollary 1.

To prove Corollary 2, we suppose $y \in A_i(x) \cap C(x)$; then $M(y)$ is a closed connected subset of $X \setminus \{x\}$, so that $M(y) \subset A_i(x)$. But $M(y) \neq A_i(x)$ since $x \in \overline{A_i(x)}$, so that $p_x(A_i(x)) \neq \{y\}$. By the lemma, $A_i(x) \cap C(x) = p_x(A_i(x))$ and is connected since p_x is continuous; it is clearly not contained in any larger connected subset of $C^0(x)$.

2. Application to functions whose values are connected closed sets. We recall that a set-valued function is upper-semi-continuous (u.s.c.) iff, given any x_0 and any open set G containing $f(x_0)$, there exists a neighbourhood N of x_0 such that $f(x) \subset G$ for all $x \in N$. The following proposition follows at once from the definition and lemmas 4 and 5.

LEMMA 6. *If f is an upper-semi-continuous function defined on a set X (with the properties assumed in § 1), mapping points of X to non-empty closed connected subsets of X, then $p_x \circ f$ and $p_{i,x} \circ f$ are u.s.c. and map points of X to closed connected subsets of $C(x)$, $C_i(x)$ respectively.*

(Here, as usual, we use the same symbol for the point-to-point function p_x and the set-to-set function defined by $p_x(E) = \{p_x(y);\ y \in E\}$.)

We next prove a lemma (again under the same conditions on X) corresponding to lemma 7 of (5). We remark that if we assumed X regular (not merely Hausdorff) we could give an alternative proof much more like that of the lemma mentioned.

LEMMA 7. *Let f, X be as in lemma 6. If $x_1 < y_1$, $f(x_1) \subset M^0(x_1)$, and $x_2 = \sup E$, where $E = \{x;\ x \in M(x_1) \cap L(y_1)$ and $f(x) \subset M^0(x)\}$, then $x_2 \in E \cup f(x_2)$.*

The supremum x_2 exists (in $L(y_1)$) as $L(y_1)$ is a compact chain. Since the order-topology and induced topology agree on $L(y_1)$, and $X \setminus f(x_2)$ is open, if $x_2 \notin f(x_2)$ there exists z_1 in $L(x_2) \setminus \{x_2\}$ such that $z_1 < z_2 < x_2$ implies $z_2 \notin f(x_2)$. Take any such z_2 and suppose that, if possible, $f(x_2) \cap (X \setminus M(z_2)) \neq \emptyset$. Since $X \setminus M(z_2) = (X \setminus \{z_2\}) \setminus M^0(z_2)$ is open and closed in $X \setminus \{z_2\}$, which contains the connected set $f(x_2)$, we must have $f(x_2) \subset X \setminus M(z_2)$. By the u.s.c. property, there exists z_3 (depending on z_2) in $L(x_2) \setminus \{x_2\}$ such that $z_3 < z \leqslant x_2$ implies $f(z) \subset X \setminus M(z_2)$. But this is impossible, since, as $x_2 = \sup E$, there exists z in E such that $\max(z_2, z_3) < z \leqslant x_2$ ($\max(z_2, z_3)$ exists by the chain property), so that $f(z) \subset M^0(z) \subset M(z_2)$. Thus $f(x_2) \subset M^0(z_2)$ for all z_2 such that $z_1 < z_2 < x_2$. Let $y \in f(x_2)$; then $y > z_2$ for all z_2 as above, and hence (as $L(x_2)$ is a chain) for all $z_2 < x_2$, in particular for all $z_2 \in E$ if we suppose $x_2 \notin E$.

That is, if $x_2 \notin E$, $y \geqslant \sup E = x_2$ (now considering the sup in the compact chain $L(y) \cap L(y_1)$) so that as y is arbitrary we have $f(x_2) \subset M(x_2)$, contradicting our suppositions that $x_2 \notin f(x_2)$ and $x_2 \notin E$.

We are now in a position to state and prove our main theorem; we recapitulate all the conditions.

THEOREM 1. *Let X be a connected, locally connected and (asymmetrically) partially ordered Hausdorff space, such that, for all $x, y \in X$,* (a) *$M(x)$ is closed;* (b) *$M^0(x)$ is open;* (c) *$L(x) \cap L(y)$ is a compact chain. Let f be an upper-semicontinuous function, defined on X, whose values are non-empty closed connected subsets of X, such that $p_x \circ f | C(x)$ has the fixed-point property for every $C(x)$ (or alternatively, $p_{i,x} \circ f | C_i(x)$ has the fixed-point property for every $C_i(x)$, $x \in X$, $i \in I(x)$). Then either f has the fixed-point property on X (i.e. $x \in f(x)$ for some x) or there exists a sequence $\{x_n; n = 1, 2, \ldots\}$ in X such that*

(i) *$x_p > x_n$ whenever $p > n$;*

(ii) *$f(x_n) \subset M^0(x_n)$, all n;*

(iii) *$\cap M(x_n) = \emptyset$;*

(iv) *either $x_p \in f(x_n)$ for all n, all $p > n$, or $f(x_n) \cap M(x_p) = \emptyset$, all n, all $p > n$.*

COROLLARY. *If X is enumerably compact, then f has the fixed-point property.*

Suppose that, for all $x \in X$, $x \notin f(x)$.

We first form a sequence $\{x_n; n = 0, 1, 2, \ldots\}$ having properties (i), (ii) and (iii) above (for $n \geqslant 0$) and also the following property (iv)*: for $n \geqslant 1$, $x > x_n$ implies that either (a) $f(x) \cap M(x) = \emptyset$ or (b) $f(x_{n-1}) \cap M(x) = \emptyset$ or (c) $x \in f(x_{n-1})$. We begin with x_0, the least element of X, which satisfies (ii) since we assume that $x_0 \notin f(x_0)$. Suppose that x_n has been defined for $0 \leqslant n \leqslant m$ in such a way that (i), (ii), hold for $0 \leqslant n \leqslant m$, $1 \leqslant p \leqslant m$ and (iv)* for $1 \leqslant n \leqslant m$; we proceed to define x_{m+1}.

Set $a = x_m$ and let $b > a$ be any point selected from $f(x_m)$. By lemma 7, the set $\{x; a \leqslant x \leqslant b$ and $f(x) \subset M^0(x)\}$ has a greatest member c say (since $x \notin f(x)$ all x). If $c \neq a$, we put $x_{m+1} = c$, so that (i) and (ii) remain satisfied. If $x > x_{m+1} = c$ and if $f(x) \cap M(x) = f(x) \cap M^0(x) \neq \emptyset$, then as $f(x)$ is connected and $M^0(x)$ open and closed in $X \setminus \{x\}$ we have $f(x) \subset M^0(x)$, so by definition of c, $b \notin M(x)$; as b is in the connected set $f(x_m)$ the same argument shows that either $f(x_m) \cap M(x) = \emptyset$ or $x \in f(x_m)$. Thus (iv)* also holds for $n = m+1$.

We now consider the possibility that $c = a$. We suppose first that $L(b) \cap M^0(a)$ has no least member, and show that this is in fact impossible. For if so, since the order-topology of $L(b)$ agrees with its subspace topology, we have $a \in \big(L(b) \cap M^0(a)\big)^-$. As $a \in X \setminus f(a)$ (open), we can find $y \in L(b) \cap M^0(a) \cap \big(X \setminus f(a)\big)$. That is, $f(a) \subset M^0(y)$ (again by connectedness); by the u.s.c. condition we can find a neighbourhood U of a such that $x \in U$ implies $f(x) \subset M^0(y)$. Now by the same argument, since $L(y) \cap M^0(a)$ has no least member there exists $x \in U \cap L(y) \cap M^0(a)$. This however is impossible as $x \leqslant y \leqslant b$ and $f(x) \subset M^0(y) \subset M^0(x)$ so that, by definition of c, $a = c \geqslant x > a$.

Let therefore (if $c = a$) b' be the least member of $L(b) \cap M^0(a)$, so that, for some $i \in I(a)$, we have $b' \in C_i^0(a)$ and $p_a(b) = p_{i,a}(b) = b'$. By lemma 5, Corollary 1, $p_a[f(a)] = p_{i,a}[f(a)] \subset C_i^0(a)$, not containing a. By the hypotheses there exists some y in $C(a)$ with $y \in p_a \circ f(y)$ (or some y in $C_i(a)$ with $y \in p_{i,a} \circ f(y)$) so that $y > a = x_m$ and (since we assume $y \notin f(y)$), $f(y) \subset M^0(y)$. We can therefore put $x_{m+1} = y$ and satisfy (i) and (ii). As above, (iv)* is also satisfied since $x > x_{m+1}$ implies $x > x_m = c$.

We can therefore suppose x_n defined for all n, satisfying conditions (i), (ii) and (iv)*. We say that condition (iii) must also be satisfied. For if $y \in \cap M(x_n)$, in the compact chain $L(y)$ there exists $s = \sup x_n$, and $s \in \bar{S}$ where S is the set $\{x_n; n = 1, 2, \ldots\}$. By lemma 7, condition (ii), with $s \notin f(s)$, implies that $f(s) \subset M^0(s)$. Then by upper-semicontinuity, since $s \in \bar{S}$, we have $f(x_{n-1}) \subset M^0(s) \subset M^0(x_{n+1})$, for some n, but this, with (i) and (ii), contradicts (iv)*.

Finally, we can choose a subsequence $\{x_{r(n)}, n = 1, 2, \ldots\}$ of $\{x_n\}$ which (with the obvious change of notation) satisfies condition (iv) as well as (i), (ii) and (iii). Suppose first that for all N there exists $p = p(N) > N$ such that $x_q \in f(x_p)$ for all $q \geqslant q(p)$ say (necessarily $q(p) > p$). Take in succession $r(1) = p(1)$, $r(m+1) = p[q(r(m))]$; then $f(x_{r(n)})$ contains $x_{r(m)}$ for all $m > n$. The other possibility is that, for some N_0, to every $p \geqslant N_0$ corresponds some $q = q(p)$, as large as we like, such that $x_q \notin f(x_p)$; we need require only that $q \geqslant p+2$ and then condition (iv)* requires that $f(x_p) \cap M(x_q) = \emptyset$, so that $f(x_p) \cap M(x_n) = \emptyset$ for all $n \geqslant q$. If we put $r(1) = N_0$ and $r(n+1) = q[r(n)]$, we obtain a sequence satisfying the second condition of (iv).

Alternative statement of the theorem. Let Q_0 denote the class of upper-semi-continuous functions whose values are nonempty connected closed subsets of the domain of definition of the function. Then, in view of lemmas 4 and 5, we can state our theorem as follows:

THEOREM 1A. *If X satisfies the conditions of Theorem* 1, *and if each $C(x)$ (or alternatively each $C_i(x)$) has the fixed-point property for functions of class Q_0 (defined on $C(x)$ or $C_i(x)$ respectively), then given a function f of class Q_0 on X, either f has the fixed-point property or a sequence $\{x_n\}$ exists with the properties* (i) *to* (iv).

In a similar way, we can state a theorem for functions possessing some property Q (cf. Theorem 1 of [5]), provided that if any function f has property Q on X and p is an order-preserving retraction on X, then

$p \circ f | p(X)$ has property Q whenever it is of class Q_0. (As we have shown, the retractions we actually use preserve the property of being of class Q_0, but it is easily seen that this is not in general true of all order-preserving retractions, since $p \circ f(x)$ need not be closed.) The detailed statement of this form of the theorem is somewhat tedious and is left to the reader.

3. Application to the cut-point order.

THEOREM 2. *If X is a connected and locally connected Hausdorff space, then the cut-point order on X (with respect to any $x_0 \in X$) satisfies the conditions* (a), (b) *and* (c) *of Theorem* 1. *The non-degenerate sets $C_i(x)$ are the maximal non-degenerate cut-point-free connected subsets of X (Whyburn's E_0-sets).*

We recall that the cut-point order with respect to x_0 is defined by writing $x < y$ iff either $x = x_0$, $y \neq x_0$ or x separates x_0 from y; that is, there exist in $X \setminus \{x\}$ open complementary sets A, B such that $x_0 \in A$, $y \in B$. (If X is locally connected this is equivalent to saying that y does not lie in that component $A_0(x)$ of $X \setminus \{x\}$ which contains x_0.) It is indeed a partial order, and has the properties: (i) for every $x, y \in X$, $L(x)$, and hence also $L(x) \cap L(y)$, is a chain containing x_0; (ii) if $y < x$, then z separates x from y iff $y < z < x$. It is clear that if X is locally connected, then $M(x) = X \setminus A_0(x)$, closed, while $M^0(x) = X \setminus \overline{A_0(x)}$, open.

It remains to prove that $L(x)$ is compact. This was proved by Whyburn, in [6], chapter III, Theorem 4.2, in the case when X is metric; the basic ideas of his proof can, as we shall briefly show, easily be adapted to our more general case. We note first that if $y \notin L(x)$, then (X being locally connected and Hausdorff) every $z \in A_0(y)$ has a connected neighbourhood $U(z)$ such that $y \notin \overline{U(z)}$, so that $\overline{U(z)} \subset A_0(y)$. By Whyburn's "chain lemma" (loc. cit., II, Theorem 3.1), it follows that there exists a connected closed subset of $X \setminus \{y\}$ which contains x_0 and x. We deduce (cf. loc. cit., III, Theorem 4.12) that $L(x)$ is closed ([2]).

Now let $\{G_i, i \in I\}$ be any relatively open cover of $L(x)$. Since X is locally connected and $L(x)$ closed, X can be covered by connected open sets $V(y)$ such that $L(x) \cap V(y)$ is (empty or) contained in some G_i. Again by the chain lemma, there exists a connected set, $V = \bigcup (V(y_r);\ r = 1, 2, \ldots, n)$ say, containing both x_0 and x. If $y \notin V$, then $A_0(y) \supset V$; that is, $L(x) \subset V$, and hence is covered by a finite subset of $\{G_i\}$.

Now let E be any E_0-set of X; as E is connected and cut-point free (in itself) it is easily seen that no point of X can separate two points of E. Let $a, b \in E$ $(a \neq b)$ and let c be the greatest point of $L(a) \cap L(b)$.

([2]) This part of the result is stated (for X locally connected and Hausdorff) by Hocking and Young [2], Theorem 3.8. The proof there given, while essentially the same as that sketched above, appears however to assume X regular.

Then (using the notation of § 1, applied to the cut-point order) all points of E other than c lie in the same component A_i of $X \setminus \{c\}$; as one at least of a, b is in $M^0(c)$ we have $E \subset A_i \cup \{c\} \subset M(c)$. In the same way, no point x can separate c from any point of E, as if so it must separate c from one at least of a, b, so that (e.g.) $c < x < a$, $c < x \leqslant b$, contrary to the definition of c; hence $E \subset C(c)$ so that $E \subset C_i(c)$.

Conversely, $C_i(x)$ is always connected, being $p_{i,x}(X)$ where $p_{i,x}$ is continuous. We show that it is cut-point-free (in itself). Now $C_i(x) \setminus \{x\} = C_i^0(x)$ is connected by lemma 5, Corollary 2, while if $y \in C_i^0(x)$, we have $C_i(x) \setminus \{y\} = p_{i,x}\big(X \setminus M(y)\big) = p_{i,x}\big(A_0(y)\big)$, again connected as $p_{i,x}$ is continuous.

On combining these results, together with the obvious fact that distinct $C_i(x)$, $C_j(x')$ intersect in at most one point, even if $x = x'$, we see that every non-degenerate $C_i(x)$ is an E_0-set, and conversely.

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UNIVERSITY OF WASHINGTON, SEATTLE, WASH., U.S.A.
EMMANUEL COLLEGE, CAMBRIDGE, ENGLAND

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