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ON SELFREPRODUCTIVE SETS

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Selfreproductive continua, as defined below, arise naturally in the study of some problems concerning ε-maps. This note gives a simple characterization of the compact selfreproductive sets of dimensions 0 and 1 and shows that there are no higher dimensional selfreproductive compact sets. An application to a simple problem is given following Theorem 4. The methods used here are mostly basic theorems of dimension theory which can be found in [2].

The first version of this paper did not settle the question of whether there are any selfreproductive compact sets of dimension greater than 1. The author is indebted to Dr. Molski who pointed out that his example [3] of a collection of n-dimensional AR-sets no one of which contains an n-cell provided an answer to this question. Molski's example has many additional desirable properties which are unnecessary for my purposes and introduce considerable complications into his construction. In considering just what features of his construction are needed to answer the question, it became clear that a simpler construction would suffice. This example appears as Theorem 7.

If ε is a positive number, a map f from the compact metric space (X, d) is said to be an *ε-map* if $f(x) = f(y)$ implies $d(x, y) < \varepsilon$. Since X is compact, this is equivalent to the assertion that $f^{-1}f(x)$ has diameter less than ε for each x in X. A metrizable compact set M is called *selfreproductive* if there is a positive number ε such that for each ε-map f whose domain is M, $f(M)$ contains a subset homeomorphic to M. In this note all simplexes are assumed to be compact and a polyhedron is only the union of a finite collection of simplexes and not necessarily connected. We will use the covering definition of dimension. All spaces are assumed to be metric.

THEOREM 1. *Each n-dimensional compact selfreproductive set M is homeomorphic to a subset of an n-dimensional polyhedron.*

Proof. Suppose ε is a positive number. There is a finite open cover $\mathscr{U}$ of M of order not more than n such that each member of $\mathscr{U}$ has diameter less than $\varepsilon/2$. There is a map f from M into a geometric realization of the nerve of $\mathscr{U}$ (which is a polyhedron of dimension not more than n) sending each member U of $\mathscr{U}$ into the star of the vertex corresponding to U ([2], p. 70). Then f is an ε-map. For each positive number ε, there is an ε-map of M into an n-dimensional polyhedron.

COROLLARY. *A 0-dimensional set is selfreproductive if and only if it is finite.*

THEOREM 2. *If n is a positive integer, each n-dimensional compact selfreproductive set contains an n-cell.*

Proof. An n-dimensional compact subset of an n-dimensional polyhedron must contain an open subset of the interior of some n-cell, by Theorem IV, 3 of [2].

Remark. There are no infinite dimensional compact selfreproductive sets since each compact set can be mapped into a polyhedron (whose dimension may change with ε) by an ε-map for each positive number ε.

LEMMA ([1], p. 103, Satz VI). *Suppose f is an ε-map from a compact set X into a compact set Y. There is a positive number δ such that if g is a δ-map from Y into Z, then gf is an ε-map from X into Z.*

THEOREM 3. *Each 1-dimensional selfreproductive continuum is homeomorphic to a 1-dimensional polyhedron with no points of order higher than 3.*

Proof. Suppose M is a 1-dimensional polyhedron, p is a point of M of order higher than 2, and δ is a positive number. There are points a and b of M and arcs α and β from a to p and b to p respectively such that $\alpha \cap \beta$ is $\{p\}$ and $\alpha \cup \beta$ is a subset of the δ-neighborhood of p, and each point of $\alpha \cup \beta$ other than p has order 2 in M. There is a homeomorphism h from α onto β such that $h(p)$ is p. There is a map f from M onto a 1-dimensional polyhedron such that $f(x) = f(h(x))$ for each x in α and such that if x is not in $\alpha \cup \beta$, $f^{-1}(fx)$ is $\{x\}$. Then f is a (2δ)-map, $f(p)$ has order one less than the order of p, $f(a)$ is of order 3, $f(x)$ is of order 2 for x a non-endpoint of α, and the order of $f(x)$ is the order of x if x is in M but not in $\alpha \cup \beta$.

Now suppose M is a 1-dimensional continuum and ε is a positive number. We will reduce the order of points of M to 3 or less in steps, using the construction just made. There is an ε-map f from M onto a 1-dimensional polyhedron. Suppose $f(M)$ has a point p of order higher than 3. There is a map g_1 of $f(M)$ into a 1-dimensional polyhedron such that g_1f is an ε-map, $g_1(p)$ has order less than the order of p in $f(M)$, and the order of $g_1(x)$ is no more than the greater of 3 and the order of x in $f(M)$, for x in M, x not p. Since $f(M)$ has only a finite number of points with order more than 2, there is a sequence $g_2, \ldots, g_n$ of maps such that $g_n g_{n-1} \ldots g_2 g_1 f$ is an ε-map from M onto a 1-dimensional polyhedron with no points of order higher than 3. Therefore, each selfreproductive 1-dimensional continuum is homeomorphic to a subcontinuum of a 1-dimensional polyhedron with no points of order higher than 3, which proves the theorem.

THEOREM 4. *Each 1-dimensional polyhedron with no points of order higher than 3 is selfreproductive.*

Proof. It will be sufficient to show that each connected 1-dimensional polyhedron with no points of order higher than 3 is selfreproductive. Suppose M is a connected 1-dimensional polyhedron with no points of order higher than 3, M is the union of the simplexes of the complex S and S contains more than one 1-simplex. Let ε be a positive number such that any vertex of S is at least 3ε from any 1-simplex which does not contain it. Suppose f is an ε-map with domain M. We will construct a subcontinuum of $f(M)$ homeomorphic to M. For each 1-simplex e of S let $T(e)$ be the arc which is the middle third of e. The endpoints of $T(e)$ are at least ε apart. For any two 1-simplexes e and e' of S, $fT(e)$ and $fT(e')$ are disjoint non-degenerate locally connected continua. For each vertex p of S let $U(p)$ denote the union of the (at most three) arcs of the 1-simplexes having p as a vertex which run exactly one third of the distance from p to the other vertex. If e is a 1-simplex and p and p' are two vertices of S, then $fU(p)$ and $fU(p)'$ are disjoint continua and $fT(e)$ intersects $fU(p)$ if and only if p is a vertex of e. There is a function a from S into the collection of subcontinua of $f(M)$ such that $a(p)$ is an arc, an arc, or a simple triod of $fU(p)$ if p is a 0-simplex of order $1, 2$ or 3 in M respectively and $a(e)$ is a subarc of $fT(e)$ if e is a 1-simplex of S and such that $a(p) \cap a(e)$ is exactly one point, an endpoint of each, if p is a 0-simplex which is a vertex of the 1-simplex e. Then M is homeomorphic to $\cup \{a(s)\colon s \in S\}$, and M is selfreproductive.

There is a well known non-planar 1-dimensional polyhedron Y which has 6 points of order 3 that can be divided into two sets A and B such that Y is the union of 9 non-overlapping arcs, one from each point of A to each point of B. By Theorem 4, Y is selfreproductive and so each continuum that has a subcontinuum homeomorphic to Y cannot be ε-mapped into the plane for each positive number ε. This fact implies that if M is an n-dimensional polyhedron, $n > 2$, then there is a positive number ε such that there is no ε-map of M into the plane. Of course, this is also a simple consequence of elementary theorems in dimension theory.

It might have been remarked earlier that the property of being selfreproductive is a topological property for compact sets. However, this immediate consequence of the Lemma was not needed before the preceding example.

THEOREM 5. *Each component of a selfreproductive* 1-*dimensional compact set is selfreproductive.*

Proof. A selfreproductive 1-dimensional set is homeomorphic to a subset of a 1-dimensional polyhedron with no points of order more than 3. Each component is either a point or a selfreproductive 1-dimensional continuum by Theorem 4.

The converse of Theorem 5 is not true. The Cantor set crossed with an arc is a set each of whose components is selfreproductive, but which is not itself selfreproductive. However, a less elegant characterization of 1-dimensional selfreproductive compact sets can be given as follows:

THEOREM 6. *Each selfreproductive* 1-*dimensional compact set* M *has these properties*:

(1) *Each component of* M *is selfreproductive.*

(2) *All but finitely many of the components of* M *are arcs or points.*

(3) *Each point of* M *which is not an end point of the component of* M *which contains it is an interior point of that component.*

(4) *Either some* 1-*dimensional component of* M *has an endpoint or* M *has only finitely many* 0-*dimensional components.*

Conversely, any 1-*dimensional compact set with properties* (1), (2), (3) *and* (4) *is selfreproductive.*

Proof. The proof of the first part follows easily from the fact that each selfreproductive 1-dimensional set M is homeomorphic to a subset of a 1-dimensional selfreproductive polyhedron each of whose points of order 3 is interior to M.

Suppose on the other hand that M is a 1-dimensional compact set which satisfies properties (1), (2), (3) and (4). If M has only finitely many 0-dimensional components and no component which is an arc, then M is clearly selfreproductive. Suppose M has a component with an endpoint. The interiors of finitely many simple triods and arcs can be removed from M to give a set M' which can be embedded in an arc. But then M can be embedded in a 1-dimensional polyhedron with no points of order higher than 3. The subset of M consisting of all points not in a component with a point of order 3 can be embedded in an arc. If this is all of M, M is clearly selfreproductive since each compact 1-dimensional subset of an arc is selfreproductive. If M is not homeomorphic to a subset of an arc, the union M'' of the components of M with a point of order 3, the simple closed curves of M, and a component of M having an endpoint is itself selfreproductive and also contains a proper subset homeomorphic to M. Therefore, M is selfreproductive.

The theorem which follows will provide an easy proof that there are no compact selfreproductive sets of dimension 2 or higher.

THEOREM 7. *Suppose M is an n-cell, n not less than 2, and ε is a positive number. Then there is an ε-map f defined on M which is a homeomorphism on the boundary of M and is such that $f(M)$ contains no n-cell.*

Proof. There is a null-sequence of disjoint sets of two points each, $A(1), A(2), \ldots$ such that $\{A(i): i \geqslant 1\}$ is a subset of the interior of M which is dense in M and such that each set $A(i)$ has diameter less than ε. The projection f from M onto the decomposition space of the upper semi-continuous decomposition of M whose only non-degenerate elements are $A(1), A(2), \ldots$ is clearly an ε-map. Suppose $f(M)$ contains an n-cell M'. Since f is one-to-one on the complement of a countable set, one may assume that the boundary of M' is mapped homeomorphically back to M by f^{-1}. If $f^{-1}(M')$ does not contain a complementary domain of f^{-1} of the boundary of M', then there is a compact proper subset F of an $(n-1)$-sphere which lies in a complementary domain of f^{-1} of the boundary of M', separates $f^{-1}(M')$ and misses all of $A(1), A(2), \ldots$ But then $f(F)$ together with $A(1), A(2), \ldots$ (considered as points of $f(M)$) separates M' which is absurd since F is not homeomorphic to a set separating euclidean n-space. Therefore $f^{-1}(M')$ contains an n-cell. But this is also a contradiction since there are then uncountably many $(n-1)$-spheres in M' which miss the boundary of M' but do not separate M'.

COROLLARY 8. *If M is an n-dimensional polyhedron, n not less than 2, and ε is a positive number, there is an ε-map f defined on M such that $f(M)$ contains no n-cell.*

COROLLARY 9. *There is no n-dimensional selfreproductive compact set if n is not less than 2.*

Proof. By Theorem 2, an n-dimensional selfreproductive compact set contains an n-cell. But the previous corollary shows that this is impossible.

The conclusion that there are no n-dimensional compact selfreproductive sets when n is larger than 1 is disappointing. Such sets, if they did exist, would have applications to some problems involving ε-maps. However, a more restrictive definition like the one below might prove useful. One could define a set M to be selfreproductive in the class of spaces $\mathscr{M}$ if there is a positive number ε such that if f is an ε-map from M into a member of the class $\mathscr{M}$, then $f(M)$ contains a subset homeomorphic to M. By suitable restrictions on the class $\mathscr{M}$, the collection of compact sets selfreproductive in the class $\mathscr{M}$ becomes considerably richer.

REFERENCES

[1] P. Alexandroff und H. Hopf, *Topologie*, Berlin 1935.

[2] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton 1941.

[3] R. Molski, *On a family of AR-sets*, Fundamenta Mathematicae 57 (1965), p. 135-145.

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