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ON MAPPINGS OF CANTORIAN MANIFOLDS

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Let us recall that a *Cantorian manifold* is understood to mean a finite-dimensional compact connected metric space X such that X remains connected after removing any subset of dimension less than $\dim X-1$. A question from my paper [2] concerning mappings of Euclidean spheres has been answered by Skljarenko [3] and Jung [1] who proved that if f is a non-constant mapping of a Cantorian manifold X, then

$$0 < \dim\{y\colon \dim X - \dim f(X) \leqslant \dim f^{-1}(y)\}. \tag{i}$$

The aim of the present paper is to strengthen this inequality by localizing the dimension of $f(X)$. Namely, we use the inductive definition of dimension and instead of $\dim f(X)$ we take $\dim_y f(X)$, i.e. the dimension of the image $f(X)$ at the point y (see Theorem 1 below). Observe that $\dim_x X = \dim X$ for all $x \in X$ because X is a Cantorian manifold. Outside the class of Cantorian manifolds an analogue can be proved under the strong restriction that $\dim X \leqslant 2$ (see Theorem 2). That this restriction is necessary is shown by an example at the end of the paper.

We start with a lemma which modifies a lemma of Vaĭnšteĭn [4].

LEMMA. *Suppose $X_i \subset X$ is an F_σ in a separable metric space X $(i=0,1,2,\ldots)$ and X_0 is 0-dimensional. If A, B are disjoint closed subsets of X, then there exists a closed separator C of X between A and B such that $C \cap X_0 = \emptyset$ and, for every $i=1,2,\ldots$ and $x \in X$, the inequalities*

$$0 < \dim_x [X_i \cup \{x\}] < \infty$$

imply the inequality

$$\dim_x[(C \cap X_i) \cup \{x\}] < \dim_x [X_i \cup \{x\}].$$

Proof. Denote by X_{ij} the set of all points $x \in X$ such that $X_i \cup \{x\}$ is j-dimensional at x $(i,j=1,2,\ldots)$. Then there exists a collection β_{ij} of open subsets of X such that the intersection of X_i with the boundary of each set from β_{ij} is at most $(j-1)$-dimensional, and every point of X_{ij} possesses a local open basis in X consisting of sets from β_{ij}. Since X is separable metric, there exists a countable collection $\gamma_{ij} \subset \beta_{ij}$ with the latter property. Let us take all non-empty intersections X_{ijk} $(k = 1, 2, \ldots)$ of X_i with the boundaries of the sets from γ_{ij}, and put $X_{ijk} = X_0$ provided $i \cdot j \cdot k = 0$. Since X_{ijk} are closed subsets of X_i, each X_{ijk} is an F_σ in X $(i, j, k = 0, 1, 2, \ldots)$. According to Vaĭnšteĭn's lemma (see [4], p. 176), there exists a closed separator C of X between A and B such that

$$\dim(C \cap X_{ijk}) < \dim X_{ijk}$$

for $i, j, k = 0, 1, 2, \ldots$ It readily follows that C fulfils requirements of our lemma.

THEOREM 1. *If f is a non-constant mapping of a Cantorian manifold X, then*

$$\text{(ii)} \qquad 0 < \dim\{y\colon \dim X - \dim_y f(X) \leqslant \dim f^{-1}(y)\}.$$

Proof. Inequality (ii) trivially holds if $f(X)$ is infinite-dimensional. Thus we can prove the theorem inductively on $k = \dim f(X)$. Since f is non-constant and X is connected, we have $k \geqslant 1$ and (ii) follows from (i) provided $k = 1$. Assume the theorem is true for $k < n$ and we are given a non-constant mapping f of a Cantorian manifold X such that $\dim f(X) = n$.

Put $m = \dim X$, and $Y = \{y\colon m-1 \leqslant \dim f^{-1}(y)\}$. The set $f(X)$ is positive-dimensional at each of its points, and therefore (ii) holds if Y is positive-dimensional. Assuming $\dim Y \leqslant 0$, let us consider two points p, q of $f(X)$. Since Y is an F_σ in $f(X)$, it follows from the lemma that there exists a closed separator C of $f(X)$ between p and q such that

$$C \cap Y = \emptyset, \qquad \dim_y C < \dim_y f(X)$$

for $y \in C$. Consequently, we have $\dim C < n$ and $m-1 \leqslant \dim f^{-1}(C)$ because $f^{-1}(C)$ cuts the m-dimensional Cantorian manifold X. Thus $f^{-1}(C)$ contains a Cantorian manifold X' of dimension at least $m-1$. Its image $f(X')$ cannot be a single point as $f(X')$ lies in C and C is disjoint with Y. By the inductive hypothesis, the theorem applies to $f|X'$ and X'; but

$$\dim X - \dim_y f(X) \leqslant m - \dim_y C - 1 \leqslant \dim X' - \dim_y f(X')$$

for $y \in f(X')$, and the proof of Theorem 1 is completed.

Remark. An attempt to strengthen inequality (i) by localizing the dimension of $f^{-1}(y)$ will probably fail. Related to this is the following question (**P 614**): does there exist a Cantorian manifold X and a mapping f of X such that $\dim f(X) < \dim X$ and each counter image $f^{-1}(y)$, where $y \in f(X)$, contains a point x at which it is 0-dimensional, i.e. $\dim_x f^{-1}(y) = 0$?

THEOREM 2. *If f is a non-constant mapping of an at most 2-dimensional continuum X and $m(y) = \max\{\dim_x X\colon x \in f^{-1}(y)\}$, then*

$$\text{(iii)} \qquad 0 < \dim\{y\colon m(y) - \dim_y f(X) \leqslant \dim f^{-1}(y)\}.$$

Proof. Let Y denote the set $\{\ \}$ from inequality (iii). We have $\dim Y > 0$ provided $\dim X = 1$ or $\dim f(X) > 1$, so let us assume that $\dim X = 2$ and $\dim f(X) = 1$. If $m(y) = 2$ implies $\dim f^{-1}(y) > 0$, for each $y \in f(X)$, then $Y = f(X)$. Thus we can also assume that there exists a point $y_0 \in f(X)$ such that $m(y_0) = 2$ and $f^{-1}(y_0)$ is 0-dimensional.

Put $Y' = \{y\colon 1 \leqslant \dim f^{-1}(y)\}$. Since $Y' \subset Y$, it suffices to prove that $\dim Y' > 0$. Suppose on the contrary that $\dim Y' \leqslant 0$, and consider a point $x_0 \in f^{-1}(y_0)$ such that $\dim_{x_0} X = 2$. Let U be an arbitrary neighbourhood of x_0 in X. The set $f^{-1}(y_0)$ being 0-dimensional, there exists a neighbourhood V of x_0 in X such that $\overline{V} \subset U$ and the boundary $\overline{V} \setminus V$ of V is disjoint with $f^{-1}(y_0)$. Consequently, the sets $\{y_0\}$ and $f(\overline{V} \setminus V)$ are disjoint. Since Y' is an F_σ in $f(X)$, it follows from the lemma above that there exists a closed separator C of $f(X)$ between y_0 and $f(\overline{V} \setminus V)$ such that

$$C \cap Y' = \emptyset, \qquad \dim C < 1$$

which yields

$$\dim f^{-1}(C) \leqslant \dim C + \dim f|f^{-1}(C) = 0,$$

according to the Hurewicz theorem. But $X' = f^{-1}(C) \cap \overline{V}$ is a separator of $\overline{V}$ between $f^{-1}(y_0) \cap \overline{V}$ and $\overline{V} \setminus V$. Thus X' is a 0-dimensional separator of X between x_0 and $X \setminus V$. This contradicts the equality $\dim_{x_0} X = 2$, and Theorem 2 is proved.

EXAMPLE. *There exists a non-constant mapping f of a 3-dimensional continuum X such that*

$$\dim_x X = 3, \qquad \dim_y f(X) = 1$$

for $x \in X$ and $y \in f(X)$, and the set $\{y\colon 2 \leqslant \dim f^{-1}(y)\}$ is countable.

Proof. Denote by I the unit segment of the real line and by $J_1, J_2, \ldots$ the segments which are closures of components of the complement of the Cantor set in I. Let D_i be a plane circular disk whose diameter is J_i $(i = 1, 2, \ldots)$. We define X by the formula

$$X = (I \cup D_1 \cup D_2 \cup \ldots) \times I,$$

and we determine f as a mapping induced by the upper semi-continuous decomposition of X into sets $D_i \times I$, where $i = 1, 2, \ldots$, and the sets $\{p\} \times I$, where $p \in I \setminus (D_1 \cup D_2 \cup \ldots)$.

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