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ON HEREDITARILY LOCALLY CONNECTED SPACES AND ONE-TO-ONE CONTINUOUS IMAGES OF A LINE

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An instructive exercise in an elementary course in topology is to construct one-to-one continuous mappings from the reals to a metric space which are not homeomorphisms. The problem of classifying all continuous mappings from the reals to a metric space is more difficult. An image of the reals under a continuous mapping may fail to be both locally connected and locally compact at each point. Its closure may be an indecomposable continuum on the plane.

Certain flows in topological dynamics [1] have orbits which are one-to-one continuous images of the reals. And, there is considerable interest in problems involving such mappings. Let E^n denote the Euclidean n-space. There are five topologically different locally connected and locally compact images of the line E^1 under a one-to-one continuous mapping (Theorem 1 of this paper). However, we give an example of a locally connected one-to-one continuous image of E^1 in E^3 which is not locally compact and whose closure is not locally connected (Example 2).

Hereditarily locally connected spaces play a role in the classification of continua [7]. By a *continuum* we mean a connected compact metric space, and we say a connected space X to be *hereditarily locally connected* provided all connected subsets of X are locally connected. Sometimes, for continua, another definition is used: a continuum C is called hereditarily locally connected if all subcontinua of C are locally connected. But, as shown by Wilder's theorem (see [5], p. 199), these two definitions are equivalent. We obtain a result which constitutes a possible general sum theorem for the class of hereditarily locally connected continua (Theorem 2).

Some of the simpler examples of one-to-one continuous images of the line are the following: (I) open interval, (II) figure eight, (III) dumbbell, (IV) letter theta, (V) noose (fig. 1).

THEOREM 1. *If a locally connected and locally compact metric space X is a one-to-one continuous image of the line, then X is homeomorphic to one of the five objects* (I)-(V) *listed above.*

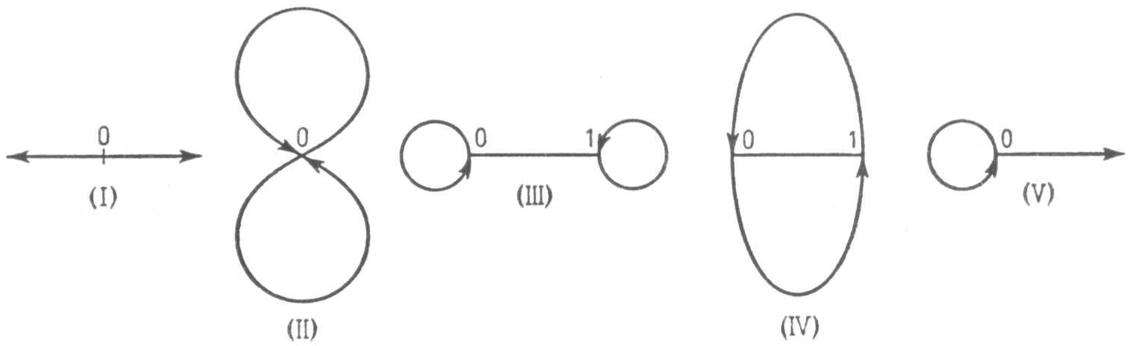

Fig. 1

Proof. Let $f\colon E^1 \to X$ be a one-to-one continuous mapping of the line onto X. First we prove that each one of the sets

$$H^+ = \bigcap_{n=1}^{\infty} \mathrm{cl}\{f(t)\colon t \geqslant n\}, \qquad H^- = \bigcap_{n=1}^{\infty} \mathrm{cl}\{f(t)\colon t \leqslant -n\}$$

either is empty or consists of a single point. By symmetry, it suffices to prove this only for H^+. Suppose on the contrary that H^+ contains at least two points. Let us take a point $p \in H^+$ such that in case H^- consists of exactly one point this point is not p. Then we can find a point $q \neq p$ such that $q \in H^+$, and a point $r \neq p$ such that $r \in H^-$ provided $H^- \neq \emptyset$. Since X is locally connected and locally compact, there exists a continuum $C \subset X$ containing p in its interior and containing neither q nor r. It follows that there exists a countable set of real numbers

$$\ldots < t_{-2} < t_{-1} < t_0 < t_1 < t_2 < \ldots$$

such that t_i tend to $+\infty$ (or $-\infty$) when i tend to $+\infty$ (or $-\infty$, respectively), and C contains no one of the points $f(t_i)$. Consequently, the continuum C is the union of compact sets $C \cap f([t_i, t_{i+1}])$ which are pairwise disjoint and infinitely many of them are non-empty. The last statement contradicts Sierpiński's theorem (see [5], p. 113). Thus both H^+ and H^- are either empty or single points.

From what is already proved it follows, by local compactness of X, that if $H^+ = \{p^+\}$ or $H^- = \{p^-\}$, and $u_n \in E^1$ are numbers tending to $+\infty$ (or $-\infty$), then their images $f(u_n)$ converge to p^+ (or p^-, respectively). Now, the conclusion is clear. If both H^+ and H^- are empty, f is a homeomorphism and X is of type (I). If only one of the sets H^+ and H^- is empty, X is of type (V). If both H^+ and H^- are non-empty, X is of type (II), or (III), or (IV), depending on whether $p^+ = p^-$, or $p^+ > p^-$, or $p^+ < p^-$, respectively.

Remark. Easy examples (fig. 2) show that the local connectedness is an essential condition in Theorem 1. We shall see, by Example 0, that

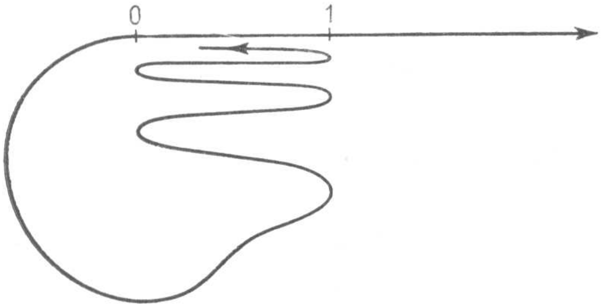

Fig. 2

also the local compactness cannot be omitted. However, we do not know if local compactness in Theorem 1 can be replaced by the condition that X lies on the plane (compare problem P 615 below) (1).

LEMMA. *Suppose Z is a zero-dimensional compact subset of a metric space X. There exists, for every $\varepsilon > 0$, a number $\eta > 0$ such that if $K \subset X$ is a continuum with* $\operatorname{diam} K \geqslant \varepsilon$, *then $K \setminus Z$ contains a continuum Q with* $\operatorname{diam} Q \geqslant \eta$.

Proof. Since Z is zero-dimensional and compact, there exist open subsets $U_1, \ldots, U_m$ of X such that

$$Z \subset U_1 \cup \ldots \cup U_m, \qquad \operatorname{diam} U_i < \varepsilon/4, \qquad \operatorname{cl} U_i \cap \operatorname{cl} U_j = \emptyset$$

for $i \neq j$. Then the number

$$\eta = \min_{i \neq j} \{\varepsilon/4, \ \operatorname{dist}(U_i, U_j)\}$$

is positive and satisfies all requirements. Indeed, let $K \subset X$ be a continuum such that $\operatorname{diam} K \geqslant \varepsilon$. We have two points $p, q \in K$ with $\operatorname{dist}(p, q) \geqslant \varepsilon$. If $K \cap Z = \emptyset$, we take $Q = K$. If $K \cap Z \neq \emptyset$ and K meets more than one of the sets U_i, a certain component Q of the compact set $K \setminus (U_1 \cup \ldots \cup U_m)$ must join boundaries of a pair of sets U_i, whence $\operatorname{diam} Q \geqslant \eta$. If $K \cap Z \neq \emptyset$ and K intersects only one of the sets U_i, say U_k, let us take a point $r \in K \cap Z$. Then $r \in U_k$ and at least one of the distances $\operatorname{dist}(p, r)$, $\operatorname{dist}(r, q)$ is greater than or equal to $\operatorname{dist}(p, q)/2$.

Assuming that

$$\operatorname{dist}(p, r) \geqslant \operatorname{dist}(p, q)/2 \geqslant \varepsilon/2,$$

we get $\operatorname{diam} Q \geqslant \varepsilon/4$ for the component Q of $K \setminus U_k$ which contains p; but $K \setminus U_k \subset K \setminus Z$.

(1) *Added in proof.* We are informed by J. Burton Jones that the answer to the problem is "yes". A sketch of an argument has been submitted to this journal.

THEOREM 2. *If a continuum C admits a countable decomposition $C = C_1 \cup C_2 \cup \ldots$ such that*

$$\text{(i)} \qquad \dim \operatorname{cl} \bigcup_{i \neq j} C_i \cap C_j = 0,$$

$$\text{(ii)} \qquad \lim_{i \to \infty} \operatorname{diam} C_i = 0,$$

and C_i are hereditarily locally connected $(i = 1, 2, \ldots)$, then C is hereditarily locally connected.

Proof. Suppose on the contrary that C is not hereditarily locally connected. Then by Zarankiewicz's theorem (see [5], p. 196), there exists in C an infinite sequence of pairwise disjoint continua $K_0, K_1, K_2, \ldots$ such that K_m converge to K_0, and $\operatorname{diam} K_m \geqslant \varepsilon > 0$ for $m = 0, 1, \ldots$

Applying (i) and the lemma we get continua

$$Q_m \subset K_m \setminus \bigcup_{i \neq j} C_i \cap C_j$$

such that $\operatorname{diam} Q_m \geqslant \eta > 0$ for $m = 0, 1, \ldots$ Hence each Q_m is contained in exactly one of the continua C_i, and it follows from (ii) that a certain continuum C_{i_0} contains infinitely many of the continua Q_m. This means, by Zarankiewicz's theorem again, that C_{i_0} cannot be hereditarily locally connected, and the proof is completed.

Remark. Decompositions of the condensed sinusoid (compare fig. 2) into arcs are good examples to argue that neither (i) nor (ii) can be removed from Theorem 2.

EXAMPLE 0. *A bounded one-to-one continuous image Y of E^1 in E^3 such that both Y and its closure $\operatorname{cl} Y$ in E^3 are hereditarily locally connected, and Y is not locally compact.*

Let $p_1, p_2, \ldots$ be the increasing sequence of all prime numbers greater than 2. Denote by A_i the arc in E^3 composed of $p_i - 2$ demi-circles which successively join $p_i - 1$ points

$$(1/p_i, 0, 0), \quad (2/p_i, 0, 0), \quad \ldots, \quad \big(1-(1/p_i), 0, 0\big)$$

on the half-plane $y \geqslant 0$, $z = y/i$. Denote by B_0 the half-open interval $x = 1/p_1$, $-1 < y \leqslant 0$, $z = y$, and take demi-circles B_{0j} and B_{1j} joining the points

$$(1/p_{2j}, 0, 0), \quad (1/p_{2j+1}, 0, 0)$$

and

$$\big(1-(1/p_{2j-1}), 0, 0\big), \quad \big(1-(1/p_{2j}), 0, 0\big),$$

respectively, on the half-plane $y \leqslant 0$, $z = 0$. It is readily verifiable that the bounded set

$$Y = \bigcup_{i=1}^{\infty} A_i \cup B_0 \cup \bigcup_{i=0}^{1} \bigcup_{j=1}^{\infty} B_{ij}$$

can be represented as a one-to-one continuous image of E^1. Moreover, the closure $\mathrm{cl}\,X$ of the union $X = A_1 \cup A_2 \cup \ldots$ (compare [2], p. 235) is well known Urysohn's hereditarily locally connected continuum (see [6], p. 46). According to Theorem 2, the continuum

$$\mathrm{cl}\,Y = \mathrm{cl}\,X \cup \mathrm{cl}\,B_0 \cup \bigcup_{i=0}^{1} \bigcup_{j=1}^{\infty} B_{ij}$$

is hereditarily locally connected, and so the connected set Y must be hereditarily locally connected. On the other hand, Y is not locally compact at each point from the intersection of Y with the x-axis.

Now, let $f\colon E^1 \to E^n$ be a one-to-one continuous mapping. The image $X = f(E^1)$ is called *smooth* if, for each triple $a < b < c$, the arc $f([a, c])$ has a tangent straight line at the point $f(b)$. In case X is smooth, we say X to be of *bounded curvature* provided, for each $\varepsilon > 0$, there exists $\eta > 0$ such that if $\mathrm{diam} f([a, b]) < \eta$, the angle between tangent straight lines at the points $f(a)$ and $f(b)$ is less than ε. There exist, on the plane E^2, bounded smooth one-to-one continuous images X of E^1 such that X is of bounded curvature, and X is neither locally connected nor locally compact at each point (see [3], p. 258). In the latter case, the complement $E^2 \setminus \mathrm{cl}\,X$ must consist of at least 4 components (see [4], p. 147).

P 615. Is it true that if a set X on the plane is a one-to-one continuous image of the line, and X is locally connected, then X is locally compact? (See footnote (1))

P 616. Is it true that if a set X in a Euclidean space is a smooth one-to-one continuous image of the line, X is of bounded curvature, and X is locally connected, then X is locally compact?

The set in the following example contains arcs of arbitrarily small circles, and consequently it is not of bounded curvature.

EXAMPLE 1. *A bounded smooth one-to-one continuous image Y' of E^1 in E^3 such that Y' is hereditarily locally connected, and Y' is not locally compact.*

We get Y' as a homeomorphic image of Y from Example 0. The set Y is the union of the half-open interval B_0 and the arcs

$$A_1, B_{11}, A_2, B_{01}, A_3, B_{12}, A_4, B_{02}, A_5, \ldots$$

which form an infinite chain. Its part $B_0 \cup A_1$ admits tangent straight lines at interior points. In order to assure smoothness at the interior points of all arcs A_i, we have to reflect some of demi-circles forming A_i symmetrically in the x-axis. We obtain new arcs A_i', and in order to assure smoothness at their end points, we need to replace the demi-circles B_{ij} by segments B_{ij}' of certain screw lines, possibly also reflected in the

x-axis. Since the diameters of involved demi-circles converge to zero when i or j tend to the infinity, the set

$$Y' = \bigcup_{i=1}^{\infty} A_i' \cup B_0 \cup \bigcup_{i=0}^{1} \bigcup_{j=1}^{\infty} B_{ij}'$$

is homeomorphic with Y. Actually, a homeomorphism comes to light in the process of reflecting and screwing demi-circles.

EXAMPLE 2. *A bounded one-to-one continuous image Y'' of E^1 in E^3 such that Y'' is hereditarily locally connected,* $\mathrm{cl}\, Y''$ *is not locally connected, and Y'' is not locally compact.*

This set also will be a topological copy of the set Y from Example 0. If $(x_1, 0, 0)$ and $(x_2, 0, 0)$ are end points of the demi-circle B_{ij}, we denote by B_{ij}'' the union of two straight line segments which successively join the points

$$(x_1, 0, 0), \quad (x_1, -1, 0), \quad (x_2, 0, 0),$$

and define

$$Y'' = \bigcup_{i=1}^{\infty} A_i \cup B_0 \cup \bigcup_{i=0}^{1} \bigcup_{j=1}^{\infty} B_{ij}''$$

where A_i and B_0 are the same as in Example 0. Since the points $(0, 0, 0)$ and $(1, 0, 0)$ do not belong to Y, the projections of B_{ij} onto B_{ij}'' from the centres of B_{ij} establish a homeomorphism of Y onto Y''. The continuum $\mathrm{cl}\, Y''$ is not locally connected, e.g., at the point $(0, -1, 0)$.

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