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TWO REMARKS CONCERNING RETRACTS OF ANR-SPACES

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By an *ANR-space* we understand here any compact metric space which if embedded in a metric space is a retract of one of its neighborhoods. By an *r-map* ([1], p. 7) of a space X onto another space Y we understand a continuous function $f\colon X \to Y$ for which there exists a continuous map $g\colon Y \to X$ which is a right inverse of f, that is $fg\colon Y \to Y$ is the identity map.

In this note, we prove two elementary theorems on ANR-spaces. The first of them gives a positive answer to a question raised by A. Kirkor, the other — a positive answer to a question raised by R. Molski.

THEOREM 1. *If A is a retract of an ANR-space B, then there exists a positive number ε such that for every r-map $f\colon A \to f(A) \subset B$ satisfying the condition*

$$(1) \qquad \varrho\big(x, f(x)\big) < \varepsilon \quad \textit{for every point } x \in A,$$

the set $f(A)$ is a retract of B.

Proof. We can assume that B is a subset of the Hilbert cube Q^ω. Since $B \in \mathrm{ANR}$, there exists a neighborhood U of B in Q^ω and a retraction $r\colon U \to B$. Consider now an r-map $f\colon A \to f(A) \subset B$ satisfying (1) and let $g\colon f(A) \to A$ be a right inverse to f. If $x = f(y) \in f(A)$, then (1) implies that $\varrho\big(x, g(x)\big) = \varrho\big(fg(x), g(x)\big) < \varepsilon$. Hence, for $\varepsilon > 0$ sufficiently small, the segment (in Q^ω) joining the point $x \in f(A)$ with the point $g(x)$ lies in U, i.e.

$$tx + (1-t)g(x) \in U \quad \text{for every } x \in f(A) \text{ and } 0 \leqslant t \leqslant 1.$$

Setting

$$\varphi(x, t) = r\big(tx + (1-t)g(x)\big) \quad \text{for } x \in f(A) \text{ and } 0 \leqslant t \leqslant 1,$$

one gets a homotopy $\varphi\colon f(A) \times \langle 0, 1\rangle \to B$. Moreover, there exists a retraction $s\colon B \to A$. Setting

$$\psi(x, t) = s\varphi(x, t) \quad \text{for every } x \in f(A) \text{ and } 0 \leqslant t \leqslant 1,$$

we get a homotopy $\psi\colon f(A) \times \langle 0, 1\rangle \to A$ joining the map $sg\colon f(A) \to A$ with the restriction $s|f(A)\colon f(A) \to A$. But $s\colon B \to A$ is a continuous extension of the map $s|f(A)$ and thus the homotopy extension property of ANR-space ([2], p. 103) implies that there exists a continuous extension $\bar{g}\colon B \to A$ of the map sg. Setting

$$\bar{s}(x) = f\bar{g}(x) \quad \text{for every point } x \in B,$$

one obtains a continuous map $\bar{s}\colon B \to f(A)$. If $x \in f(A)$, then $g(x) \in A$ and $\bar{g}(x) = sg(x) = g(x)$. Hence $\bar{s}(x) = f\bar{g}(x) = fg(x) = x$, that is $\bar{s}$ is a retraction of B to $f(A)$.

THEOREM 2. *Let A and B be two ANR-sets lying in a metric space C. If A is both a retract of C and a deformation retract of B, then B is a retract of C.*

Proof. Let $r\colon C \to A$ be a retraction. Setting

$$f(x) = r(x) \quad \text{for every point } x \in B,$$

one gets a continuous map $f\colon B \to B$. Moreover, since A is a deformation retract of B, there exists a continuous map $\varphi\colon B \times \langle 0, 1\rangle \to B$ such that

$$\varphi(x, 0) = x \text{ and } \varphi(x, 1) \in A \text{ for every point } x \in B,$$

$$\varphi(x, 1) = x \text{ for every point } x \in A.$$

Setting

$$\psi(x, t) = \begin{cases} \varphi(x, 2t) & \text{for} \quad 0 \leqslant t \leqslant \frac{1}{2} \text{ and } x \in B, \\ f\varphi(x, 2-2t) & \text{for} \quad \frac{1}{2} \leqslant t \leqslant 1 \text{ and } x \in B, \end{cases}$$

we get a homotopy $\psi\colon B \times \langle 0, 1\rangle \to B$ joining the identity map $i_B\colon B \to B$ with the map f. Hence the maps $f\colon B \to B$ and $i_B\colon B \to B$ are homotopic. Setting

$$\bar{f}(x) = r(x) \quad \text{for every point } x \in C,$$

we get a continuous extension $\bar{f}\colon C \to B$ of the map f. Since $B \in \mathrm{ANR}$, we infer by the homotopy extension property of ANR-spaces ([2], p. 103) that i_B has a continuous extension $g\colon C \to B$. Manifestly g is a retraction, and consequently B is a retract of C.

REFERENCES

[1] K. Borsuk, *Theory of retracts*, Monografie Matematyczne 44, Warszawa 1966.

[2] — *Sur les prolongements des transformations continues*, Fundamenta Mathematicae 28 (1937), p. 99-110.

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