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ON UNCONDITIONAL CONVERGENCE IN LINEAR METRIC SPACES

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Let X be a linear metric complete space. We say that a series $\sum_{n=1}^{\infty} x_n$ is *unconditionally convergent* if it satisfies one of the following conditions:

(a) For each permutation p_n of positive integers the series $\sum_{n=1}^{\infty} x_{p_n}$ is convergent.

(b) For an arbitrary sequence ε_n, where $\varepsilon_n = \pm 1$ or 0, the series $\sum_{n=1}^{\infty} \varepsilon_n x_n$ is convergent.

(c) For an arbitrary sequence ε_n, where $\varepsilon_n = +1$ or -1, the series $\sum_{n=1}^{\infty} \varepsilon_n x_n$ is convergent.

Conditions (a), (b) and (c) are equivalent. If the space X is locally convex we can add (see [3], p. 59) another equivalent condition:

(d) For an arbitrary bounded sequence λ_n the series $\sum_{n=1}^{\infty} \lambda_n x_n$ is convergent.

In this note we show that the equivalence (a) $\equiv$ (b) $\equiv$ (c) $\equiv$ (d) is true for locally bounded spaces. The note contains also an example of a linear metric complete space such that (a) $\equiv$ (b) $\equiv$ (c) does not imply (d). (Let us remark that (d) trivially implies (b).)

THEOREM 1. *Let X be a locally bounded complete space. Then* (b) *implies* (d).

Proof. Without loss of generality we can assume that the topology in X is determined by a p-homogeneous norm $\| \;\|$ (see [1] or [6]).

Let us remark that there is a constant C such that

$$(*)\qquad \sup_{0\leqslant \lambda_i\leqslant 1} \|\lambda_1 x_1 + \ldots + \lambda_n x_n\| \leqslant C \sup_{\varepsilon_i = 0 \text{ or } 1} \|\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n\|.$$

Indeed, let us expand λ_i in the dyadic form:

$$\lambda_i = \sum_{j=1}^{\infty} \frac{\varepsilon_{i,j}}{2^j}, \qquad \varepsilon_{i,j} = 0 \text{ or } 1.$$

Then

$$\Big\|\sum_{i=1}^{n} \lambda_i x_i\Big\| = \Big\|\sum_{i=1}^{n} \sum_{j=1}^{\infty} \frac{\varepsilon_{i,j} x_i}{2^j}\Big\| = \Big\|\sum_{j=1}^{\infty} \frac{1}{2^j}\Big(\sum_{i=1}^{n} \varepsilon_{i,j} x_i\Big)\Big\|$$

$$\leqslant \sum_{j=1}^{\infty} \frac{1}{2^{jp}} \sup_{\varepsilon_i = 0 \text{ or } 1} \|\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n\| \leqslant C \sup_{\varepsilon_i = 0 \text{ or } 1} \|\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n\|,$$

where

$$C = \sum_{j=1}^{\infty} \frac{1}{2^{jp}} = \frac{2^p}{2^p - 1}.$$

Condition (*) trivially implies the thesis of Theorem 1, q.e.d.

Consider a linear metric space X. Let A be a star set. The number $c(A) = \inf\{s > 0\colon A + A \subset sA\}$ is called the *modulus of concavity* of the set A (see [6]). Obviously there are sets with an infinite modulus of concavity. We say that the space X is *pseudolocally convex* [8] if each neighbourhood of zero U contains a neighbourhood of zero U_1 with a finite modulus of concavity, i.e. such that $c(U_1) < +\infty$. Locally bounded spaces are pseudolocally convex.

Repeating the construction of p-homogeneous norm given in [6] we can construct a sequence of p_i-homogeneous pseudonorms (p_i can be different for different pseudonorms) determining topology.

Using the method of the proof of Theorem 1 we can prove

THEOREM 1′. *If X is a pseudolocally convex complete space, then* (b) *implies* (d).

We do not know if the class of pseudolocally convex spaces is the largest class of spaces possessing the property that (b) implies (d). We do not know even if it is true or not for the space S or the spaces $N(L)$ and $N(l)$ (for definition and properties of the spaces $N(L)$ and $N(l)$, see [5] and [7]) (**P 617**).

We can only give an example of a linear metric space such that (b) does not imply (d). This construction is based on the following lemmas.

LEMMA 1. *In the n-dimensional euclidean space there is a symmetric open star set A containing all points $p_1, \ldots, p_{3^n}$ of type $(\varepsilon_1, \ldots, \varepsilon_n)$, where $\varepsilon_i = +1$ or -1 or 0 such that the set*

$$A^{n-1} = \underbrace{A + A + \ldots + A}_{n-1 \text{ times}}$$

does not contain the unit cube $C = \{(\xi_1, \ldots, \xi_n) : |\xi_i| \leqslant 1, i = 1, 2, \ldots, n\}$.

Proof. Let A_0 be the set constituted by all segments connecting point 0 with the points $p_1, \ldots, p_{3^n}$. Let A_ε be a ball (in the euclidean sense) of radius ε, where ε will be determined later. Obviously, the set $A = A_0 + A_\varepsilon$ is a symmetric open star set and it contains all points $p_1, \ldots, p_{3^n}$.

We will show that for sufficiently small ε the set A^{n-1} does not contain the cube C. Obviously

$$A^{n-1} = A_0^{n-1} + A_\varepsilon^{n-1}.$$

But A_0^{n-1} is an $(n-1)$-dimensional set. Hence, for sufficiently small ε, A^{n-1} cannot contain the cube C.

LEMMA 2. *Let E be an n-dimensional euclidean space. We can introduce an F-norm (i.e. a subadditive, but not necessarily homogeneous norm, see [2]) such that $\|p_i\| \leqslant 1$, where p_i are points determined in preceding lemma, yet there is a point $p = (\lambda_1, \ldots, \lambda_n)$ in the unit cube such that*

$$\|p\| \geqslant n-1.$$

Proof. We will construct the norm $\| \; \|$ by the Kakutani method [4]. By $U(1)$ we denote the set A constructed in the preceding lemma. By $U(n)$ we denote the set A^n. Similarly as in the Kakutani paper [4], we define a set $U(1/2^q)$ such that

$$U\left(\frac{1}{2^{q+1}}\right) + U\left(\frac{1}{2^{q+1}}\right) \subset U\left(\frac{1}{2^q}\right).$$

If r is a dyadic number, that is, if

$$r = n + \sum_{i=1}^{m} \frac{\varepsilon_i}{2^m} \qquad (\varepsilon_i = +1 \text{ or } 0),$$

then

$$U(r) = U(n) + \varepsilon_i U\left(\frac{1}{2}\right) + \ldots + \varepsilon_j U\left(\frac{1}{2^j}\right) + \ldots + \varepsilon_m U\left(\frac{1}{2^m}\right).$$

Obviously,

$$U(r+s) \supset U(r) + U(s),$$

where r and s are two dyadic numbers.

Let

$$\|x\| = \inf\{r\colon\ x \in U(r)\}.$$

The function $\| \ \|$ is an F-norm (see [4]). Since $A = U(1)$ and $p_i \in A$, $\|p_i\| \leqslant 1$. Now, because of Lemma 1, there exists a point $p = (\lambda_1, \ldots, \lambda_n)$ which does not belong to $A^{n-1} = U(n-1)$. We have for it

$$\|p\| \geqslant n-1.$$

We denote by X_k a 2^k-dimensional space with a norm $\| \ \|_k = 2^{-k} \| \ \|$, where $\| \ \|$ is a norm constructed in Lemma 2.

Let X be the space of all sequences $x = \{\xi_n\}$ such that

$$|x| = \sum_{k=1}^{\infty} \|(\xi_{2^{k-1}+1}, \xi_{2^{k-1}+2}, \ldots, \xi_{2^k})\|_{k-1} + |\xi_1| < +\infty.$$

It is easy to check that the set X is a linear metric complete space with respect to the norm $| \ |$.

Let $x_n = \{\delta_i^n\}$, where δ_i^n is a Kronecker symbol. Let ε_i be an arbitrary sequence of numbers $+1$ or -1. The series $\sum\limits_{n=1}^{\infty} \varepsilon_n x_n$ is convergent. Indeed, from definition,

$$\Big|\sum_{i=m}^{l} \varepsilon_i x_i\Big| = |y_0| + |y_{k+1}| + |y_{k+2}| + \ldots + |y_k'| + |y_0'|,$$

where k is the smallest integer such that $m < 2^k$, k' is the largest integer such that $l > 2^{k'}$,

$$Y_0 = \sum_{i=m}^{2^k} \varepsilon_i x_i, \quad Y_j = \sum_{i=2^{j-1}+1}^{2^j} \varepsilon_i x_i \quad \text{and} \quad Y_0' = \sum_{i=2^{k'}+1}^{l} \varepsilon_i x_i.$$

Now, due to Lemma 2, we have

$$|Y_0| = \|(0, \ldots, 0, \varepsilon_m, \varepsilon_{m+1}, \ldots, \varepsilon_{2^k})\|_{k-1} \leqslant \frac{1}{2^{k-1}},$$

$$|Y_i| = \|(\varepsilon_{2^{i-1}+1}, \ldots, \varepsilon_{2^i})\|_{i-1} \leqslant \frac{1}{2^{i-1}},$$

and

$$|Y_0'| = \|(\varepsilon_{2^{k'}+1}, \ldots, \varepsilon_l, 0, \ldots 0)\|_{k'} \leqslant \frac{1}{2^{k'}},$$

whence

$$\Big|\sum_{i=m}^{l} \varepsilon_i x_i\Big| \leqslant \frac{1}{2^{k-1}} + \frac{1}{2^k} + \ldots + \frac{1}{2^{k'}} \leqslant \frac{1}{2^{k-2}} \leqslant \frac{8}{m}.$$

Hence the series $\sum_{n=1}^{\infty}\varepsilon_n x_n$ is convergent. Thus the series $\sum_{n=1}^{\infty}x_n$ satisfies condition (c).

We will now show that the series $\sum_{n=1}^{\infty}x_n$ does not satisfy condition (d). By Lemma 2, for each k there exists a point $p^k = (\lambda_1^k, \ldots, \lambda_{2^k}^k) \in x_k$ such that $|\lambda_i^k| \leqslant 1$ and

$$\|(\lambda_1^k, \ldots, \lambda_{2^k}^k)\| \geqslant \frac{1}{2^k}(2^k-1) \geqslant \frac{1}{2}.$$

Let $\tilde{\lambda}_m = \lambda_i^k$, where k is the smallest number such that $m > 2^k$ and $i = m-2^k$. The sequence λ_m is bounded, but for an arbitrary k we have

$$\Big|\sum_{i=2^k+1}^{2^{k+1}} \tilde{\lambda}_i x_i\Big| = \|(\lambda_1^k, \ldots, \lambda_{2^k}^k)\| \geqslant \frac{1}{2}.$$

Therefore the series $\sum_{n=1}^{\infty}\tilde{\lambda}_m x_m$ is not convergent. Thus we can formulate the following

THEOREM 2. *There exist a linear metric complete space X and a series $\sum_{n=1}^{\infty}x_n$ satisfying conditions* (a), (b) *and* (c) *which does not satisfy condition* (d).

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