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ON NON-SYMMETRIC MODULAR SPACES

BY

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1. Introduction. This paper is an outgrowth of several ideas, but it has been most directly inspired by a paper [2] concerning modular spaces written by J. Musielak and W. Orlicz.

Musielak and Orlicz defined a modular as follows: given a real linear space X, a functional $\varrho(x)$ with values $-\infty < \varrho(x) \leqslant +\infty$ defined on X is said to be a modular if the following conditions hold:

A1. $\varrho(x) = 0$ if and only if $x = 0$.

A2. $\varrho(x) = \varrho(-x)$.

A3. $\varrho(\alpha x+\beta y) \leqslant \varrho(x)+\varrho(y)$ for every $\alpha, \beta \geqslant 0$, $\alpha+\beta = 1$.

If $\varrho(x)$ satisfies the condition $\varrho(0) = 0$ instead of A1, then $\varrho(x)$ is said to be a pseudomodular. After establishing basic properties of modulars, the authors proceed to define a new type of functional using the modular concept by writing

$$\|x\| = \inf\left\{\varepsilon > 0\colon\ \varrho\left(\frac{x}{\varepsilon}\right) \leqslant \varepsilon\right\}.$$

The functional $\|x\|$ is then shown to be an F-norm. In § 2 of the present paper, the above condition A2 is removed and functionals on X satisfying A1, A3 and perhaps also additional conditions are studied. In § 3 a non-symmetric norm-like functional $\|x\|$ is defined in the sense of Musielak and Orlicz and some of its properties are studied. Examples of spaces possessing these new types of functionals are given at convenient intervals within the exposition. In § 4 a sufficient condition for a special type of non-symmetric modular defined on the space ω of all sequences of real numbers to be convergence-equivalent to a symmetric Musielak-Orlicz modular on ω is established. An outline of research presently being conducted is sketched in § 5.

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2. Non-symmetric modulars.

Definition 2.1. A functional $\varrho(x)$ with values $-\infty < \varrho(x) \leqslant +\infty$ defined on a real linear space is said to be a (non-symmetric) *modular* if it satisfies the following conditions:

M1. $\varrho(x) = 0$ iff $x = 0$.

M2. $\varrho(\alpha x + \beta y) \leqslant \varrho(x) + \varrho(y)$ for all $\alpha, \beta \geqslant 0$, where $\alpha + \beta = 1$.

$\varrho(x)$ is called a *pseudomodular* if instead of M1 it satisfies only the condition

M1′. $\varrho(0) = 0$.

Analogous to some basic results in [2], the following can be established using only the properties M1′ and M2:

LEMMA 2.1. *If $\varrho(x)$ is a pseudomodular on X, then*

(a) $\varrho(x) \geqslant 0$;

(b) *for each $x \in X$, $\varrho(\alpha x)$ is a non-decreasing function of $\alpha \geqslant 0$;*

(c) *for $\alpha_i \geqslant 0$ $(i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} \alpha_i = 1$, $\varrho\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leqslant \sum_{i=1}^{n} \varrho(x_i)$;*

(d) *if $X_\varrho = \{x \in X\colon \varrho(x) < +\infty$ and $\varrho(-x) < +\infty\}$, then X_ϱ is an origin-symmetric, convex subset of X.*

If $Y \neq \emptyset$ is an origin-symmetric, convex subset of X, denote by Y^* the set of all $x \in X$ such that $kx \in Y$ for some positive constant k depending on X. It is easily shown that such a Y^* is a linear subspace of X. In particular, referring to Lemma 2.1 (d), X_ϱ^* is a linear subspace of X.

Definition 2.2. A sequence $\{x_n\} \subset X$ is said to be *ϱ^+-convergent* to $x \in X$ (in symbols: ϱ^+-lim $x_n = x$ or $x_n \xrightarrow{\varrho^+} x$) if there exists a constant $k > 0$ depending on the sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} \varrho\{k(x_n - x)\} = 0.$$

$\{x_n\}$ is *strongly* ϱ^+-convergent to x if this limit is zero independent of the choice of $k > 0$.

Definition 2.3. A sequence $\{x_n\} \subset X$ is said to be *ϱ^--convergent* to $x \in X$ (ϱ^--lim $x_n = x$, $x_n \xrightarrow{\varrho^-} x$) if there exists a constant $k > 0$ depending on the sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} \varrho\{k(x - x_n)\} = 0.$$

$\{x_n\}$ is *strongly* ϱ^--convergent to x if this limit is zero independent of the choice of $k > 0$.

Definition 2.4. A sequence $\{x_n\} \subset X$ is said to be *ϱ-convergent* to $x \in X$ (ϱ-lim $x_n = x$, $x_n \xrightarrow{\varrho} x$) if ϱ^+-lim $x_n = x$ and ϱ^--lim $x_n = x$. *Strong* ϱ-convergence is similarly defined.

LEMMA 2.2. *If $\varrho(x)$ is a modular, then the ϱ-limit of a sequence $\{x_n\}$ exists and is uniquely determined provided both the ϱ^+-limit of $\{x_n\}$ and the ϱ^--limit of $\{x_n\}$ exist.*

Proof. Assume ϱ^--lim $x_n = x$ and ϱ^+-lim $x_n = y$. Then

$$(\forall \varepsilon > 0)(\exists n_\varepsilon)(\forall n \geqslant n_\varepsilon)\, \varrho\{k(x - x_n)\} < \frac{\varepsilon}{2}, \qquad k > 0$$

and also

$$(\forall \varepsilon > 0)(\exists n'_\varepsilon)(\forall n \geqslant n'_\varepsilon)\, \varrho\{k'(x_n - y)\} < \frac{\varepsilon}{2}, \qquad k' > 0.$$

Let $k'' = \min(k, k')$ and $n''_\varepsilon = \max(n_\varepsilon, n'_\varepsilon)$; then

$$(\forall \varepsilon > 0)(\exists n''_\varepsilon)(\forall n \geqslant n''_\varepsilon)\, \varrho\{k''(x - x_n)\} < \frac{\varepsilon}{2} \text{ and } \varrho\{k''(x_n - y)\} < \frac{\varepsilon}{2}.$$

Thus, for $n \geqslant n''_\varepsilon$,

$$\varrho\left\{\frac{k''}{2}(x - y)\right\} \leqslant \varrho\{k''(x - x_n)\} + \varrho\{k''(x_n - y)\} < \varepsilon.$$

Since ε is arbitrary, $\varrho\{\frac{1}{2}k''(x - y)\} = 0$, hence $\frac{1}{2}k''(x - y) = 0$, and thus $x = y$ because $k'' > 0$. This means that ϱ-lim $x_n = x$. This limit is unique because ϱ-lim $x_n = x'$ and ϱ-lim $x_n = y'$ yields $x' = y'$ by essentially the same argument.

If $\varrho(x)$ is a modular, one cannot (in contrast to the situation with Musielak-Orlicz modulars) generally establish that the ϱ^+-limit (or the ϱ^--limit) of a sequence is unique if it exists, because a sequence may have a ϱ^+-limit but no ϱ^--limit (or conversely) as follows:

EXAMPLE 2.1. Let f be the convex real continuous function defined by $f(x) = x^2$ if $x \geqslant 0$ and $f(x) = -x$ if $x < 0$. Let X be the space ω of all sequences of real numbers. In the space ω introduce the modular ϱ_f defined by

$$\varrho_f(x) = \varrho_f(\{x_i\}) = \sum_{i=1}^{\infty} f(x_i)$$

for any $x = \{x_i\} \in \omega$; the conditions M1 and M2 can be verified easily for ϱ_f. In ω consider the sequence of elements $\{x_n\}$, where

$$x_k = \Big(\underbrace{0, 0, 0, \ldots, 0}_{k\,\text{terms}}, \frac{1}{k+1}, \frac{1}{k+2}, \ldots\Big).$$

For all k,

$$\varrho_f(x_k) \leqslant \sum_{i=1}^{\infty} \frac{1}{i^2} < 2$$

but $\varrho_f(-x_k) = +\infty$. Clearly,

$$\lim_{n\to\infty} \varrho_f(x_n - \{0\}) = 0,$$

but for any constant $k' > 0$,

$$\lim_{n\to\infty} \varrho_f\{k'(\{0\} - x_n)\} = +\infty.$$

Hence, ϱ_f^+-lim $x_n = 0 \in \omega$ but ϱ_f^--lim x_n does not exist: if it existed it would also have to be $0 \in \omega$ (Lemma 2.2) but this is not the case.

In general, a *single function modular* ϱ_f will refer to a modular defined by a single function f on the space ω, as in Example 2.1.

Definition 2.5. Two modulars ϱ and ϱ' on a space X are said to be *equivalent* (in symbols: $\varrho \sim \varrho'$) provided that, for any sequence $\{x_n\} \subset X$, ϱ^+-lim $x_n = x$ iff ϱ'^+-lim $x_n = x$, and ϱ^--lim $x_n = y$ iff ϱ'^--lim $x_n = y$.

THEOREM 2.1. *ϱ^+- and ϱ^--convergence coincide for a modular $\varrho(x)$ on a space X iff there exists a modular $\varrho'(x)$ on X such that $\varrho' \sim \varrho$ and ϱ' is symmetric*, i.e., *$\varrho'(x) = \varrho'(-x)$ for every $x \in X$.*

Proof. If ϱ' is symmetric and $\varrho' \sim \varrho$, then ϱ'^+- and ϱ'^--convergence coincide by definitions 2.2 and 2.3. Hence, ϱ'^+-lim $x_n = x$ iff ϱ'^--lim $x_n = x$, for any sequence $\{x_n\} \subset X$. Since $\varrho' \sim \varrho$, this means ϱ^+-lim $x_n = x$ iff ϱ^--lim $x_n = x$ by definition 2.5; hence ϱ^+- and ϱ^--convergence coincide. Conversely, if ϱ^+-convergence and ϱ^--convergence coincide, define $\varrho'(x) = \varrho(x) + \varrho(-x)$; then ϱ' is symmetric, and from the inequality

$$\varrho'\{[\min(k_1, k_2)]\cdot(x - x_n)\} \leqslant \varrho\{k_1(x - x_n)\} + \varrho\{k_2(x_n - x)\} \qquad \text{with } k_1, k_2 > 0,$$

which follows from the definition of ϱ', it is clear that $\varrho' \sim \varrho$.

Definition 2.6. If, for a given modular ϱ, there exists a modular $\varrho' \sim \varrho$ such that ϱ' is symmetric on X, then ϱ is called *symmetrizable*.

Example 2.1 exhibits a non-symmetrizable ϱ in view of Theorem 2.1 and the fact that ϱ^+- and ϱ^--convergence do not coincide in the example.

Definition 2.7. A point $x \in X$ is *proper* for a modular ϱ if for any real sequence $\{a_n\}$, $\lim_{n\to\infty} a_n = 0$ implies $\lim_{n\to\infty} \varrho(a_n x) = 0$. ϱ is proper on $Y \subset X$ if every $x \in Y$ is proper for ϱ. If ϱ is proper on X, then ϱ is simply called proper.

The modular in Example 2.1 is not proper, but there do exist non-symmetrizable proper modulars:

EXAMPLE 2.2. Let $\{f_n\}$, $n = 1, 2, \ldots$, be the sequence of convex real continuous functions defined by $f_n(x) = x$ if $x \geqslant 0$, $f_n(x) = -nx$

if $x < 0$. Let $X = \omega$ as in Example 2.1. In ω introduce the modular ϱ_{f_n} defined by

$$\varrho_{f_n}(x) = \varrho_{f_n}(\{x_i\}) = \sum_{i=1}^{\infty} f_i(x_i).$$

Clearly, ϱ_{f_n} satisfies M1 and M2. Now,

$$X^*_{\varrho_{f_n}} = \Big\{\{x_i\} \in X\colon (\exists \alpha, \beta, \beta < 0 < \alpha) \sum_{i=1}^{\infty} f_i(\alpha x_i) < +\infty$$

$$\text{and } \sum_{i=1}^{\infty} f_i(\beta x_i) < +\infty\Big\}.$$

It is obvious that ϱ_{f_n} is proper on $X^*_{\varrho_{f_n}}$. In $X^*_{\varrho_{f_n}}$ consider the sequence of elements $\{x_n\}$, where

$$x_k = \Big(\underbrace{0, 0, 0, \ldots, 0}_{k-1\text{ terms}}, \frac{1}{k}, 0, 0, \ldots\Big).$$

$$\lim_{n\to\infty} \varrho_{f_n}(x_n - \{0\}) = \lim_{n\to\infty} \frac{1}{n} = 0,$$

but

$$\lim_{n\to\infty} \varrho_{f_n}\{k'(\{0\} - x_n)\} = k'$$

for any constant $k' > 0$. Hence, $\varrho^+_{f_n}$-lim $x_n = 0 \in \omega$ but $\varrho^-_{f_n}$-lim x_n does not exist and thus ϱ_{f_n} is not symmetrizable on $X^*_{\varrho_{f_n}}$.

In general, a *function sequence modular* ϱ_{f_n} will refer to a modular defined by a sequence of functions $\{f_n\}$ on the space ω, as in Example 2.2.

LEMMA 2.3. *Let $\overline{X}_\varrho$ denote the set of all $x \in X_\varrho$ for which x is proper for ϱ. $\overline{X}_\varrho$ is convex and origin-symmetric.*

Proof. That $\overline{X}_\varrho$ is convex follows since for any real sequence $\{a_n\}$ such that $\lim\limits_{n\to\infty} a_n = 0$ it is also true that $\lim\limits_{n\to\infty} 2a_n = 0$, hence for $x, y \in \overline{X}_\varrho$ and $0 \leqslant a \leqslant 1$,

$$\varrho\big(a_n\{(1-a)x + ay\}\big) \leqslant \varrho\{2a_n(1-a)x\} + \varrho(2a_n ay) \leqslant \varrho(2a_n x) + \varrho(2a_n y),$$

so that

$$\lim_{n\to\infty} \varrho(2a_n x) = \lim_{n\to\infty} \varrho(2a_n y) = 0$$

and $(1-a)x + ay$ is proper for ϱ. The origin-symmetry of $\overline{X}_\varrho$ is obvious from the definition of $\overline{X}_\varrho$.

As the following example shows, Lemma 2.3 cannot be improved to assert that the linear space $\overline{X}^*_\varrho$ is closed with respect to ϱ^+-convergence (or with respect to ϱ^--convergence) in general. This fact again contrasts the more general non-symmetric modulars with the Musielak-Orlicz modulars (cf. Theorem 1.11 in [2]).

EXAMPLE 2.3. Let $X = l_1$, the sequence space of absolutely summable real sequences with the usual norm

$$\|x\|_1 = \sum_{i=1}^{\infty} |a_i|, \qquad \text{where} \qquad \{a_i\} = x \in l_1.$$

Now, define a non-symmetric modular ϱ on l_1 by setting $\varrho(x) = \|x\|_1 + 1$ if infinitely many of the a_i are positive, and $\varrho(x) = \|x\|_1$ otherwise. It is easy to verify that ϱ satisfies the conditions M1 and M2. Also, it is evident that $X_\varrho = X = l_1$ and that $x = \{a_i\} \in \overline{X}_\varrho$ iff all but a finite number of the a_i are equal to zero. Now put

$$x_0 = \left(1, \frac{1}{2}, \frac{1}{2^2}, \ldots, \frac{1}{2^n}, \frac{1}{2^{n+1}}, \ldots\right),$$

$$x_n = \left(1, \frac{1}{2}, \frac{1}{2^2}, \ldots, \frac{1}{2^n}, 0, 0, \ldots\right)$$

for all $n = 1, 2, \ldots$; then

$$\lim_{n\to\infty} \varrho(x_n - x_0) = \lim_{n\to\infty} \|x_n - x_0\|_1 = \lim_{n\to\infty} \sum_{i=n+1}^{\infty} \left|\frac{-1}{2^i}\right| = 0,$$

and all x_n are proper for ϱ, but x_0 is not proper for ϱ.

LEMMA 2.4. *If ϱ^+-lim $x_n = x$ and ϱ^+-lim $y_n = y$, then we have ϱ^+-$\lim(\alpha x_n + \beta y_n) = \alpha x + \beta y$ provided $\alpha \geqslant 0$ and $\beta \geqslant 0$.*

Proof. By hypothesis, there exist $k_1 > 0$ and $k_2 > 0$ such that

$$\lim_{n\to\infty} \varrho\{k_1(x_n - x)\} = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \varrho\{k_2(y_n - y)\} = 0.$$

If $\alpha, \beta > 0$, by M2,

$$\varrho\left[\tfrac{1}{2}\min\left(\frac{k_1}{\alpha}, \frac{k_2}{\beta}\right)\{(\alpha x_n + \beta y_n) - (\alpha x + \beta y)\}\right] \leqslant \varrho\left\{\frac{k_1}{\alpha}(\alpha x_n - \alpha x)\right\} +$$

$$+ \varrho\left\{\frac{k_2}{\beta}(\beta y_n - \beta y)\right\} = \varrho\{k_1(x_n - x)\} + \varrho\{k_2(y_n - y)\} \to 0 \quad \text{as} \quad n \to \infty,$$

i.e., ϱ^+-$\lim(\alpha x_n + \beta y_n) = \alpha x + \beta y$. If α or β or both are zero, the proof is trivial.

By means of a closely parallel proof one obtains

LEMMA 2.5. *If ϱ^--$\lim x_n = x$ and ϱ^--$\lim y_n = y$, then we have ϱ^--$\lim(\alpha x_n + \beta y_n) = \alpha x + \beta y$ provided $\alpha \geqslant 0$ and $\beta \geqslant 0$.*

THEOREM 2.2. *Let $X_f^* \neq \{0\}$ be a finite-dimensional linear subspace of $\bar{X}_\varrho^*$. If $e_1, \ldots, e_m$ is a basis of X_f^*, then*

$$x_n = \sum_{t=1}^{m} \lambda_{t_n} e_t$$

is ϱ^+-convergent (and also ϱ^--convergent) to

$$x = \sum_{t=1}^{m} \lambda_t e_t$$

iff $\lim\limits_{n\to\infty} \lambda_{t_n} = \lambda_t$ for $t = 1, 2, \ldots, m$.

Proof. The proof is by induction on the dimension d of X_f^*. Assume $d = 1$. Then there exists $e_m \neq 0 \in X_f^*$ such that $X_f^* = \{a'e_m : a' \text{ is a real number}\}$. $x_n' = \lambda_{m_n} e_m$ is ϱ^+-convergent to $x' = \lambda_m e_m$ if there exists $k > 0$ such that

$$\lim_{n\to\infty} \varrho\{k(\lambda_{m_n} e_m - \lambda_m e_m)\} = \lim_{n\to\infty} \varrho\{k(\lambda_{m_n} - \lambda_m) e_m\} = 0.$$

Assume further that

$$\lim_{n\to\infty} k(\lambda_{m_n} - \lambda_m) \neq 0;$$

then there exists a subsequence $\{\lambda_{m_{n_\nu}}\}$ of $\{\lambda_{m_n}\}$ such that

$$\lim_{\nu\to\infty} k(\lambda_{m_{n_\nu}} - \lambda_m) = a \neq 0.$$

Then

$$\lim_{\nu\to\infty} \{a - k(\lambda_{m_{n_\nu}} - \lambda_m)\} = 0.$$

Since e_m is proper for ϱ,

$$\lim_{\nu\to\infty} \varrho\{ae_m - k(\lambda_{m_{n_\nu}} - \lambda_m) e_m\} = 0.$$

Since $\{\lambda_{m_{n_\nu}}\}$ is a subsequence of $\{\lambda_{m_n}\}$ and

$$\lim_{n\to\infty} \varrho\{k(\lambda_{m_n} - \lambda_m) e_m\} = 0,$$

it follows that

$$\lim_{\nu\to\infty} \varrho\{k(\lambda_{m_{n_\nu}} - \lambda_m) e_m\} = 0$$

because any subsequence of a convergent sequence of real numbers converges to the limit of the original sequence. By M2,

$$\varrho\{\tfrac{1}{2} ae_m\} \leqslant \varrho\{ae_m - k(\lambda_{m_{n_\nu}} - \lambda_m) e_m\} + \varrho\{k(\lambda_{m_{n_\nu}} - \lambda_m) e_m\}.$$

Letting $\nu \to \infty$ on the right-hand side, it follows that $\varrho\{\frac{1}{2}ae_m\} = 0$. By M1, this means $\frac{1}{2}ae_m = 0$ and since $\frac{1}{2}a \neq 0$, it follows that $e_m = 0 \in X_f^*$, which is a contradiction. Hence $\lim_{n\to\infty} k(\lambda_{m_n} - \lambda_m) = 0$ and since $k \neq 0$, $\lim_{n\to\infty} \lambda_{m_n} = \lambda_m$. Conversely, if $\lim_{n\to\infty} \lambda_{m_n} = \lambda_m$, then it follows (because e_m is proper for ϱ) that, for any $k > 0$,

$$\lim_{n\to\infty} \varrho\{k(\lambda_{m_n} - \lambda_m)e_m\} = 0.$$

This establishes the theorem for $d = 1$. By the induction assumption, in any $(m-1)$-dimensional subspace of X_f^* having basis $e_1, e_2, \ldots, e_{m-1}$,

$$x_n'' = \sum_{t=1}^{m-1} \lambda_{t_n} e_t$$

is ϱ^+-convergent to

$$x'' = \sum_{t=1}^{m-1} \lambda_t e_t$$

iff $\lim_{n\to\infty} \lambda_{t_n} = \lambda_t$ for $t = 1, 2, \ldots, m-1$. By use of Lemma 2.4 it follows that $\varrho^+\text{-}\lim(x_n' + x_n'') = \varrho^+\text{-}\lim x_n = x = x' + x''$, which completes the induction proof for ϱ^+-convergence. The analogous proof for ϱ^--convergence uses Lemma 2.5 and is almost identical.

Theorem 2.2 shows that, on $X_f^* \subset \bar{X}_\varrho^*$, ϱ^+-convergence and ϱ^--convergence always coincide. By Theorem 2.1, this means that any modular on any finite-dimensional subspace of $\bar{X}_\varrho^*$ is symmetrizable. It is also clear from Theorem 2.2 that, given any two modulars ϱ_1 and ϱ_2 on X and a finite-dimensional linear subspace $X_f^* \subset X$ with $X_f^* \subset \bar{X}_{\varrho_1}^* \cap \bar{X}_{\varrho_2}^*$, then $\varrho_1 \sim \varrho_2$ on X_f^*.

3. Non-symmetric modular norms.

Definition 3.1. For a given modular ϱ on X, the *ϱ-norm* associated with ϱ is the functional

$$\|x\| = \inf\left\{\varepsilon > 0 \colon \varrho\left(\frac{x}{\varepsilon}\right) \leqslant \varepsilon\right\}.$$

If ϱ is a pseudomodular, $\|x\|$ is called the *ϱ-pseudonorm* associated with ϱ.

Analogous to similar basic results in [2], the following can be derived using only the properties M1, M2 and definition 3.1:

LEMMA 3.1. *If $\|x\|$ is a ϱ-norm on X, then*

(a) $\|x\| \geqslant 0$;

(b) $\|x\| = 0$ *iff* $x = 0 \in X$;

(c) $\|x+y\| \leqslant \|x\| + \|y\|$;

(d) *for each* $x \in X$, $\|ax\|$ *is a non-decreasing function of* $a \geqslant 0$;

(e) *for* $\|x\| < 1$, $\varrho(x) \leqslant \|x\|$;

(f) *if* $\varrho(x/a) = a$ *for some* $a > 0$, *then* $\|x\| = a$;

(g) *if* $\varrho(ax)$ *is a continuous function of* $a \geqslant 0$ *for every* $x \in \bar{X}_\varrho^*$, *then* $\varrho(x/\|x\|) = \|x\|$ *for every* $x \neq 0 \in \bar{X}_\varrho^*$.

Definition 3.2. For a given modular ϱ on X, a sequence $\{x_n\} \subset X$ is said to be *(ϱ-norm)$^+$-convergent* to $x \in X$ (in symbols: $\| \|^+$-$\lim x_n = x$) if $\lim\limits_{n\to\infty} \|x_n - x\| = 0$. Similarly, $\{x_n\}$ is said to be *(ϱ-norm)$^-$-convergent* to $x \in X$ (in symbols: $\| \|^-$-$\lim x_n = x$) if $\lim\limits_{n\to\infty} \|x - x_n\| = 0$. Finally, $\{x_n\}$ is said to be *(ϱ-norm)-convergent* to $x \in X$ if both $\| \|^+$-$\lim x_n = x$ and $\| \|^-$-$\lim x_n = x$.

By part (e) of Lemma 3.1, $\| \|^+$-convergence implies ϱ^+-convergence and $\| \|^-$-convergence implies ϱ^--convergence. Parts (a), (c), (d), (e) and (f) of Lemma 3.1 also hold for ϱ-pseudonorms; part (b) becomes $\|x\| = 0$ iff $\varrho(x) = 0$ and part (g) holds for every $x \in \bar{X}_\varrho^*$ such that $\|x\| \neq 0$ in the case of a ϱ-pseudonorm.

It should be noted that the condition that, given a sequence $\{x_n\} \subset X$, there exists $k > 0$ and $x \in X$ such that

$$\lim_{n\to\infty} \|k(x_n - x)\| = 0,$$

is equivalent to $\| \|^+$-$\lim x_n = x$. This follows directly from definition 3.1. Similarly,

$$(\exists k > 0) \lim_{n\to\infty} \|k(x - x_n)\| = 0$$

is equivalent to $\| \|^-$-$\lim x_n = x$. This situation differs markedly from that encountered with modular convergence and strong modular convergence (definitions 2.2 to 2.4). The connection between strong modular convergence and modular norm convergence is shown by

LEMMA 3.2. *For a given modular ϱ on X, strong ϱ^+-convergence is equivalent to $\| \|^+$-convergence, strong ϱ^--convergence is equivalent to $\| \|^-$-convergence, and therefore strong ϱ-convergence is also equivalent to $\| \|$-convergence.*

Proof. Strong ϱ^+-$\lim x_n = x$ iff for every $k > 0$,

$$\lim_{n\to\infty} \varrho\{k(x_n - x)\} = 0$$

iff (putting $k = 1/\varepsilon$) for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \varrho\left\{\frac{x_n - x}{\varepsilon}\right\} = 0$$

iff $(\forall\,\varepsilon>0)(\exists\,n_\varepsilon)(\forall\,n\geqslant n_\varepsilon)\quad \varrho\{(x_n-x)/\varepsilon\}\leqslant\varepsilon$ iff $\lim\limits_{n\to\infty}\|x_n-x\|=0$ and $\|\ \|^+\text{-}\lim x_n=x$. The second equivalence is proved quite similarly.

In the remaining discussion it is convenient to use

Definition 3.3. A subset $Y\subset X$ is said to be *ϱ-strong* (with respect to a modular ϱ defined on X) if, for any sequence $\{x_n\}\subset Y$, $\varrho^+\text{-}\lim x_n=x$ implies strong $\varrho^+\text{-}\lim x_n=x$ and, for any sequence $\{y_n\}\subset Y$, $\varrho^-\text{-}\lim y_n=y$ implies strong $\varrho^-\text{-}\lim y_n=y$.

This condition is the generalization of condition B2 in [2] to non-symmetric modular spaces.

LEMMA 3.3. *For a given modular ϱ on X, ϱ^+-convergence is equivalent to $\|\ \|^+$-convergence and ϱ^--convergence is equivalent to $\|\ \|^-$-convergence on a subset $Y\subset X$ iff Y is ϱ-strong.*

The proof is immediate from Lemma 3.1 (e) and definition 3.3.

LEMMA 3.4. *If the modular ϱ is proper on $Y\subset X$, then, for the associated ϱ-norm $\|\ \|$ and for a real sequence $\{a_n\}$, $\lim\limits_{n\to\infty}a_n=0$ implies $\lim\limits_{n\to\infty}\|a_nx\|=0$ for all $x\in Y$.*

Proof. If $\lim\limits_{n\to\infty}a_n=0$, then for any $\varepsilon>0$ and for any $x\in Y$,

$$\lim_{n\to\infty}\varrho\left(a_n\cdot\frac{x}{\varepsilon}\right)=0$$

because x is proper for ϱ. Hence

$$\lim_{n\to\infty}\|a_nx\|=\lim_{n\to\infty}\left[\inf\left\{\varepsilon>0\colon\ \varrho\left(\frac{a_nx}{\varepsilon}\right)\leqslant\varepsilon\right\}\right]=0.$$

LEMMA 3.5. *If $\{a_n\}$ is a sequence of non-negative real numbers such that $\lim\limits_{n\to\infty}a_n=a$, where a is a real number, then*

(a) *if $\lim\limits_{n\to\infty}\|x_n-x\|=0$ and x is proper for ϱ, then*

$$\lim_{n\to\infty}\|a_nx_n-ax\|=0;$$

(b) *if $\lim\limits_{n\to\infty}\|x-x_n\|=0$ and x is proper for ϱ, then*

$$\lim_{n\to\infty}\|ax-a_nx_n\|=0.$$

Proof. (a) By Lemma 3.1 (c),

$$\|a_nx_n-ax\|\leqslant\|a_n(x_n-x)\|+\|(a_n-a)x\|.$$

Since $\lim\limits_{n\to\infty}(a_n-a)=0$ and x is proper for ϱ, it follows by Lemma 3.4 that

$$\lim_{n\to\infty}\|(a_n-a)x\|=0.$$

Since $\{a_n\}$ is a convergent sequence, it is bounded from above, i.e., there exists a positive real number K such that $0 \leqslant a_n \leqslant K$ for all n.

Then $\|a_n(x_n - x)\| \leqslant \|K(x_n - x)\|$ for all n. But, by definition 3.1, $\lim\limits_{n\to\infty} \|x_n - x\| = 0$ implies $\lim\limits_{n\to\infty} \|K(x_n - x)\| = 0$, so that

$$\lim_{n\to\infty} \|a_n(x_n - x)\| = 0.$$

Hence

$$\lim_{n\to\infty} \|a_n x_n - ax\| = 0.$$

The proof of (b) is quite similar and is therefore omitted.

In non-symmetric modular spaces, the uniqueness of the norm $\|x\|$ cannot be established as in Theorem 1.22 of [2]. The difficulty is the same as that encountered in trying to prove that the ϱ^+-limit of a sequence is unique, namely that ϱ^+-convergence does not imply ϱ^--convergence and conversely.

4. Symmetrizability of single function modulars. As stated before, if ω is the linear space of all sequences of real numbers, a single function modular on ω is a modular ϱ_f defined by

$$\varrho_f(x) = \varrho_f(\{x_i\}) = \sum_{i=1}^{\infty} f(x_i)$$

for all $x = \{x_i\} \in \omega$. Here f is a real-valued function and it is understood that ϱ_f as defined satisfies modular conditions M1 and M2. A set of necessary and sufficient conditions on functions f which give rise to modulars ϱ_f on ω is easily shown to be (with reference to M1 and M2) that (a) $-\infty < f(x) \leqslant +\infty$; (b) $f(x) = 0$ iff $x = 0$; and (c) $f(ax+\beta y) \leqslant f(x)+f(y)$ provided $a+\beta = 1$ and $a, \beta \geqslant 0$.

In view of Example 2.3 and other pathologies of non-symmetrizable modulars, it appears useful to derive concrete sufficient (or necessary and sufficient) conditions for a given modular on a given space to be symmetrizable. In that case, the given modular is convergence-equivalent to a symmetric modular to which the Musielak-Orlicz theory in [2] applies. The general symmetrizability problem appears to be very difficult. However, the following special result can be established:

THEOREM 4.1. *If ϱ_f is a single function modular on ω and f satisfies the conditions*

(i) $f(x) < +\infty$ *for* $|x| \leqslant M$, $M > 0$ *a constant, and*

(ii) $0 < K_1 = \varliminf\limits_{x\to 0^+} \dfrac{f(x)}{f(-x)} \leqslant \varlimsup\limits_{x\to 0^+} \dfrac{f(x)}{f(-x)} = K_2 < +\infty,$

then ϱ_f is symmetrizable.

Proof. Since $0 < K_1 = \varliminf_{x\to 0+} f(x)/f(-x)$,

$$\sup_k \inf_{0<x\leqslant k} \frac{f(x)}{f(-x)} = K_1 > 0.$$

Hence

$$(\forall \varepsilon > 0)(\exists K' = K'_\varepsilon \leqslant M) \inf_{0<x\leqslant K'} \frac{f(x)}{f(-x)} > K_1 - \varepsilon,$$

i.e.,

$$(\forall x\colon\ 0 < x \leqslant K' \leqslant M) \frac{f(x)}{f(-x)} > K_1 - \varepsilon,$$

which implies $(\forall x\colon\ 0 < x \leqslant K')\ f(x) > (K_1 - \varepsilon)\cdot f(-x)$. Also, since $\varlimsup_{x\to 0^+} f(x)/f(-x) = K_2 < +\infty$,

$$\inf_k \sup_{0<x\leqslant k} \frac{f(x)}{f(-x)} = K_2 < +\infty.$$

Hence

$$(\forall \varepsilon > 0)(\exists K'' = K''_\varepsilon \leqslant M) \sup_{0<x\leqslant K''} \frac{f(x)}{f(-x)} < K_2 + \varepsilon,$$

i.e.,

$$(\forall x\colon\ 0 < x \leqslant K'' \leqslant M) \frac{f(x)}{f(-x)} < K_2 + \varepsilon,$$

which implies $(\forall x\colon\ 0 < x \leqslant K'')\ f(x) < (K_2 + \varepsilon)\cdot f(-x)$. Now let $K = \min\{K', K''\}$; then

$$(\forall \varepsilon > 0)(\exists K)(\forall x\colon\ 0 < x \leqslant K)(K_1 - \varepsilon)\cdot f(-x) < f(x) < (K_2 + \varepsilon)\cdot f(-x).$$

Choose $\varepsilon_0 = K_1/2 < K_2$. Then

$$(\exists K_0 = K_0(\varepsilon_0) > 0)(\forall x\colon\ 0 < x \leqslant K_0) \frac{K_1}{2} f(-x) < f(x)$$
$$< \left(K_2 + \frac{K_1}{2}\right)\cdot f(-x) < 2K_2\cdot f(-x),$$

so that $(\forall x\colon\ 0 < x \leqslant K_0) f(x) < 2K_2\cdot f(-x)$ and also

$$(\forall x\colon\ 0 < x \leqslant K_0) f(-x) < \frac{2}{K_1}\cdot f(x).$$

Let $\{x_n\} = \{x_n^{(i)}\} \subset \omega$ be any sequence. To prove that ϱ_f is symmetrizable it suffices (by Theorem 2.1 and definition 2.6) to show that

$\varrho_f^+\text{-}\lim x_n = 0$ implies $\varrho_f^-\text{-}\lim x_n = 0$ and that $\varrho_f^-\text{-}\lim x_n = 0$ implies $\varrho_f^+\text{-}\lim x_n = 0$. Assume that $\varrho_f^+\text{-}\lim x_n = 0$, i.e.,

$$(\exists k > 0) \lim_{n\to\infty} \varrho_f(\{kx_n^{(i)}\}) = \lim_{n\to\infty} \sum_{i=1}^{\infty} f(kx_n^{(i)}) = 0.$$

Hence

$$(\forall \delta > 0)(\exists n_\delta)(\forall n \geqslant n_\delta) \sum_{i=1}^{\infty} f(kx_n^{(i)}) < \delta.$$

Now choose $\delta = \delta_0$ so small that $(\exists x_1 < 0) f(x_1) \geqslant \delta_0$ and that $(\exists x_2 > 0) f(x_2) \geqslant \delta_0$; this is possible because $f(x) = 0$ iff $x = 0$. It follows that

$$(\exists n_{\delta_0})(\forall n \geqslant n_{\delta_0})(\forall i)\, |kx_n^{(i)}| < \max\{-x_1, x_2\}.$$

Define $\mu = \max\{-x_1, x_2\}/K_0' > 0$, where $K_0' < K_0$ is chosen so small that $\mu > 1$; this can be done because no lower bound on K_0 was specified earlier. Thus

$$(\exists n_{\delta_0})(\forall n \geqslant n_{\delta_0})(\forall i) \left| \frac{k}{\mu} x_n^{(i)} \right| < K_0'$$

and therefore

$$(\forall n \geqslant n_{\delta_0})(\forall i) \sum_{i=1}^{\infty} f\left(\frac{-k}{\mu} x_n^{(i)}\right)$$

$$< \max\left[2K_2, \frac{2}{K_1}\right] \cdot \sum_{i=1}^{\infty} f\left(\frac{k}{\mu} x_n^{(i)}\right) \leqslant \max\left[2K_2, \frac{2}{K_1}\right] \cdot \sum_{i=1}^{\infty} f(kx_n^{(i)}),$$

the last inequality following because $\mu > 1$ and f satisfies the inequality $f(\alpha x + \beta y) \leqslant f(x) + f(y)$ for $\alpha + \beta = 1$ and $\alpha, \beta \geqslant 0$. Since $\max[2K_2, 2/K_1]$ is a constant depending only on the nature of f, and

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} f(kx_n^{(i)}) = 0$$

was assumed, one can conclude that

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} f\left(-\frac{k}{\mu} x_n^{(i)}\right) = 0,$$

and hence $\varrho_f^-\text{-}\lim x_n = 0$. An almost identical argument shows that $\varrho_f^-\text{-}\lim x_n = 0$ implies $\varrho_f^+\text{-}\lim x_n = 0$.

5. Outline of related research. There are three main directions to this research. Firstly, symmetrizability conditions for function sequence modulars and other types of modulars on the space ω and on other spaces are sought. Secondly, the properties of modulars which in addition to satisfying M1 and M2 are defined on a universally continuous semi-ordered linear space are being investigated. This work is based on a paper by Koshi and Shimogaki [1]. Finally, modulars which satisfy all conditions of Nakano's modulars [3] except for symmetry are being studied.

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