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THE HILBERT TRANSFORM IN WEIGHTED SPACES OF INTEGRABLE VECTOR-VALUED FUNCTIONS

BY

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Let D be the open unit disc in the complex plane C. It is well known that to every real function u harmonic in D there corresponds exactly one real harmonic function v such that $v(0) = 0$ and $u+iv$ is analytic in D. The function v is called the *harmonic conjugate* or the *Hilbert transform* of u and denoted by $\tilde{u}$ or Hu. It is evident that H can be extended to a linear operator on the space of all complex-valued harmonic functions in D.

If f is a trigonometric polynomial

$$f(re^{it}) = \sum_{k=-n}^{n} x_k r^{|k|} e^{ikt}, \qquad x_k \in C,$$

then

$$Hf(re^{it}) = -i \sum_{k=-n}^{n} \operatorname{sgn}(k) x_k r^{|k|} e^{ikt}.$$

If we restrict ourselves to the unit circle T then

$$Hf(e^{it}) = -i \sum_{k=-n}^{n} \operatorname{sgn}(k) x_k e^{ikt} \tag{1}$$

and, by M. Riesz theorem (cf. e.g. [4] p. 54),

$$\|Hf\|_p \leqslant A_p \|f\|_p, \quad 1 < p < \infty, \tag{2}$$

where $\|f\|_p$ denotes the L^p-norm with respect to the normalized Lebesgue measure m on T. In other words, H extends to a continuous operator on $L^p(m)$. It is easily seen that this extension coincides with boundary values of the harmonic conjugate of the harmonic extension to D of a function from $L^p(m)$.

(1) makes perfectly good sense if we allow $x_k \in X$, X being a complex Banach space. If X is a Hilbert space then the M. Riesz theorem holds just

as in the scalar case but, on the other hand, if for instance X contains l_n^1 or l_n^∞ uniformly then H is bounded for no p (cf. [1]). In this way one arrives at the problem of determining all Banach spaces X for which the Hilbert transform is bounded. Some equivalent formulations of this problem may be found in [1]. Recently D. L. Burkholder [2] has found a geometric condition on X that implies the boundedness of H. It is, so far, an open problem whether his condition is also necessary.

In this note we consider weighted L^p-spaces of vector-valued functions and show that admitting weights does not change the class of Banach spaces for which the Hilbert transform is bounded. Our result generalizes the Hunt–Muckenhoupt–Wheeden theorem [5] to the vector-valued case. The proof is based on the proof of the scalar case presented in [3]. The main difficulty in the formal extension was the proof of weak 1–1 type of a maximal singular operator H^* corresponding to H. It was shown by Burkholder in [2] that under his geometrical assumption on the space X, the operator H^* is of weak type 1–1, so if it were shown that his condition was also necessary then our theorem could be easily derived from his results.

Before stating the theorem we will need some notation.

We will call a function w: $T \to \boldsymbol{R}$ a *weight function* if w is nonnegative, m-integrable and not identically zero.

For a weight function w and a Banach space X by $L^p(X, w)$ we will denote the space of all functions f: $T \to X$, Bochner-integrable with respect to the measure $w\,dm$ and such that

$$\int_T \|f(t)\|^p w(t)\,dm(t) < \infty .$$

We will also use: $L^p(X) = L^p(X, 1)$, $L^p(w) = L^p(C, w)$, $L^p = L^p(C, 1)$.

For a subset $E \subset T$ we will use $|E| = m(E)$. T will be identified with the interval $[-\pi, \pi)$ and functions on T with 2π-periodic functions on $\boldsymbol{R}$.

Now we can state our result.

THEOREM. *Let w_0 be a weight function and assume that for some $p_0 \in (1, \infty)$ the Hilbert transform H is bounded in $L^{p_0}(X, w_0)$. Then for every weight function w and $p \in (1, \infty)$, H is bounded in $L^p(X, w)$ if and only if w satisfies the Muckenhoupt condition*

$$(\mathrm{A_p}) \qquad \sup_I \Big(|I|^{-1} \int_I w\,dm\Big)^{1/p} \Big(|I|^{-1} \int_I w^{-p'/p}\,dm\Big)^{1/p'} < +\infty$$

where I ranges over all subarcs of T and $1/p + 1/p' = 1$.

For the proof we will need the following two theorems:

THEOREM A (Muckenhoupt [7], cf. also [3]). *The Hardy–Littlewood maximal function is an operator of strong type p-p with respect to the measure $w\,dm$ if and only if w satisfies $(\mathrm{A_p})$.*

(The H–L maximal function for $f \in L^1$ is defined as

$$M_f(t) = \sup_{h \in (0,\pi)} \Big| \int_{t-h}^{t+h} f(s)\,ds \Big|.)$$

THEOREM B (Hunt–Muckenhoupt–Wheeden [5], cf. also [3]). *The Hilbert transform is bounded in $L^p(w)$ if and only if w satisfies* (A_p).

Evidently Theorem B implies that if H is bounded in $L^p(X, w)$ then w satisfies (A_p) so we only have to prove the inverse implication.

LEMMA 1. *Under the assumptions of the Theorem, H is bounded in $L^{p_0}(X)$.*

Proof. For $s \in T$ denote by W_s the rotation operator $(W_s f)(t) = f(t-s)$. If f is a trigonometric polynomial then for every $s \in T$, $W_s f \in L^{p_0}(X, w_0)$ and $H(W_s f) = W_s Hf$. Then

$$\|w_0\|_1 \|Hf\|^{p_0}_{L^{p_0}(X)} = \int_T \int_T \|(W_s Hf)(t)\|^{p_0} w_0(t)\,dm(t)\,dm(s) \leqslant C^{p_0} \|w_0\|_1 \|f\|^{p_0}_{L^{p_0}(X)},$$

where C is the norm of H in $L^{p_0}(X, w_0)$.

Hence it is enough to prove the Theorem under the assumption $w_0 = 1$.

Before we proceed with the proof we are going to need some more notation.

In the sequel we will always assume that f is an X-valued trigonometric polynomial.

For $\varepsilon \in (0, \pi)$ let

$$(H_\varepsilon f)(t) = (2\pi)^{-1} \int_\varepsilon^{2\pi-\varepsilon} f(e^{i(t-s)}) \operatorname{ctg} \tfrac{1}{2} s\,ds$$

and

$$(H^* f)(t) = \sup_{\varepsilon \in (0,\pi)} \|(H_\varepsilon f)(t)\|.$$

For $g \in L^1(X)$, by $g(re^{it})$ we will denote the harmonic extension of g into D (by the Poisson kernel) and by N_g the radial maximal function

$$N_g(t) = \sup_{r \in (0,1)} \|g(re^{it})\|.$$

C will stand for any constant and it may change during the proof.

LEMMA 2. *The subadditive operator H^* is of strong type p_0-p_0.*

Proof. Put $E(r, t) = \|(H_\varepsilon f)(e^{it}) - (Hf)(re^{it})\|$ where $\varepsilon = 1-r$. Estimating as in [6], p. 77, we get $E(r, t) \leqslant CM_{\|f\|}(t)$. So

$$(H^* f)(t) \leqslant CM_{\|f\|}(t) + N_{Hf}(t). \tag{3}$$

Since $N_g(t) \leqslant N_{\|g\|}(t)$ then, by Lemma 2.4 of [6], we have

$$N_g(t) \leqslant M_{\|g\|}\, t. \tag{4}$$

From (3) and (4) we get

$$\|H^* f\|_{p_0} \leqslant C\,\|M_{\|f\|}\|_{p_0} + |M_{\|Hf\|}\|_{p_0}. \tag{5}$$

By the strong type p_0-p_0 of the H–L maximal function, from (5) follows

$$\|H^* f\|_{p_0} \leqslant C\,\|f\|_{L^{p_0}(X)} + \|Hf\|_{L^{p_0}(X)}$$

and thus, by the assumption,

$$\|H^* f\|_{p_0} \leqslant C\,\|f\|_{L^{p_0}(X)}.$$

LEMMA 3 (Calderón–Zygmund; cf. e.g. [8]). *Let $g \in L^1$, $g \geqslant 0$ and $\alpha > \|g\|_1$. Then there exist disjoint subsets F, $\Omega \subset T$, $F \cup \Omega = T$ such that $g(t) \leqslant \alpha$ for almost all $t \in F$, $\Omega = \bigcup I_n$ where I_n are pairwise disjoint arcs such that $\alpha \leqslant |I_n|^{-1} \int_{I_n} g \leqslant 2\alpha$.*

LEMMA 4. *The operator H^* is of weak type 1–1.*

Sketch of the proof. Using Lemma 3 for the function $\|f(t)\|$ and $\alpha > \|f\|_{L^1(X)}$ we split T into F and Ω and put

$$f_1(t) = \begin{cases} f(t) & \text{for } t \in F, \\ |I_n|^{-1} \int_{I_n} f & \text{for } t \in I_n, \end{cases} \qquad f_2 = f - f_1.$$

Then $H^* f \leqslant H^* f_1 + H^* f_2$, so

$$m\{H^* f > \alpha\} \leqslant m\{H^* f_1 > \alpha/2\} + m\{H^* f_2 > \alpha/2\}.$$

For f_1 we have

$$\|f_1\|^{p_0}_{L^{p_0}(X)} = \int_F \|f_1\|^{p_0} + \int_\Omega \|f\|^{p_0} \leqslant \alpha^{p_0-1} \int_F \|f\| + (2\alpha)^{p_0}\, m(\Omega).$$

Since $|I_n| \leqslant \alpha^{-1} \int_{I_n} \|f\|$, $|\Omega| \leqslant \alpha^{-1} \int_\Omega \|f\|$ and so

$$\|f_1\|^{p_0}_{L^{p_0}(X)} \leqslant \alpha^{p_0-1} \int_F \|f\| + 2^{p_0} \alpha^{p_0-1} \int_\Omega \|f\| = C\alpha^{p_0-1} \|f\|_{L^1(X)}.$$

By Lemma 2 and the above,

$$m\{H^* f_1 > \alpha/2\} \leqslant (2/\alpha)^{p_0}\, C\alpha^{p_0-1} \|f\|_{L^1(X)} = (C/\alpha)\,\|f\|_{L^1(X)}.$$

The estimate of $m\{H^* f_2 > \alpha/2\}$ is done exactly as in the proof of Theorem 4b of [8] p. 42 with the change of moduli to norms.

LEMMA 5 (Coifman–Fefferman [3]). *Condition* (A_p) *implies*

(A_∞) *There exist constants C, $\delta > 0$ such that for every subarc $I \subset T$ and* *every measurable subset* $E \subset I$ *we have*

$$\Big(\int_E w\,dm\Big)\Big(\int_I w\,dm\Big)^{-1} \leqslant C(|E|\,|I|^{-1})^{\delta}.$$

LEMMA 6. *Assume that* w *satisfies* (A_∞). *Then to every* $p \in (1, \infty)$ *there corresponds* C_p *such that*

$$\|H^* f\|_{L^p(w)} \leqslant C_p \|M_{\|f\|}\|_{L^p(w)}.$$

Proof of Lemma 6 is the formal extension of the proof of Theorem III of [3] to the vector-valued case. The only problem is the weak type 1–1 of the operator H^* which was proved in Lemma 4.

Lemma 6 together with Theorem A imply

$$\|H^* f\|_{L^p(w)} \leqslant C'_p \|f\|_{L^p(X,w)}.$$

To end the proof it is enough to show that for almost all $t \in T$ we have

$$(H^* f)(t) \geqslant \|(Hf)(t)\|. \tag{6}$$

This can be done by extending to the vector-valued case the theorem about almost everywhere convergence of $H_\varepsilon f$ (Th. 4c [8]) or, simplier, in the following way: For every $x^* \in X^*$, $\|x^*\| = 1$ and for almost every $t \in T$ we have

$$\begin{aligned}(H^* f)(t) &\geqslant \sup_\varepsilon |x^*((H_\varepsilon f)(t))| = \sup_\varepsilon |(H_\varepsilon(x^* \circ f))(t)| \\ &\geqslant \lim_{\varepsilon \to 0} |(H_\varepsilon(x^* \circ f)(t)| = |(H(x^* \circ f))(t)| = |x^*((Hf)(t))|.\end{aligned} \tag{7}$$

Now, since Hf is continuous, $Hf(T)$ is compact, and so we may choose a countable sequence of norming functionals for this set and thus prove (6) by (7).

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Added in proof. After this paper was accepted for publication, it was proved by J. Bourgain (in: *Some remarks on Banach spaces in which martingale difference sequences are unconditional*, Ark. Mat. 21 (1983), p. 163–168) that Burkholder's condition is equivalent to the boundedness of the Hilbert transform; hence results of this paper can now be essentially derived from [2].

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