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CONTINUOUS MEASURES AND ANALYTIC SETS*

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0. Introduction. Let Γ be a countably infinite abelian group, and G its dual group. A subset S of Γ is called a *w-set* in Γ if there is a continuous complex-valued measure μ in G such that $|\hat{\mu}| \geqslant 1$ everywhere in S. (The name refers to the work of Wiener [11], apparently the first on Fourier–Stieltjes transforms of continuous measures; by definition, $\hat{\mu}(\gamma) = \int \bar{\gamma} d\mu$.) Regarding the space $M_c(G)$ as a subset of a dual space $C^*(G)$, it is of type $F_{\sigma\delta}$ in the w*-topology. This gives the easy half of our main result.

THEOREM. *In the metric space 2^Γ, the class* wΓ *of all* w*-sets is an analytic set but not a Borel set.*

A previous work on classes contained in 2^Γ concerns the class $\mathscr{R}$ of *Riesz sets* [9]; the class $\mathscr{R}$, which plays the role of negligible sets, is *co-analytic* but not Borel. The present theorem depends on the harmonic analysis on certain non-locally compact topological groups established by Varopoulos [10], pp. 112–131 (see also [7]). For certain groups Γ (type I) this dependence is explicit, and for the remaining groups Γ, the analysis of [10] is adapted by a ruse.

Remark. Every Sidon set is a w-set [2]. On the other hand, it is easy to prove that Sidon sets are a class of type F_σ in 2^Γ. So our Theorem shows that not every w-set is Sidon. This is known and can be proved in many ways (cf., e.g., [6]).

1. Preliminaries. We shall first reduce the main result to certain special cases.

LEMMA 1. *Let φ be a homomorphism of Γ onto a group Γ_1, and $S_1 \subseteq \Gamma_1$. Then S_1 is a* w*-set in Γ_1 if and only if $\varphi^{-1}(S_1)$ is a* w*-set in Γ.*

Proof. Suppose that $\mu \in M_c(G_1)$ and $|\hat{\mu}| \geqslant 1$ on S_1. (We have written G_1 for the dual of Γ_1.) Then the dual mapping φ^* is a homeomorphism of G_1 into G, and

$$(\varphi^*\mu)^\wedge(s) = \hat{\mu}(\varphi s),$$

whence $|(\varphi^*\mu)^\wedge| \geqslant 1$ on $\varphi^{-1}(S_1)$, and of course $\varphi^*\mu$ belongs to $M_c(G)$. In the opposite direction, we begin with $\mu \in M_c(G)$ and replace μ by $\lambda = \mu * \tilde{\mu}$, so that $\hat{\lambda} \geqslant 0$ on Γ and $\hat{\lambda} \geqslant 1$ on $\varphi^{-1}(S_1)$. Let Λ be the kernel of φ,

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let Lim be a Banach limit over Λ, and observe that there is a measure σ such that

$$\hat{\sigma}(\chi) \equiv \operatorname*{Lim}_{\gamma} \hat{\lambda}(\chi+\gamma).$$

What we need to know about σ is that

$$\sigma(E) = \lambda(E \cap \Lambda^{\perp})$$

for each Borel set E, and $\hat{\sigma}(\chi)$ is contained in the convex hull of the set $\hat{\lambda}(\chi+\Lambda)$ for each χ. Identifying $\Lambda^{\perp}$ with G_1, we conclude that S_1 is a w-set in Γ_1.

We now divide the groups Γ into two classes:

(I) For every integer $m = 1, 2, 3, \ldots$, $\Gamma/m\Gamma$ is finite.

(II) For some integer $m \geqslant 1$, $\Gamma/m\Gamma$ is infinite.

2. Groups of type I. In this case the subgroup of G defined by the equation $mg = 0$ is finite for each m. Therefore, G is an I-group and contains a perfect Kronecker set K (see [3], pp. 566–570, and [8], pp. 99–102). We suppose in fact that K is a Cantor set. We define a map from closed sets E of K into 2^{Γ} as follows:

$$\chi \in B(E) \Leftrightarrow |\chi - 1| < 1/3 \quad \text{on } E.$$

The mapping is lower semi-continuous in the following sense: if $\lim E_n = E$ in the Hausdorff metric, then

$$B(E) \subseteq \liminf B(E_n).$$

As for real-valued lower semi-continuous functions, whenever U is open in 2^{Γ}, the inverse image $\{E : B(E) \in U\}$ is then of type F_σ in 2^K, and in particular the inverse image is Borelian. We shall show that $B(E)$ is a w-set if and only if E is uncountable, or in different terms: $B(E) \in \mathrm{w}\Gamma \Leftrightarrow E$ is uncountable. By a theorem of Hurewicz [4], the class of uncountable closed sets in 2^K is analytic but not Borelian, whence $\mathrm{w}\Gamma$ is not Borelian. Clearly, $B(E)$ is a w-set when E is uncountable, since E supports a continuous probability measure. For the more difficult implication, we first summarize the necessary results from [7] and [10].

Suppose that X is a 0-dimensional compact metric space and that $S(X)$ is the (metric) group of all unimodular, continuous functions on X.

(a) Every continuous character of $S(X)$ is in the subgroup generated algebraically by the evaluations at the elements of X.

(b) Bochner's theorem is valid for $S(X)$: every continuous positive-definite function on $S(X)$ is represented as an integral over continuous characters on S. Strictly speaking, the continuous characters have to be turned into a measurable set; this quibble does not affect the remaining argument. See also [1].

(c) Let H be an abelian group provided with an invariant pseudo-metric d;

let (Y, μ) be a finite measure space and $S^*(Y)$ the group of unimodular, μ-measurable functions on Y. Let T be an algebraic homomorphism from H into $S^*(Y)$. One of the following two cases must occur:

(c_1) There is a measurable subset $Y_1 \subseteq Y$, $\mu(Y_1) > 0$, so that T is continuous as a mapping from H to $S^*(Y_1)$.

(c_2) For every neighborhood V of the identity of H, the convex hull $\mathrm{co}(T(V))$ contains the function 0 in its closure (in $L^1(\mu)$, for example).

We can now prove that when $B(E)$ belongs to wΓ, then E must be uncountable. Suppose, then, that μ is a continuous signed measure such that $\hat{\mu} \geqslant 1$ in $B(E)$. In Γ we introduce a pseudo-metric d by the formula

$$d(\gamma, 0) \equiv \sup\{|\gamma(g)-1|:\ g \in E\}.$$

We apply (c) to the measure space $(G, |\mu|)$. Now $B(E)$ is a neighborhood of 0 in Γ (using the pseudo-metric d) and for every function f in the convex hull of $B(E)$ we have

$$\int |f|\,|d\mu| \geqslant \mathrm{Re}\int \bar{f} d\mu \geqslant 1$$

(since the last inequality is true for characters in $B(E)$). Therefore, alternative (c_2) must be rejected, and (c_1) accepted. We now define

$$p(\chi) = \int_{Y_1} \bar{\chi}|d\mu|, \qquad \chi \in \Gamma,$$

so that p is positive definite on Γ, and continuous for the pseudo-metric d. Since K is a 0-dimensional Kronecker set, E is one as well, and p determines (by uniform approximation) a continuous positive-definite function on $S(E)$. Comparison with (b) in the summary above shows that

$$p(\chi) = \int \bar{\chi} d\lambda$$

with a measure λ concentrated in the algebraic subgroup generated by E. If E were countable, then we would have $|\mu|\,|Y_1 = 0$, whence

$$p(1) = |\mu|(Y_1) = 0;$$

this concludes the proof for groups of type I.

3. Groups of type II. In this case there is a prime p such that $\Gamma/p\Gamma$ is infinite, for the inequality

$$o(\Gamma/m_1 m_2 \Gamma) \leqslant o(\Gamma/m_1\Gamma)\cdot o(\Gamma/m_2\Gamma)$$

is valid for all pairs of integers $m_1, m_2 \geqslant 1$. Then $\Gamma/p\Gamma$ is an infinite sum Z_p^∞, and, by Lemma 1, we can assume that Γ is one of these groups.

In this case G contains a perfect K_p-set F_1 (see [3] and [8]); this means that every continuous function on F_1 to the group of p-th roots of unity is the restriction to F_1 of a continuous character of G. Now F_1 is homeomorphic to a Cantor set, and therefore to a union of three disjoint Cantor sets. Thus F_1 can be represented as a product

$$F_1 = F \times \{1, 2, 3\},$$

F being also a Cantor set. To each closed set E of F we attach objects $\alpha(E)$, $\beta(E)$, and $B(E)$.

(i) $\alpha(E)$ is the subgroup of G generated algebraically by $E \times \{1, 2, 3\}$, and $\beta(E)$ is the closure of $\alpha(E)$ in G.

(ii) $B(E)$ is the subset of Γ defined by this condition: for each x in E, $\gamma(x \times i) = 1$ for at least two numbers $i = 1, 2, 3$. $B(E)$ takes the place of the set $B(E)$ used before. (Attempting to follow the method used for groups of type I, we would consider the characters on a certain group — but that group is discrete.) The analytical part of the proof is contained in

LEMMA 2. *There exists a sequence (λ_k) of probability measures in $B(F)$ such that $\hat{\lambda}_k(g) \to 0$ on $\beta(F) \backslash \alpha(F)$.*

Proof. For each $k = 1, 2, 3, \ldots$ let $(A_1, \ldots, A_r)$ be a partition of F into disjoint closed sets of diameter $< k^{-1}$. Let $\gamma(i, j)$ be a continuous character of G such that $\gamma(i, j) = \omega_p = \exp(2\pi i p^{-1})$ in $A_j \times i$ and 1 in the remainder of $F \times \{1, 2, 3\} = F_1$ $(i = 1, 2, 3,\ 1 \leqslant j \leqslant r)$. This recipe determines the value $(\gamma(i, j), g)$ for each g in $\beta(F)$. Finally, let

$$\lambda_k = \mathop{*}_{j=1}^{r} \tfrac{1}{4}\big[\delta(0) + \delta(\gamma(1, j)) + \delta(\gamma(2, j)) + \delta(\gamma(3, j))\big].$$

A constant $c_p < 1$ is defined by

$$c_p^2 = (7 + \cos 2\pi/p)/8.$$

Suppose that $g \in \beta(F)$ and $\limsup |\hat{\lambda}_k(g)| \geqslant \eta > 0$, while $c_p^M < \eta$ for some integer $M \geqslant 1$. For infinitely many $k = 1, 2, 3, \ldots$, fewer than M of the factors of $\hat{\lambda}_k$ have modulus $< c_p$ at g. Since the value of $\gamma(i, j)$ is always a p-th root of 1, the equations $(\gamma(i, j), g) = 1$ for $i = 1, 2, 3$ are valid for every j with at most M exceptions. Let Γ_k be the subgroup of Γ generated by the characters $\gamma(i, j)$ introduced at the k-th step. Then there is an identity

$$(\gamma, g) = \prod_{1}^{M} \gamma(g_n)^{e_n} \quad \text{for every } \gamma \in \Gamma_k,$$

with elements g_n of F_1 and numbers $e_n = 0, 1, \ldots, p-1$. This holds for infinitely many integers k, but since $g \in \beta(F)$, it is clear that a single relation of this kind must hold for every γ in Γ, i.e., $g \in \alpha(F)$.

We can now complete the main theorem for II. The mapping $E \to B(E)$ is continuous, in fact homeomorphic from 2^F to 2^Γ. When E is uncountable, we take a continuous probability measure ν in E and set

$$\mu = \nu \otimes (\delta(0) + \delta(1) + \delta(2)),$$

whence $\mathrm{Re}\,\hat{\mu} \geqslant 1$ in $B(E)$. Suppose, in the opposite direction, that $B(E)$ is in $\mathrm{w}\Gamma$ and $\hat{\mu} \geqslant 1$ in $B(E)$ for some continuous measure μ in G. Since $B(E)+\beta(E)^{\perp} = B(E)$, this will remain true for a continuous measure concentrated in $\beta(E)$. We apply Lemma 2, replacing F by E throughout. For the sequence (λ_k) of probability measures in $B(E)$,

$$\int_G \hat{\lambda}_k(g)\mu(dg) = \int_\Gamma \hat{\mu}(X)\lambda_k(d\chi) \geqslant 1$$

and Lemma 2 confirms that $|\mu|$ has positive mass in $\alpha(E)$, whence $\alpha(E)$ – and consequently E itself – must be uncountable. The theorem of Hurewicz cited earlier then shows that $\mathrm{w}\Gamma$ cannot be a Borel set in 2^{Γ}.

In the proof just concluded, the mapping of E to $B(E)$ is a homeomorphism, but for groups of type I it is possibly discontinuous. (That has no effect on the succeeding argument.) To remove this defect, let σ be a continuous measure on K, and t a number in $(0, 1/3)$ such that

$$\sigma\{g\colon |\chi(g)-1| = t\} = 0 \quad \text{for each } \chi \text{ in } \Gamma.$$

There is a closed set $K_1 \subseteq K$ such that $\sigma(K_1) > 0$ and $\chi - 1 \neq t$ in K_1 for each χ. We then define $B'(E)$ for $E \subseteq K_1$ by the inequality $|\chi-1| < t$ in E. This is a homeomorphism from 2^{K_1} into 2^{Γ} and the remaining steps are the same.

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