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ALGEBRAS OVER THEORIES

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Introduction. In [5], Lawvere mentions the following problem:

Let $f: A \to B$ be a map of theories. When does the forgetful functor $f^b: B^b \to A^b$ from B-models to A-models have a right adjoint?

The purpose of this article is to throw some light on this question by extending an interesting analogy which emerged in Lawvere's paper. This analogy may be loosely summarized as

$$\text{Rings, modules} \sim \text{Theories, models.}$$

A great variety of definitions and theorems generalize in this way, e.g. bimodels, tensor products, matrix rings, Morita equivalence, the centre of a ring, commutative rings, the tensor algebra of a module, Azumaya algebras etc. A whole programme of research suggests itself; namely, take a theorem about rings, and try to prove its analogue about theories. This article is intended as a piece of propaganda for this programme. For this reason, and for the sake of brevity I have omitted proofs, and have restricted myself to definitions and statements of results. For further details, the reader is referred to reference [7].

The main part of this article concerns the problem of Lawvere mentioned above. The analogues of the following notions turn out to be relevant:

We may look at a ring homomorphism $f: R \to S$ in two ways:

1° as a ring homomorphism,

2° as a homomorphism of R-algebras.

The coincidence of these two notions is not so entirely trivial as at first appears. By an R-algebra we mean an (R, R)-bimodule X equipped with an associative multiplication $X \otimes_R X \to X$ and a two-sided unit $R \to X$.

If we translate notions 1° and 2° by means of the analogy mentioned above we no longer get the same thing. It turns out that they do coincide if and only if f is a "good" map as far as Lawvere's problem is concerned, i.e. if f^b has a right adjoint (as well as the left adjoint which always exists).

As a corollary of this, if $f\colon A \to B$ is a good map then B is obtained from A by adding in unary operations and further axioms. As an example of such a situation [6], the embedding

$$\text{(commutative rings)} \subseteq \text{(special } \lambda\text{-rings)}$$

is good.

1. Algebraic theories. An *algebraic theory* A is a category with coproducts in which every object is a coproduct of copies of a fixed fundamental object A^1. We denote by A^S the coproduct of an S-indexed family of A^1's, for any set S.

An *A-model* is a product preserving functor

$$A^{\text{op}} \to \text{Ens}$$

where A^{op} denotes the opposite, or dual, of the category A, and Ens denotes the category of sets and functions. A *homomorphism* of A-models is a natural map. We denote by A^b the category of A-models and homomorphisms. We leave it to the reader to check that these notions correspond to the classical concepts of universal algebra, except that operations are allowed to have arities of arbitrary cardinal.

The category A^b is complete, i.e. left and right limits exist. We denote by

$$I_A\colon A \to A^b$$

the functor given by $a \mapsto \text{Hom}_A(-, a)$. It is full and faithful. The following theorem is very useful:

THEOREM 1. *Let A be an algebraic theory, C a category with right limits, and $T\colon A \to C$ a coproduct preserving functor. Then there exists a unique (up to natural isomorphism) functor $T\colon A^b \to C$ such that $T.I_A = T$.*

If A and B are algebraic theories, a functor $f\colon A \to B$ is a *map of theories* if it preserves coproducts and fundamental objects. In this way we get a category *Th* of algebraic theories and their maps. The category *Th* is complete.

If $f\colon A \to B$ is a map of theories, composition with f gives a "forgetful" functor $f^b\colon B^b \to A^b$. Theorem 1 tells us there is a functor $f_*\colon A^b \to B^b$ making the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle I_A}\downarrow & & \downarrow{\scriptstyle I_B} \\ A^b & \xrightarrow{f_*} & B^b \end{array}$$

commute. The functor f_* is left adjoint to f^b, and this adjoint pair is strongly tripleable [1], [4], [7].

The category Ens is clearly an algebraic theory. It is trivial in the sense that its models are simply sets with no extra structure, i.e. $\mathrm{Ens}^b = \mathrm{Ens}$, and I_{Ens} is the identity functor. Further, Ens is initial in *Th*. If $j_A\colon \mathrm{Ens} \to A$ is the unique map of theories, we denote $(j_A)^b$, $(j_A)_*$ by U_A, F_A respectively. The functor U_A is the "underlying set" functor and is given by $X \mapsto X(A^1)$, for X an A-model. The functor F_A is the "free A-model" functor, and is equal to the composite

$$\mathrm{Ens} \xrightarrow{j_A} A \xrightarrow{I_A} A^b.$$

The fact that I_A is full and faithful means that we may identify A by means of I_A with the full subcategory of free A-models. Theorem 1 is essentially a consequence of the fact that every model is a right limit of free models.

Let R be a ring (with unit, of course). Let $\tilde{R}$ be the category of free left R-modules and R-homomorphisms. Then R is an algebraic theory, and $\tilde{R}$-models are simply left R-modules. We call a theory of the form $\tilde{R}$ *annular*. Because $\tilde{R}$ and R uniquely determine each other we will use the same symbol for both theory and ring. Thus, Z is the theory of abelian groups. It is a happy coincidence that the words "model" and "module" are so alike (I am indebted to Jon Beck for this remark). This is the basis of the rings — theories, modules — models analogy mentioned in the introduction. We also have (ring homomorphism) — (map of theories). Indeed, if A and B are annular theories, the maps of theories $f\colon A \to B$ are in bijective correspondence with the ring homomorphisms $f\colon A \to B$. The functor f^b is "pullback along f" and f_* is $B \otimes_A (-)$.

2. Bimodels. Let A and B be algebraic theories. A "co-A-model in the category of B-models" is given by a coproduct preserving functor

$$X\colon A \to B^b.$$

We call this an (A, B)-*bimodel.* A map of bimodels is to be a natural map, and we denote the category of (A, B)-bimodels by $[A, B]$. Evaluation at A^1 gives a forgetful functor

$$[A, B] \to B^b\colon X \mapsto X(A^1)$$

which forgets the co-A-model structure.

If A and B are annular, an (A, B)-bimodel X is simply an (A, B)-bimodule, i.e. an abelian group X which is simultaneously a left B-module and a right A-module such that

$$(b \cdot x) \cdot a = b \cdot (x \cdot a)$$

for $a \in A$, $b \in B$, $x \in X$.

If A is the theory of groups, and B the theory of commutative rings, an (A, B)-bimodel is a Hopf algebra.

According to Theorem 1, an (A, B)-bimodel

$$X\colon A \to B^b$$

determines a functor $\tilde{X}\colon A^b \to B^b$. We denote this by $X \otimes_A (-)$, for obvious reasons.

If M is a B-model, the composite

$$A^{\mathrm{op}} \xrightarrow{X} (B^b)^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{B^b}(-, M)} \mathrm{Ens}$$

is an A-model, which we denote by $\mathrm{Hom}_B(X, M)$. Its underlying set is the set of homomorphisms from the underlying B-model $X(A^1)$ of X to M. The functor

$$\mathrm{Hom}_B(X, -)\colon B^b \to A^b$$

is right adjoint to $X \otimes_A (-)\colon A^b \to B^b$.

Let X be an (A, B)-bimodel and Y an (B, C)-bimodel. The composite

$$A \xrightarrow{X} B^b \xrightarrow{Y \otimes_B (-)} C^b$$

is an (A, C)-bimodel which we denote by $Y \otimes_B X$. Note that

$$(Y \otimes_B X) \otimes_A (-) = Y \otimes_B \big(X \otimes_A (-)\big).$$

This tensor product operation on bimodels is coherently naturally associative.

Note that for any theory A, the functor I_A is an (A, A)-bimodel, and that $I_A \otimes_A (-)$ and $\mathrm{Hom}_A(I_A, -)$ are both identity functors. We shall adopt the usual abuse of notation prevalent in ring theory by denoting I_A by simply A. Thus $A \otimes_A X \simeq X$ and $\mathrm{Hom}_A(A, X) \simeq X$, evidently a satisfactory state of affairs. We refer to I_A as the *fundamental bimodel* of A. Its underlying A-model is $F_A(1)$, the free A-model on one generator. The fundamental bimodels act as units for the tensor product.

The following theorem is useful (and easy):

THEOREM 2. *Let A and B be algebraic theories. A functor $T\colon A^b \to B^b$ is of the form $X \otimes_A (-)$ for an (A, B)-bimodel X if and only if T has a right adjoint. Any natural map between such functors is induced by a unique map of bimodels.*

3. Algebras over theories. An *algebra* X over an algebraic theory A is an (A, A)-bimodel equipped with maps of (A, A)-bimodels

$$\pi\colon X \otimes_A X \to X \qquad \eta\colon A \to X$$

which we call "multiplication" and "unit" respectively for which the diagrams

$$\begin{array}{ccc} X\otimes_A X\otimes_A X & \xrightarrow{1_X\otimes_A\pi} & X\otimes_A X \\ {\scriptstyle \pi\otimes_A 1_X}\downarrow & & \downarrow{\scriptstyle \pi} \\ X\otimes_A X & \xrightarrow{\pi} & X \end{array}$$

$$\begin{array}{ccccc} A\otimes_A X & \xrightarrow{\eta\otimes_A 1_X} & X\otimes_A X & \xleftarrow{1_X\otimes_A\eta} & X\otimes_A A \\ & \searrow{\scriptstyle \sim} & \downarrow{\scriptstyle \pi} & {\scriptstyle \sim}\swarrow & \\ & & X & & \end{array}$$

commute. These diagrams just assert that π is associative and that η is a two-sided unit.

If X and X' are algebras over A a map Φ: $X \to X'$ of (A, A)-bimodels is a map of algebras if the diagrams

$$\begin{array}{ccc} X\otimes_A X & \xrightarrow{\Phi\otimes_A\Phi} & X'\otimes_A X' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ X & \xrightarrow{\Phi} & X' \end{array} \qquad \begin{array}{ccc} & A & \\ {\scriptstyle \eta}\swarrow & & \searrow{\scriptstyle \eta'} \\ X & \xrightarrow{\Phi} & X' \end{array}$$

commute. In this way we get a category $\mathrm{Alg}(A)$ of algebras over A, which has an obvious forgetful functor $\mathrm{Alg}(A) \to [A, A]$. This functor has a left adjoint T: $[A, A] \to \mathrm{Alg}(A)$. If M is an (A, A)-bimodel, $T(M)$, the "tensor algebra" of M has for its underlying (A, A)-bimodel

$$T(M) = \coprod_{n\geqslant 0} M^n,$$

where $M^0 = A$ and $M^{n+1} = M\otimes_A M^n$.

If X is an algebra over A, an X-*module* M is an A-model together with a map of A-models μ: $X\otimes_A M \to M$ such that the diagrams

$$\begin{array}{ccc} X\otimes_A X\otimes_A M & \xrightarrow{\pi\otimes_A 1_M} & X\otimes_A M \\ {\scriptstyle 1_X\otimes_A\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ X\otimes_A M & \xrightarrow{\mu} & M \end{array} \qquad \begin{array}{ccc} & & X\otimes_A M \\ & {\scriptstyle \eta\otimes_A 1_M}\nearrow & \downarrow{\scriptstyle \mu} \\ A\otimes_A M & \xrightarrow{\sim} & M \end{array}$$

commute. If M, M' are X-modules, a homomorphism Θ: $M \to M'$ is a map of X-modules if the diagram

$$\begin{array}{ccc} X \otimes_A M & \xrightarrow{1_X \otimes_A \theta} & X \otimes_A M' \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu'} \\ M & \xrightarrow{\quad \theta \quad} & M' \end{array}$$

commutes.

In this way we get a category X^b of X-modules, which has an obvious forgetful functor η^b: $X^b \to A^b$. This functor has a left adjoint η_*: $A^b \to X^b$ given by $M \mapsto X \otimes_A M$, where the structure map $X \otimes_A (X \otimes_A M) \to X \otimes_A M$ of $\eta_*(M)$ is $\pi \otimes_A 1_M$.

Let $\tilde{X}$ denote the full subcategory of X^b of X-modules of the form $\eta_*(F_A(S))$ for some set S. Then X is an algebraic theory and $\tilde{X}^b$ is just X^b.

For this reason we abuse language by denoting $\tilde{X}$ by simply X. Thus an algebra over A defines a theory which we denote by the same symbol; the functor η_*: $A^b \to X^b$, restricted to free models, induces a map of theories which we denote by η: $A \to X$, i.e. we use the same symbol as for the unit of the algebra. It is not hard to see that this is a procedure entirely consistent with the notations η^b, η_* which we already have. Hence, η: $A \to X$ may be interpreted either in Alg(A) or *Th*.

We may now ask: given a map of theories f: $A \to B$, when is B the theory associated with an algebra over A?

Answer: if and only if f^b has a right adjoint.

First, suppose X is an algebra over A. We may define a functor η_+: $A^b \to X^b$ by defining $\eta_+(M) = \mathrm{Hom}_A(X, M)$. The structure map of $\eta_+(M)$ is to be the map $X \otimes_A \mathrm{Hom}_A(X, M) \to \mathrm{Hom}_A(X, M)$ adjoint to the map $\mathrm{Hom}_A(X, M) \to \mathrm{Hom}_A(X, \mathrm{Hom}_A(X, M))$ given by

$$\mathrm{Hom}_A(X, M) \xrightarrow{\mathrm{Hom}_A(\pi, 1_M)} \mathrm{Hom}_A(X \otimes_A X, M) \xrightarrow{\sim} \mathrm{Hom}_A(X, \mathrm{Hom}_A(X, M)).$$

It is not hard to check that η_+ is right adjoint to η^b. Conversely, suppose f: $A \to B$ is a map of theories such that f^b: $B^b \to A^b$ has a right adjoint f_+. Then the composite

$$A^b \xrightarrow{f_*} B^b \xrightarrow{f^b} A^b$$

has a right adjoint, namely $A^b \xrightarrow{f_+} B^b \xrightarrow{f^b} A^b$.

Hence, by Theorem 2, there is an (A, A)-bimodel X such that $f^b f_* = X \otimes_A (-)$. Since this is a composite of a left adjoint followed by its right adjoint, this functor has a natural monad structure. Again by Theorem 2, this implies that X has a natural algebra structure. The strong tripleability of the pair (f^b, f_*) implies that as theories $B = X$ and that f is the "unit" of X.

In particular, it follows that B is a quotient of an enlargement of A by unary operations. To see this note that a B-model M is simply an A-model together with an action μ: $X \otimes_A M \to M$. Now $X \otimes_A M$ is generated as an A-model by elements $x \otimes m$ where x is an element of the underlying set of X and m runs over the generators of M. Thus, the action is defined by the unary operations $m \mapsto \mu(x \otimes m)$ for each x. In fact these unary operations form a semigroup, with multiplication defined by π: $X \otimes_A X \to X$ and with unit defined by η: $A \to X$.

Let us call a map of theories f: $A \to B$ *good* if f^b has a right adjoint. We have the following theorems:

THEOREM 3. *The map f: $A \to B$ is good if and only if B "is" an algebra over A.*

THEOREM 4. *A composite of good maps is good.*

THEOREM 5. *If in the commutative diagram of theories*

$$\begin{array}{ccc} & A & \\ {\scriptstyle f_1}\swarrow & & \searrow{\scriptstyle f_2} \\ B_1 & \xrightarrow{g} & B_2 \end{array}$$

f_1 and f_2 are good then so is g.

Note that maps between annular theories are always good. Lawvere [5] mentions the fact that j_A: Ens $\to A$ is good if and only if A is a unary theory (i.e. a monoid — so A-models are sets on which the monoid acts). This is immediate, because monoids are precisely algebras over Ens.

The map of theories $A \to *$, where $*$ is the terminal object of *Th* (a category with only one map; its models are one element sets) is good if and only if A has precisely one nullary operation or, equivalently, if A^b has a zero object.

Unsolved problem. Let A be a theory. Consider the category whose objects are good maps $B \to A$ and whose maps are commutative diagrams.

$$\begin{array}{ccc} B & \longrightarrow & B' \\ & \searrow \quad \swarrow & \\ & A & \end{array}$$

Does this category have an initial object? (**P 738**). If A is annular the answer is yes, and the initial object is $Z \to A$(¹).

(¹) Prof. R. Isbell has arguments to show that the conjecture is unlikely.

4. Azumaya theories and matrix theories. If S is a set, we call a map $\alpha\colon A^1 \to A^S$ in a theory A an *S-ary operation* of A. If β is a T-ary operation, we say that α and β *commute* if the following condition holds for any $S\times T$-matrix of elements $\{x_{s,t}\}$ of any A-model:

First apply β to each row of the matrix $\{x_{s,t}\}$, and then apply α to the column so obtained. Call the result y_1. Now apply α to each column, and then β to the row so obtained, and call the result y_2. Then $y_1 = y_2$.

It may be remarked that this definition can be slicked up to an element-free form. Note that a unary operation commutes with itself, but for higher arity an operation does not necessarily commute with itself. A theory is commutative if every pair of operations commutes. In that case, just as for rings, every A-model has a natural (A, A)-bimodel structure. Thus, if X and Y are A-models, and A is commutative, $\mathrm{Hom}_A(X, Y)$ has a natural A-model structure, and we have an A-model $X\otimes_A Y$, which is naturally isomorphic to $Y\otimes_A X$.

For any theory A, we define the *centre* of A, $Z(A)$ to be the largest subtheory of A whose operations commute with all the operations of A.

If A and B are any two theories, we define $A\otimes B$ to be the quotient theory of $A * B$ (the coproduct of A and B in *Th*) by the congruence which, in effect, states that operations in $A*B$ which have come from A commute with those that have come from B. An $A\otimes B$-model may be interpreted as an A-model in the category of B-models, or conversely, as a B-model in the category of A-models. If A and B are commutative, then $A\otimes B$ is the coproduct of A and B in the category of commutative theories.

We say that a map of theories $f\colon A\to B$ is an *extension* if B is generated by $\mathrm{Im} f$ and a set of operations which all commute with $\mathrm{Im} f$. Alternatively, f can be factorized

$$A \to A\otimes C \to B$$

for some theory C, where $A\to A\otimes C$ is the canonical map, and $A\otimes C\to B$ is surjective. It is not hard to see that a composite of extensions is an extension, and that if f is an extension, then $f\big(Z(A)\big) \subseteq Z(B)$.

Now we come to another concept, that of a *primitive element* of a bimodel. Let A be a theory and X an (A, A)-bimodel. A map of bimodels $A\to X$ determines, by applying the forgetful functor $[A, A]\to A^b$, a homomorphism $F_A(1)\to X(A^1)$, and hence, by adjointness, an element of the underlying set of X. We call such an element *primitive*. If A is a ring, and X an (A, A)-bimodule, $x\in X$ is primitive if $ax = xa$ for all $a\in A$. We call X *primitively generated* if it is generated as an A-model by primitive elements.

THEOREM 6. *A good map of theories $f\colon A\to B$ is an extension if and only if B, as an algebra over A, is primitively generated.*

Let us denote the full subcategory of primitively generated (A, A)-bimodels by $\{A\}$. Composing the inclusion $\{A\} \subseteq [A, A]$ with the forgetful functor $[A, A] \to A^b$ gives a forgetful functor $\{A\} \to A^b$.

In general, if $f\colon A \to B$ is a map of theories, there is no induced functor from $[A, A]$ to $[B, B]$. However, we have the following theorem:

THEOREM 7. *Let $f\colon A \to B$ be an extension. Then there is a functor $f_!\colon \{A\} \to \{B\}$ making the diagram*

$$\begin{array}{ccc} \{A\} & \xrightarrow{f_!} & \{B\} \\ \downarrow & & \downarrow \\ A^b & \xrightarrow{f_*} & B^b \end{array}$$

commute.

We call a theory A *Azumaya* if $i_!$ is an equivalence of categories, where $i\colon Z(A) \to A$ is the inclusion, which is an extension, of course. The following is a characterization of Azumaya theories:

THEOREM 8. *A theory A is Azumaya if for every set S and subfunctor T of $\prod_S\colon A^b \to A^b$, having a left adjoint, there is a set S' and a pair $\prod_S \rightrightarrows \prod_{S'}$ of natural maps whose equalizer is T.*

An Azumaya ring is an Azumaya theory, and also the theory of groups is Azumaya (this follows from Kan's theorem that a cogroup in the category of groups is free).

Two theories A and B are *Morita equivalent* if A^b and B^b are equivalent categories. If one of them is Azumaya, so is the other, and they have isomorphic centres.

For any theory A, we denote by $M_S(A)$ full subcategory of A given by objects of the form $A^{S\times T}$ for some T. Then $M_S(A)$ is a theory (though not a subtheory of A unless $S = 1$, in which case $M_1(A) \simeq A$). If A is a ring, $M_S(A)$ is the ring of $S \times S$ row finite matrices with coefficients in A. We may prove the following facts about $M_S(A)$:

THEOREM 9. $M_S(A \otimes B) \simeq M_S(A) \otimes B$.

THEOREM 10. *A and $M_S(A)$ are Morita equivalent.*

$M_S(A)$-models are precisely A-models of the form $\prod_S X$ for some A-model X, and homomorphisms are of the form $\prod_S f$ for f a homomorphism of A-models.

Note that in particular we have $M_S(A) \simeq A \otimes M_S(\mathrm{Ens})$. The theory $M_S(\mathrm{Ens})$ is generated by S^S S-ary operations, subject to quite simple identities. In fact, if $\prod_S X$ is a typical $M_S(\mathrm{Ens})$ model, an S-ple of elements of $\prod_S X$ is given by an $S \times S$ matrix of elements of X. The generating operations give each one S-ple of elements by selecting an element from each row. We clearly have S^S ways of doing this. We also have the following interesting, and very simple theorem:

THEOREM 11. *Every theory Morita equivalent to* Ens *is of the form* M_S(Ens) *for some set* S.

We have not mentioned the obvious generalizations of algebraic K-theory to algebraic theories, but it looks as if there is a rich field waiting to be explored there.

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