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BOUNDARY CONVERGENCE OF FUNCTIONS IN THE MEROMORPHIC NEVANLINNA CLASS

BY

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The aim of this paper is to generalize, to meromorphic functions, a collection of results about holomorphic functions which began with Fatou, and to which extensive contributions have been made by Stein and more recently by Nagel, Stein and Wainger.

The main result (Section 4) is a maximal function estimate for meromorphic Nevanlinna functions, which is analogous to the maximal function estimate in [23] for holomorphic Nevanlinna functions. Using this estimate we can then conclude that, in domains with C^2 boundaries, functions in the meromorphic Nevanlinna class have limits at almost every point of the boundary, not only in classical non-tangential approach regions, but even in the larger admissible approach regions considered in [34] and in the still larger approach regions considered in [23] which can be defined in compact domains of finite type. In the case of admissible approach, this conclusion has already been proved by Lempert [21]; but the maximal function estimate which we derive gives a quantitative result which this paper does not provide.

Section 3 deals first with the special case of non-tangential approach, since the proof of the general result depends on estimates which are derived there. The proofs of these estimates depend in turn on an explicit upper bound on the size of the Green's function for a $\mathbf{C}^2$ domain in $\mathbf{R}^n$ ($n \geq 2$), which is derived in Section 2.

The proof of the main estimate depends also on a new theorem in value distribution theory, which is proved at the end of Section 4. This theorem puts a lower bound on the $(2n-2)$-dimensional Hausdorff measure of that part of the zero set of a holomorphic function, f, contained in a polydisc $D(z^0; r_1, \ldots, r_n)$, given that f vanishes at a particular point of D, and that f has no zeros in a slightly smaller polydisc $D' \subset D$.

1. Introduction.

Since the proof of our theorem depends on a good understanding of analytic hypersurfaces and integration over them, we present here a brief review of some of the important results. Although many of these are standard, we will at least state them, for the purpose of reference later in the paper. Following this we will review some notation and theorems about maximal functions, and define the non-tangential and admissible approach regions. Finally, we define the meromorphic Nevanlinna class.

We shall use the following standard notation. If $\Omega \subset \mathbf{C}^n$ is a domain, $\mathcal{O}(\Omega)$ will represent the class of holomorphic functions on Ω. When it is necessary to emphasize one particular coordinate we will often write $z = (z', z_n)$ for a point of $\mathbf{C}^n$ where $z' = (z_1, \ldots, z_{n-1}) \in \mathbf{C}^{n-1}$. The zero set of a holomorphic function f will be written χ_f, $B(z;r)$ will represent the ball of radius r and $D(z; r_1, \ldots, r_n)$ the polydisc with radii $r_1, \ldots, r_n$, centered at $z \in \mathbf{C}^n$. For shorthand we may write $D(z;r) = D(z;r', r_n) = D(z; r_1, \ldots, r_n)$. $\mathbf{CP}^{n-1}$ will denote $(n-1)$-dimensional complex projective space and $\pi_n : \mathbf{C}^n \to \mathbf{CP}^{n-1}$ the canonical projection. We shall also need the constants

$$\omega_n = \pi^{n/2}/\Gamma(n/2+1) \quad \text{and} \quad \sigma_{n-1} = n\omega_n .$$

These are respectively the volume of the unit ball and the area of the unit sphere in $\mathbf{R}^n$.

We begin with the following structure theorem.

THEOREM 1.1. *Suppose $\Omega \subset \mathbf{C}^n$ and $f \in \mathcal{O}(\Omega)$. Then either $\chi_f = \emptyset$ or both of the following hold:*

(1.1.1) *χ_f has Hausdorff dimension $2n-2$.*

(1.1.2) *The set of points $z \in \chi_f$ that* do not *have a neighborhood U_z such that $U_z \cap \chi_f$ is a complex $(n-1)$-dimensional manifold, has Hausdorff dimension at most $2n-4$.*

The set in 1.1.2, S_f, is the singular set of χ_f, and the set $\chi_f - S_f$, denoted by χ_f^*, is the set of regular points of χ_f. See [10] and [5]. We shall write $\mathrm{H}_k(E)$ for the Hausdorff k-dimensional measure of a set E.

Integration with respect to Hausdorff measure, although conceptually elegant, is in general computationally quite difficult. The fundamental implication of Theorem 1.1 is that integration over χ_f with respect to Hausdorff $(2n-2)$-dimensional measure is exactly the same as integration over the smooth manifold χ_f^* with respect to surface measure. For this latter quantity we have explicit expressions in terms of coordinate charts and their derivatives. This leads to the rather remarkable formula:

THEOREM 1.2 (Wirtinger). *Let $f(z)$ be a holomorphic function defined in a neighborhood of the closure of some open set $U \subset \mathbf{C}^n$. Define the exterior*

(1,1) *form*

$$\phi = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j .$$

Then the $(2n-2)$*-dimensional volume form on the manifold* χ_f^* *is exactly*

$$\phi^{n-1}/(n-1)! = (i/2)^{n-1}\sum_{k=1}^{n}\bigwedge_{j\neq k} dz_j \wedge d\bar{z}_j .$$

In particular, for every measurable function φ *on* χ_f *and any open set* $U \subset \mathbf{C}^n$ *we have*

$$\text{(1.2.1)} \qquad \int_{\chi_f\cap U} \varphi(z)\, d\mathrm{H}_{2n-2}(z) = (i/2)^{n-1}\int_{\chi_f^*\cap U} \varphi(z)\sum_{k=1}^{n}\bigwedge_{j\neq k} dz_j \wedge d\bar{z}_j .$$

Here ϕ^{n-1} *means* $\phi\wedge\phi\wedge\ldots\wedge\phi$ $(n-1$ *times) and* $\bigwedge_{j\neq k} dz_j\wedge d\bar{z}_j$ *represents the exterior* $(n-1,n-1)$ *form*

$$dz_1\wedge d\bar{z}_1\wedge\ldots\wedge dz_{k-1}\wedge d\bar{z}_{k-1}\wedge dz_{k+1}\wedge d\bar{z}_{k+1}\wedge\ldots\wedge dz_n\wedge d\bar{z}_n .$$

Theorem 1.2 can be easily generalized to analytic varieties of arbitrary dimension with any Hermitian metric. The reason that it is stated here only for varieties of codimension 1 with the standard Euclidean metric on $\mathbf{C}^n$ is that this is the only case of the theorem we shall use. For a detailed proof of our special case see [28] (p. 211). For a sketch proof of the general case see [8] (p. 5).

DEFINITION 1.3. Let $f(z)$ be holomorphic in a neighborhood of $z^0 \in \mathbf{C}^n$. The *multiplicity* of f at z^0, written $\gamma_f(z^0)$, is defined to be the degree of the smallest non-zero monomial in the power series expansion of f at z^0, i.e. if $f(z) = \sum_{|\alpha|\geq 0} a_\alpha(z-z^0)^\alpha$ near z^0, then

$$\gamma_f(z^0) = \min\{|\alpha| : a_\alpha \neq 0\} .$$

Recall $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $z^\alpha = z_1^{\alpha_1}\ldots z_n^{\alpha_n}$ and each α_i is a non-negative integer.

In order to really calculate using Theorem 1.2 we need the following:

LEMMA 1.4. *Let the function* $g(z)$ *be continuous in the polydisc* $D(0;r) = D(0;r',r_n)$*, and let the* $(n-1,n-1)$ *form* ω *be defined in* $D(0;r)$ *by*

$$\omega = \alpha(z_1,\ldots,z_{n-1})dz_1\wedge d\bar{z}_1\wedge\ldots\wedge dz_{n-1}\wedge d\bar{z}_{n-1}$$

where the function $\alpha(z_1,\ldots,z_{n-1})$ *is continuous in the polydisc* $D(0;r') \subset \mathbf{C}^{n-1}$*. Suppose that* $f(z)$ *is holomorphic in a neighborhood of the closure of* $D(0;r)$*. For each* $z^{0\prime} = (z_1^0,\ldots,z_{n-1}^0) \in D(0;r')$ *we let* $z_n^j(z^{0\prime})$ *be the points of intersection (arranged in some order) of the complex line* $\{z\in\mathbf{C}^n : z_1 = z_1^0,\ldots,z_{n-1} = z_{n-1}^0\}$ *with the set* $\chi_f\cap D(0;r)$*, and we let* $\gamma_f^n(z_n^j(z^{0\prime}))$ *be the* *multiplicity (in the sense of one complex variable) of the function* $f(z^{0'}, z_n)$ *(thought of as a function of the single variable* z_n*) at the point* $z_n^j(z^{0'})$*. Then*

$$\int_{\chi_f \cap D(0;r)} g(z)\gamma_f(z)\omega = \int_{D(0;r')} \sum_j g(z^{0'}, z_n^j(z^{0'}))\gamma_f^n(z_n^j(z^{0'}))\omega .$$

Proof. See [28].

Using Theorem 1.2 and Lemma 1.4 we can prove the following result which is essential to the proof of our main theorem. For details of the proof see [26].

THEOREM 1.5. *Let* $f(z)$ *be holomorphic in some open set* $U \subset \mathbf{C}^n$*. Then in the sense of distributions,* $(2\pi)^{-1}\Delta \log|f|$ *is the Riesz measure associated with* χ_f *"counted according to multiplicities". More precisely:*

$$\frac{1}{2\pi}\int_U \Delta\varphi(z)\log|f(z)|\,dV_{2n}(z) = \int_{\chi_f \cap U} \varphi(z)\gamma_f(z)\,d\mathrm{H}_{2n-2}(z) \tag{1.5.1}$$

for any smooth function φ *with compact support in* U*. (Here* $dV_{2n}(z)$ *is the Euclidean volume element on* $\mathbf{C}^n \approx \mathbf{R}^{2n}$*.)*

We will need one more theorem about zero sets of analytic functions. It will allow us to conclude that the $(2n-2)$-dimensional Hausdorff measure of $\chi_f \cap B(z_0; r)$ is "large" compared to the radius r if $z_0 \in \chi_f$. To state the theorem we first define some notation.

DEFINITION 1.6. Let $f(z)$ be holomorphic in the ball $B(0; r) \subset \mathbf{C}^n$. For $0 < r < R_0$ we set $\chi_f[r] = \chi_f \cap B(0; r)$ and $n_f(r) = (\omega_{2n-2})^{-1}\mathrm{H}_{2n-2}(\chi_f[r])$. We will also use the function $\nu_f(r) = r^{2-2n}n_f(r)$.

THEOREM 1.7. *With the notation above, for fixed* f *the function* $\nu_f(r)$ *is increasing in* r *and* $\lim_{r\to 0}\nu_f(r) = \gamma_f(0)$.

Proof. See [8].

COROLLARY 1.8. *If* f *is holomorphic in the closure of* $B(z^0; r)$ *and* $f(z^0) = 0$*, then*

$$\mathrm{H}_{2n-2}(\chi_f \cap B(z^0; r)) \geq \omega_{2n-2}\gamma_f(0)r^{2n-2} \geq \omega_{2n-2}r^{2n-2} . \tag{1.8.1}$$

We now leave the realm of complex varieties to review a few theorems in real analysis that we will need.

Suppose we have a measure space (X, m) which is also a separable metric space, and a collection of subsets $B(x; r)$ parameterized by $x \in X, 0 < r < \infty$. We will call this collection a *family of balls* if they satisfy the following five conditions for some fixed choice of constants $c > 1$ and $K > 1$:

(1.9.1) Each $B(x; r)$ is an open bounded set.

(1.9.2) $\mathrm{m}(B(x; r)) > 0 \quad \forall x \in X, \quad 0 < r < \infty.$

(1.9.3) $B(x;r_1) \subset B(x;r_2)$ if $r_1 \leq r_2$.
(1.9.4) If $B(x;r_1) \cap B(y;r_2) \neq \emptyset$ and $r_1 \geq r_2$ then $B(y;r_2) \subset B(x;cr_1)$.
(1.9.5) $\mathrm{m}(B(x;cr)) \leq K\mathrm{m}(B(x;r)) \quad \forall x \in X, 0 < r < \infty$.

For our purposes these balls will be subsets of a manifold defined by some metric.

DEFINITION 1.10. Let f be a locally integrable function on X. The *maximal function* of f *subordinate to the collection* $B(x;r)$ is defined by

$$(1.10.1) \qquad (Mf)(x) = \sup_{r>0}(m(B(x;r)))^{-1} \int_{B(x;r)} |f(y)|\, d\mathrm{m}(y).$$

The essential properties of a collection of balls and their maximal function are given in the following two theorems.

THEOREM 1.11. *There is a constant $A = A(c,K) > 0$ such that whenever Y is a subset of X, and Y is covered by a collection of balls, i.e. $Y \subset \bigcup_\alpha B(x_\alpha;r_\alpha)$, then there is a countable* disjoint *subcollection $B(x_i;r_i)$ $(i = 1,2,\ldots)$ satisfying*

$$(1.11.1) \qquad \mathrm{m}(Y) \leq A\mathrm{m}\Big(\bigcup_{i=1}^{\infty} B(x_i;r_i)\Big) = A\sum_{i=1}^{\infty} \mathrm{m}(B(x_i;r_i)).$$

THEOREM 1.12.

(1.12.1) *For every $1 < p \leq \infty$ there is a constant $A_p = A_p(c,K) > 0$ such that for all $f \in L^p(X)$*

$$||Mf||_p \leq A_p||f||_p$$

(i.e. $f \to Mf$ is bounded on $L^p(X)$).

(1.12.2) *There is a constant $A_1 = A_1(c,K) > 0$ such that for all $f \in L^1(X)$ and all $\lambda > 0$*

$$\mathrm{m}\{x : Mf(x) > \lambda\} \leq A_1||f||_1/\lambda$$

(i.e. $f \to Mf$ is weak-type (1,1)*).*

This will clearly remain true if we replace f by a finite measure μ, and $||f||_1$ by $||\mu||$, the total variation of μ. For proofs of these theorems see [33], Ch. 1.

Now let Ω be a bounded domain in $\mathbf{C}^n$ with boundary $\partial\Omega$ which is a manifold of class C^2. For each $z \in \Omega$ we write $\delta(z)$ for the Euclidean distance, $\mathrm{dist}(z,\partial\Omega)$, from z to the compact set $\partial\Omega$. For $\varepsilon > 0$ we define the subdomain

$$\Omega_\varepsilon = \{z \in \Omega : \delta(z) > \varepsilon\}.$$

For $z \in \Omega_\varepsilon$ we may sometimes write $\delta_\varepsilon(z) = \mathrm{dist}(z,\partial\Omega_\varepsilon)$.

Since $\partial\Omega$ is class C^2 there will be some ε_0 such that for every $0 \leq \varepsilon < \varepsilon_0$, $\partial\Omega_\varepsilon$ is also a manifold of class C^2. Moreover, if $0 \leq \varepsilon < \varepsilon_0$, and if $0 \leq \delta(z) < \varepsilon_0$, then the orthogonal projection, $\pi_\varepsilon(z)$, of z onto $\partial\Omega_\varepsilon$ is uniquely defined. (We will usually abbreviate $\pi_0(z)$ by $\pi(z)$.) For each $\zeta \in \partial\Omega$ the *classical non-tangential approach region*, or cone, with aperture $\alpha > 0$ is defined by

$$\Gamma_\alpha(\zeta) = \{z \in \Omega : |z - \zeta| < (1+\alpha)\delta(z)\}.$$

It will also be important to consider the truncated cones

$$\Gamma_\alpha^h = \Gamma_\alpha - \Omega_h \quad (0 < h < \varepsilon_0).$$

For each $z \in \Omega$ we define

$$T_\alpha(z) = \{\zeta \in \partial\Omega : z \in \Gamma_\alpha(\zeta)\}.$$

$T_\alpha(z)$ is the "non-tangential projection" of z onto $\partial\Omega$. We will denote surface measure on the manifolds $\partial\Omega$ and $\partial\Omega_\varepsilon$ by $d\sigma(\zeta)$ and $d\sigma_\varepsilon(\zeta)$ respectively.

A function $f(z)$ defined in a domain $\Omega \subset \mathbf{C}^n$ is meromorphic if corresponding to each $w \in \Omega$ there is a neighborhood U_w and two coprime holomorphic functions g_w and h_w on U_w such that $f(z) = g_w(z)/h_w(z)$ for all $z \in U_w$. Since the ring of germs of holomorphic functions at w is a unique factorization domain, g_w and h_w are uniquely defined up to a multiplicative unit in this ring. Thus the following definitions make sense:

(1.13.1) w is a *regular point* of f if $h_w(w) \neq 0$.

(1.13.2) w is a *pole* of f if $h_w(w) = 0$ and $g_w(w) \neq 0$.

(1.13.3) w is an *indeterminate point* of f if $h_w(w) = g_w(w) = 0$.

(1.13.4) $\gamma_f(w) = \gamma_{g_w}(w) - \gamma_{h_w}(w)$.

The set of regular points, poles and indeterminate points of a meromorphic function f will be denoted by $\mathcal{R}(f)$, $\mathcal{P}(f)$, and $\mathcal{I}(f)$ respectively. As in the case of holomorphic functions, $\chi_f \subset \mathcal{R}(f)$ will denote the set of points z where $f(z) = 0$.

For a general domain $\Omega \subset \mathbf{C}^n (n \geq 2)$, meromorphic functions cannot always be factored into the quotient of two globally defined holomorphic functions. Even pseudoconvexity of Ω is not enough to guarantee that this is possible. However, if Ω is convex, then any meromorphic f can be written as g/h where g and h are holomorphic in Ω (see [19], Ch. 6). Thus we can easily use a standard partition of unity argument, along with the fact that $\mathcal{I}(f)$ has zero $(2n-2)$-dimensional measure, to extend Theorem 1.5 to

COROLLARY 1.14. *Let $f(z)$ be* meromorphic *in some open set $\Omega \subset \mathbf{C}^n$. Then for any $\varphi \in C_0^\infty(\Omega)$ we have*

(1.14.1) $$\frac{1}{2\pi} \int_\Omega \Delta\varphi(z) \log|f(z)|\, dV_{2n}(z)$$

$$= \int_{(\chi_f \cup \mathcal{P}(f))\cap\Omega} \varphi(z)\gamma_f(z)\, d\mathrm{H}_{2n-2}(z)$$

$$= \int_{\chi_f\cap\Omega} \varphi(z)\gamma_f(z)\, d\mathrm{H}_{2n-2}(z) + \int_{\mathcal{P}(f)\cap\Omega} \varphi(z)\gamma_f(z)\, d\mathrm{H}_{2n-2}(z).$$

Notice that by definition 1.13.4, $\gamma_f(z) < 0$ for $z \in \mathcal{P}(f)$. With the notation above we finally define

$$\|f\|_0 = \sup_{0<\epsilon<\epsilon_0} \int_{\partial\Omega_\epsilon} \log^+ |f(\zeta)|\, d\sigma_\epsilon(\zeta), \tag{1.15.1}$$

$$P(f) = - \int_{\mathcal{P}(f)} \delta(z)\gamma_f(z)\, d\mathrm{H}_{2n-2}(z), \tag{1.15.2}$$

where $\log^+ t = \max\{\log t, 0\}$. Since $\mathcal{I}(f)$ has Hausdorff dimension at most $2n-4$, the integral in 1.15.2 is the same as if we had instead integrated over all of $\mathcal{P}(f) \cup \mathcal{I}(f)$. Notice also that $P(f)$ is positive.

DEFINITION 1.16. The *meromorphic Nevanlinna class* associated with $\Omega \subset \mathbf{C}^n$, denoted by$MN(\Omega)$, is the class of all meromorphic functions f on Ω satisfying

$$\|f\|_0 < \infty, \tag{1.16.1}$$

$$P(f) < \infty. \tag{1.16.2}$$

This class of functions has been studied before. (For example see [21].) Intuitively, 1.16.1 is a condition on "how large" the function f is, while 1.16.2 is a condition on "how many poles" it has. Surprisingly enough though, it is easy to construct examples (even in $\mathbf{C}^1$) that show that neither of these two conditions implies the other.

2. Estimates for Green's functions. For the rest of this paper, unless otherwise stated, we shall assume that the domains we consider are bounded and have boundaries which are C^2 manifolds, even though this assumption is sometimes stronger than what is needed. In this section, we shall briefly review some of the important properties of the Green's function for such domains $\Omega \subset \mathbf{R}^n$ ($n \geq 2$), as well as give a theorem estimating its size, which later will play an important role in proving the main results of this paper.

PROPOSITION 2.1. *The* fundamental solution *for the Laplacian on* $\mathbf{R}^n$ $(n \geq 2)$ *is given by*

$$\Gamma_n(x) = \begin{cases} (2\pi)^{-1}\log|x| & \text{if } n = 2, \\ -\beta_n |x|^{2-n} & \text{if } n \geq 3, \end{cases}$$

where $\beta_n = [(n-2)\omega_n]^{-1}$.

This means that for any $\varphi \in C_0^\infty(\mathbf{R}^n)$

$$\int_{\mathbf{R}^n} \Delta\varphi(x)\Gamma_n(x)\,dV(x) = \varphi(0). \tag{2.1.1}$$

It then follows that if $u(x)$ is defined by $u(x) = \int_{\mathbf{R}^n} \varphi(y)\Gamma(x-y)\,dV(y)$, then $u \in C^\infty(\mathbf{R}^n)$ and

$$\Delta u = \varphi. \tag{2.1.2}$$

For details, see [6] (p. 98).

The *Green's function* for a domain $\Omega \subset \mathbf{R}^n$ is a function $G_\Omega(x,y)$: $\Omega \times \overline{\Omega} - \{(x,x) : x \in \Omega\} \to \mathbf{R}$ which satisfies:

(2.2.1) For any fixed $x \in \Omega$, $h_x(y) \equiv G_\Omega(x,y) - \Gamma_n(x-y)$ is a *harmonic* function of y in all of Ω is continuous on $\overline{\Omega}$.

(2.2.2) For all $x \in \Omega$ and $y \in \partial\Omega$, $G_\Omega(x,y) = 0$.

These two conditions are enough to guarantee that, for a given bounded C^2 domain Ω, the Green's function always exists and is unique. For details, see [6] again. Moreover, 2.2.1 and 2.2.2 also imply the well known fact:

PROPOSITION 2.3. *For all* $x \in \Omega$, $y \in \Omega$ *with* $x \neq y$

$$G_\Omega(x,y) = G_\Omega(y,x). \tag{2.3.1}$$

The *Poisson kernel* for a domain $\Omega \subset \mathbf{R}^n$ is the function $P_\Omega(x,y)$: $\Omega \times \partial\Omega \to \mathbf{R}$ given by

$$P_\Omega(x,y) = \frac{\partial}{\partial n_y} G_\Omega(x,z)\Big|_{z=y} \tag{2.4.1}$$

where $\partial/\partial n_y$ represents differentiation with respect to z in the direction of the outward unit normal vector to $\partial\Omega$ at y.

Any easy application of Green's theorem gives the useful formula:

PROPOSITION 2.5. *Let* f *be a function which is* C^∞ *in a neighborhood of* $\overline{\Omega}$. *Then for all* $x \in \Omega$

$$f(x) = \int_\Omega G_\Omega(x,y)\Delta f(y)\,dV(y) + \int_{\partial\Omega} P_\Omega(x,y)f(y)\,d\sigma(y) \tag{2.5.1}$$

where $d\sigma(y)$ *is surface measure on* $\partial\Omega$.

For our application though, we do not want this formula to be restricted to smooth functions. We need:

PROPOSITION 2.6. *Let* $f(z)$ *be a function which is* meromorphic *in a neighborhood* U *of the closure of a domain* $\Omega \subset \mathbf{C}^n$. *Then for any point*

$z \in (\mathcal{R}(f) - \chi_f) \cap \Omega$

$$\log|f(z)| = \int_{\chi_f \cap \Omega} G_\Omega(z, z')\gamma_f(z')\, d\mathrm{H}_{2n-2}(z') \tag{2.6.1}$$
$$+ \int_{\mathcal{P}(f)\cap\Omega} G_\Omega(z, z')\gamma_f(z')\, d\mathrm{H}_{2n-2}(z')$$
$$+ \int_{\partial\Omega} P_\Omega(z, z')\log|f(z')|\, d\sigma(z') .$$

Intuitively, this proposition follows from Proposition 2.5 and Corollary 1.14. We must exercise a bit of caution though because Proposition 2.5 does not in general hold for functions, f, which are discontinuous. While a rigorous proof of Proposition 2.6 requires a good deal of work, the basic tools which are needed are all part of the standard theory of distributions, so we will not give the long, drawn out details here. The interested reader is referred to [26].

As notation for the main theorem of this section we make the following definitions.

DEFINITION 2.7. Let $\Omega \subset \mathbf{R}^n$ $(n \geq 2)$ be a bounded domain. We say that Ω is *concave of order* r if for all $y \in \partial\Omega$ there is a ball B_y of radius r contained in the complement of Ω which is tangent to $\partial\Omega$ at y. That is, $B_y \subset \mathbf{R}^n - \Omega$ and $\overline{B}_y \cap \overline{\Omega} = \{y\}$. Similarly we say that Ω is *convex of order* r if for all $y \in \partial\Omega$ there is a ball B'_y of radius r contained in Ω which is tangent to $\partial\Omega$ at y.

DEFINITION 2.8. The *diameter* of a domain Ω is given by

$$\operatorname{diam}(\Omega) = \sup\{|x - y| : x \in \Omega, y \in \Omega\} .$$

The main theorem of this section is

THEOREM 2.9. *Fix $r > 0$, $0 < \mathrm{D} < \infty$ and $n \geq 2$. There is a constant $K = K(r, \mathrm{D}, n)$ such that for all $\Omega \subset \mathbf{R}^n$ with* $\operatorname{diam}(\Omega) < \mathrm{D}$ *which are also concave of order r,*

$$|G_\Omega(x, y)| \leq \frac{K\delta(x)\delta(y)}{|x - y|^n} \tag{2.9.1}$$

for all $x \in \Omega$, $y \in \overline{\Omega}$, $x \neq y$.

Various forms of this theorem seem to be widely quoted, usually without proof, but the point that distinguishes the statement of it here is that the dependence of the constant K on the properties of the domain Ω is made much more explicit. The usual statement is that K exists, but depends on Ω in a completely unspecified manner. For our application hoewever, we will need to apply the theorem *uniformly* to a whole family of domains—the domains Ω_ε defined in the previous section. For a careful proof of Theorem 2.9, see [26]. There an explicit estimate for the constant K, as a function of the parameters r, D, and n, is calculated in gory detail. Notice that an immediate consequence of Theorem 2.9, 2.2.2 and the definition 2.4.1 is

COROLLARY 2.9.2. *For all Ω as in Theorem* 2.9

$$|P_\Omega(x,y)| \le K\delta(x)/|x-y|^n \tag{2.9.3}$$

for every $x \in \Omega$, $y \in \partial\Omega$.

We can use Theorem 2.9 to compare the Green's functions of the subdomains Ω_ε with the Green's function of Ω. Doing so we get

LEMMA 2.10. *Let G_ε be the Green's function of Ω_ε $(0 \le \varepsilon \le \varepsilon_0)$. (For the sake of definiteness, set $G_\varepsilon(x,y) = 0$ if $\delta(y) < \varepsilon$.) Then for each fixed $x \in \Omega$*

$$\lim_{\varepsilon\to 0} G_\varepsilon(x,y) = G_0(x,y) \quad \text{uniformly in } y \in \Omega\,. \tag{2.10.1}$$

Proof. Apply the maximum principle to the harmonic function $G_\varepsilon - G_0$. Use Theorem 2.9.

When a sequence of harmonic functions converges uniformly, so do their gradients. Combining this with the fact that the Green's function of a C^2 domain is a $C^{2-\varepsilon}$ function up to the boundary (see [34]) we get

LEMMA 2.11. *Let P_ε be the Poisson kernel of Ω_ε $(0 \le \varepsilon < \varepsilon_0)$. As in Section 2, for each $y \in \partial\Omega$ we let $\pi_\varepsilon(y)$ be the orthogonal projection of y onto $\partial\Omega_\varepsilon$. Then for each fixed $x \in \Omega$*

$$\lim_{\varepsilon\to 0} P_\varepsilon(x,\pi_\varepsilon(y)) = P_0(x,y) \quad \text{uniformly in } y \in \partial\Omega\,. \tag{2.11.1}$$

3. The non-tangential maximal function. We now consider the standard non-tangential maximal function. In the context of harmonic (in fact subharmonic) functions, this has been introduced before by Stein (see both [33], Ch. 7 and [34], Ch. 1). In this section we derive an estimate for the non-tangential maximal function in the context of meromorphic Nevanlinna functions. This estimate generalizes Stein's result since it is identical to his when restricted to the subclass of holomorphic Nevanlinna functions. Moreover, as we shall see, this estimate implies non-tangential convergence at almost every point of the boundary for all functions in the meromorphic Nevanlinna class. At this point the reader is urged to review the definitions and notation presented at the end of Section 1. In particular, one should remember the definitions of the "non-tangential projection" T_α and the quantities $P(f)$ and $||f||_0$. We begin by recalling the standard

DEFINITION 3.1. Let $\Omega \subset \mathbf{C}^n$ be a bounded C^2 domain and let f be a complex-valued function defined on D. For each $\alpha > 0$, the α *non-tangential maximal function* of f is defined for $\zeta \in \partial\Omega$ by

$$(M_\alpha f)(\zeta) = \sup\{|f(z)| : z \in \Gamma_\alpha(\zeta)\}.$$

Of course this definition makes sense for domains in $\mathbf{R}^n$. We only restrict our attention to $\mathbf{C}^n$ since we are interested in meromorphic functions. The estimate we shall prove is

THEOREM 3.2. *Let $\Omega \subset \mathbf{C}^n$ $(n \geq 1)$ be a bounded C^2 domain and fix $\alpha > 0$. There is a constant $A = A(\alpha, \Omega) > 0$ such that for all $f \in MN(\Omega)$ and all $\lambda > 0$*

$$\begin{aligned} (3.2.1) \qquad & \sigma(\{\zeta \in \partial\Omega : (M_\alpha \log^+ |f|)(\zeta) > \lambda\}) \\ & \leq A[P(f) + \lambda^{-1}(P(f) + \|f\|_0)]. \end{aligned}$$

(Recall that σ is surface measure on $\partial\Omega$. Notice also that $M_\alpha \log^+ f = \log^+ M_\alpha f$.)

Remark 3.3. Notice that in the special case when f is a holomorphic Nevanlinna function, $P(f) = 0$, so 3.2.1 reduces to Stein's inequality.

Remark 3.4. For $0 < \varepsilon < \varepsilon_0$ set $\Omega'_\varepsilon = \Omega - \overline{\Omega}_\varepsilon$. (These domains look like $\partial\Omega$ "fattened out".) For each fixed $\alpha > 0$ we will want the constant A to satisfy 3.2.1 not only for Ω but also for all Ω'_ε. Thus, although we shall work with one domain Ω throughout the proof of Theorem 3.2, the reader should notice that each time we pick a constant $A_1, A_2, \ldots$, we can pick it in such a way that it will work just as well if the proof were being done for Ω'_ε.

For the rest of this section it will be convenient to use the following notation:

$$(3.5.1) \qquad F_\alpha(\zeta) = (M_\alpha \log^+ |f|)(\zeta) = \log^+((M_\alpha f)(\zeta)) \quad (\zeta \in \partial\Omega),$$

$$(3.5.2) \qquad S_\alpha(\lambda) = \{\zeta \in \partial\Omega : F_\alpha(\zeta) > \lambda\},$$

$$(3.5.3) \qquad V_\alpha = T_{2\alpha}(\mathcal{P}(f)) = \{\zeta \in \partial\Omega : \Gamma_{2\alpha}(\zeta) \cap \mathcal{P}(f) \neq \emptyset\},$$

$$(3.5.4) \qquad W_\alpha(\lambda) = S_\alpha(\lambda) - V_\alpha.$$

Since $S_\alpha(\lambda) \subset V_\alpha \cup W_\alpha(\lambda)$ it will suffice to estimate $\sigma(V_\alpha)$ and $\sigma(W_\alpha(\lambda))$ separately. This is the strategy of the proof.

PROPOSITION 3.6. $\sigma(V_\alpha) \leq A_1 P(f)$.

Proof. For $\zeta \in \partial\Omega$ we write $B_1(\zeta; r) = B(\zeta; r) \cap \partial\Omega$ where $B(\zeta; r)$ is the standard Euclidean ball. That the B_1's satisfy 1.9.1–5 is standard. Moreover, since Ω is C^2 there are constants $A_2, A_3 > 0$ such that for all $\zeta \in \partial\Omega$ and all $0 \leq r \leq \operatorname{diam}(\Omega)$

$$(3.6.1) \qquad A_2 r^{2n-1} \leq \sigma(B_1(\zeta; r)) \leq A_3 r^{2n-1}.$$

It is also clear that there are constants $A_4, A_5 > 0$ such that for all $z \in \Omega$

$$B_1(\pi(z); A_4\delta(z)) \subset T_{2\alpha}(z) \subset B_1(\pi(z); A_5\delta(z)). \tag{3.6.2}$$

It follows from Theorem 1.11 that there is a sequence of poles $\{p_j\}_{j=1}^{\infty}$ such that the collection of sets $T_{2\alpha}(p_j)$ are pairwise disjoint and such that

$$\begin{aligned}\sigma(V_\alpha) &\le A_6\sigma\Big(\bigcup_{j=1}^{\infty} T_{2\alpha}(p_j)\Big) \\ &= A_6\sum_{j=1}^{\infty}\sigma(T_{2\alpha}(p_j)) \le A_7\sum_{j=1}^{\infty}\delta(p_j)^{2n-1}.\end{aligned} \tag{3.6.3}$$

On the other hand, there is a small constant A_8 such that for each j, $B(p_j; 2A_8\delta(p_j)) \subset \Gamma_{2\alpha}(\pi(p_j))$. Since the $T_{2\alpha}(p_j)$ are disjoint, it is easy to see that the Euclidean balls $\{B(p_j; A_8\delta(p_j))\}_{j=1}^{\infty}$ are also disjoint. Setting $B^j = B(p_j; A_8\delta(p_j))$ we thus conclude

$$\begin{aligned}P(f) &= -\int_{\mathcal{P}(f)} \delta(z)\gamma_f(z)\, d\mathrm{H}_{2n-2}(z) \\ &\ge -\sum_{j=1}^{\infty}\int_{B^j\cap\mathcal{P}(f)} \delta(z)\gamma_f(z)\, d\mathrm{H}_{2n-2}(z).\end{aligned} \tag{3.6.4}$$

Clearly we may assume that we picked $A_8 \le 1/2$ so that $\delta(z) \ge \delta(p_j)/2$ for all $z \in B^j$. Thus continuing 3.6.4 gives

$$\begin{aligned}P(f) &\ge -\sum_{j=1}^{\infty}\frac{\delta(p_j)}{2}\int_{B^j\cap\mathcal{P}(f)} \gamma_f(z)\, d\mathrm{H}_{2n-2}(z) \\ &\ge \sum_{j=1}^{\infty}\frac{\delta(p_j)}{2}\int_{B^j\cap\mathcal{P}(f)} d\mathrm{H}_{2n-2}(z) \\ &\qquad\qquad (\text{since } \gamma_f(z) \le -1 \text{ for all } z \in \mathcal{P}(f)) \\ &= \sum_{j=1}^{\infty}\frac{\delta(p_j)}{2}\mathrm{H}_{2n-2}(\mathcal{P}(f)\cap B^j).\end{aligned} \tag{3.6.5}$$

Euclidean balls are convex so in each B^j, f is the quotient of two holomorphic functions. In particular, we may apply Corollary 1.8 to conclude

$$P(f) \ge \sum_{j=1}^{\infty}\frac{\delta(p_j)}{2}A_9\delta(p_j)^{2n-2} = A_{10}\sum_{j=1}^{\infty}\delta(p_j)^{2n-1}. \tag{3.6.6}$$

Taking $A_1 = A_7/A_{10}$ finishes the proof.

We now define g_ε on $\partial\Omega$ by $g_\varepsilon(\zeta) = \log^+|f(\pi_\varepsilon(\zeta))|$. Since the g_ε have uniformly bounded L^1 norms (recall $f \in MN(\Omega)$), we can choose a subsequence g_{ε_j} ($\varepsilon_j \to 0$) which converges weakly to a finite positive measure μ on $\partial\Omega$. In fact, we have $||\mu|| \leq ||f||_0$, where $||\mu||$ is the total variation of μ. (This is the standard procedure for harmonic functions.) For meromorphic functions we need to define a second finite positive measure on $\partial\Omega$ associated with the function f. For each set $E \subset \partial\Omega$ we let

$$\pi^{-1}(E) = \{z \in \Omega : 0 < \delta(z) < \varepsilon_0 \text{ and } \pi(z) \in E\}$$

and define

$$\nu(E) = -\int_{\mathcal{P}(f)\cap\pi^{-1}(E)} \delta(z)\gamma_f(z)\, d\mathrm{H}_{2n-2}(z)\,.$$

Clearly $||\nu|| \leq P(f)$. We now have

PROPOSITION 3.7. *Suppose $f \in MN(\Omega)$. Then for all $\zeta \in \partial\Omega - V_\alpha$*

$$(M_\alpha \log^+ |f|)(\zeta) \leq A_{11}(M^1(\mu + \nu))(\zeta)\,. \tag{3.7.1}$$

(Here M^1 is taken to be the maximal operator with respect to the "Euclidean" family of balls, $B_1(\zeta; r)$, on $\partial\Omega$ which are defined in the beginning of the proof of Proposition 3.6. μ and ν are related to f as defined above. $A_{11} = A_{11}(\alpha, \Omega)$.)

Proof. Applying Proposition 2.6 to Ω_ε and using Lemmas 2.10 and 2.11 we have for all $z \in \Omega$

$$\begin{aligned} &\log^+ |f(z)| \\ &\quad \leq \int_{\mathcal{P}(f)\cap\Omega} G_\Omega(z, z')\gamma_f(z')\, d\mathrm{H}_{2n-2}(z') + \int_{\partial\Omega} P_\Omega(z, z')\, d\mu(z')\,. \end{aligned} \tag{3.7.2}$$

It is well known that if $z \in \Gamma_\alpha(\zeta)$ then the second term on the right is bounded above by $A_{12}(M^1\mu)(\zeta)$. (See [34], p. 11.) To control the first term, we argue in a similar fashion, but in this case we must make use of the assumption that $\zeta \notin V_\alpha$. This assumption guarantees that for all $z \in \Gamma_\alpha(\zeta)$, f has no poles in the ball $B(z;\ A_{13}\delta(z))$. For $j = 0, 1, 2, \ldots$ set

$$X_j = \mathcal{P}(f) \cap \left\{z' \in \Omega : 2^j A_{13}\delta(z) \leq |z - z'| < 2^{j+1}A_{13}\delta(z)\right\}. \tag{3.7.3}$$

We may then use Theorem 2.9 to write

$$\begin{aligned} &\int_{\mathcal{P}(f)\cap\Omega} G_\Omega(z, z')\gamma_f(z')\, d\mathrm{H}_{2n-2}(z') \\ &\qquad = \sum_{j=0}^{\infty} \int_{X_j} G_\Omega(z, z')\gamma_f(z')\, d\mathrm{H}_{2n-2}(z') \end{aligned} \tag{3.7.4}$$

$$\leq \sum_{j=0}^{\infty} \int_{X_j} K \frac{\delta(z)\delta(z')}{|z - z'|^{2n}} |\gamma_f(z')| \, d\mathrm{H}_{2n-2}(z')$$
$$\leq \sum_{j=0}^{\infty} \int_{X_j} K 2^{-2nj} \delta(z)^{-2n+1} \delta(z') |\gamma_f(z')| \, d\mathrm{H}_{2n-2}(z')$$
$$\leq \sum_{j=0}^{\infty} 2^{-j} K 2^{(-2n+1)j} \delta(z)^{-2n+1} \int_{\pi(X_j)} d\nu(\zeta').$$

But $\pi(X_j) \subset B_1(\zeta; A_{13} 2^j \delta(z))$ for some large constant A_{13} (since $z \in \Gamma_\alpha(\zeta)$). Hence

$$\text{(3.7.5)} \qquad \int_{\mathcal{P}(f) \cap \Omega} G_\Omega(z, z') \gamma_f(z') \, d\mathrm{H}_{2n-2}(z') \leq \sum_{j=0}^{\infty} A_{14} 2^{-j} (M^1 \nu)(\zeta)$$
$$= 2A_{14}(M^1\nu)(\zeta).$$

(The careful reader will notice that this argument shows that the first quantity on the right in 3.7.2 is, as a function of z, bounded above by a constant times the Poisson integral of the measure $d\nu$.) Now take $A_{11} = \max\{A_{12}, 2A_{14}\}$ to finish.

We now see why it was important in Theorem 2.9 to keep track of how the constant K depends on the domain Ω. The constant A_{11} in the statement of Proposition 3.7 clearly depends on K, which enters in 3.7.4. K will work for all Ω'_ε as well as it does for Ω, because all these domains have the same diameter, and since they will all be concave of order r, for some small $r > 0$ (recall that Ω is C^2). Notice that K enters into the proof of the bound on the Poisson integral term in 3.7.2 as well. (See Corollary 2.9 and [34], p. 11.)

Combining Propositions 3.6 and 3.7 with Theorem 1.12 proves Theorem 3.2. Note that when we are dealing with the domains Ω'_ε ($\varepsilon < \varepsilon_0$), we should be careful, in our definition of the measure ν, that we take $\pi(z)$ to be the orthogonal projection of z onto *both* pieces of $\partial\Omega'_\varepsilon = \partial\Omega \cup \partial\Omega_\varepsilon$. This will mean that $\pi^{-1}(E)$ represents the entire "slab" lying over the set $E \subset \partial\Omega'_\varepsilon$ stretching from one piece of the boundary to the other. We will have to define ν by $\nu(E) = \nu(E \cap \partial\Omega) + \nu(E \cap \partial\Omega_\varepsilon)$. This may mean that we have to double several of our constants, but this doesn't hurt. The most important consequence of Theorem 3.2 (and Remark 3.4) is

THEOREM 3.8. *If $f \in MN(\Omega)$ then f has a non-tangential limit at almost every $\zeta \in \partial\Omega$.*

Proof. We apply Theorem 3.2 to Ω'_ε. As $\varepsilon \to \infty$, $P(f) \to 0$ so we may conclude that f is non-tangentially bounded at almost every $\zeta \in \partial\Omega$. (Recall that f is *non-tangentially bounded* at ζ if for every $\alpha > 0$ there is an $h > 0$ such that f is bounded in $\Gamma_\alpha^h(\zeta)$.) We now follow an argument due to

Calderón ([33], p. 201), using the results in Section 2 of this paper, instead of the explicit formula for the Poisson kernel of the upper half plane which is used in [33], to conclude that f has a non-tangential limit at almost every point ζ where it is non-tangentially bounded. Although in [33] it is assumed in the statement of the theorem that the function is harmonic in the entire upper half plane, a close inspection of the proof reveals that one really only uses the fact that the function is harmonic in every truncated cone Γ_α^h in which it is bounded, and this is certainly the case for meromorphic functions.

4. Generalized approach regions and convergence. In the theory of harmonic functions, it turns out that non-tangential convergence at the boundary is actually the best we can hope for, even when dealing with the very special class of *bounded* harmonic functions. This is made more precise by the following definition and example.

DEFINITION 4.1. For each $p \geq 1$ the *p-th-power approach region* with aperture α at the point $y \in \partial\Omega$ is given by

$$\Gamma_\alpha(y;p) = \{x \in \Omega : \ |x-y|^p < (1+\alpha)\delta(x)\}. \tag{4.1.1}$$

Notice that near $\partial\Omega$, $\Gamma_\alpha(y;p_1) \subset \Gamma_\alpha(y;p_2)$ if $p_1 \leq p_2$ and that $\Gamma_\alpha(y;1) = \Gamma_\alpha(y)$.

EXAMPLE 4.2. There is a function $f \in h^\infty(\Omega)$ such that *for every* $p > 1$, there is no point on $\partial\Omega$ where f has a pth-power approach limit. That is, for all $p > 1$, $\alpha > 0$, and *all* $y \in \partial\Omega$, the limit

$$\lim_{\substack{x \to y \\ x \in \Gamma_\alpha(y;p)}} f(x)$$

does not exist. (See [38], p. 280 for details.)

It is a remarkable fact, then, that for domains $\Omega \subset \mathbf{C}^n$ $(n \geq 2)$, holomorphic Nevanlinna functions have boundary limits almost everywhere in approach regions which are considerably larger than the non-tangential ones associated with harmonic functions. The first of these generalized approach regions to be considered were the "admissible" regions defined by

$$\mathcal{A}_\alpha(\zeta) = \left\{ z \in \Omega : \begin{array}{l} |(z-\zeta)\bullet\overline{\nu}_\zeta| < (1+\alpha)\delta_\zeta(z) \text{ and} \\ |z-\zeta|^2 < \alpha\delta_\zeta(z) \end{array} \right\}. \tag{4.3}$$

(Here ν_ζ is the unit normal vector to $\partial\Omega$ at ζ, $(z-\zeta)\bullet\overline{\nu}_\zeta$ represents the complex inner product on $\mathbf{C}^n$, and $\delta_\zeta(z)$ is the smaller of the two distances from z to $\partial\Omega$ and from z to the tangent plane (to $\partial\Omega$) at ζ.) Notice that these approach regions have quadratic profiles in the tangent directions which are orthogonal under the complex inner product to the unit normal, and they have non-tangential profiles in the remaining tangent direction. Stein introduced these regions in [34], where he shows that holomorphic Nevanlinna functions have admissible limits at almost every point of the boundary. Lempert ([21]) has since shown (for meromorphic functions) that one need not restrict these regions to lying above the tangent plane as Stein does, and most recently Nagel, Stein and Wainger have generalized these approach regions (for holomorphic functions) considerably further in the case where the domain Ω is of finite type (see [23]). Their regions will have profiles which are higher order than quadratic ($p > 2$) in the complex tangential directions at points where the Levi form degenerates.

Our aim now is to prove that meromorphic functions have boundary limits almost everywhere, both in the admissible approach regions (for all C^2 domains) and in the larger approach regions in [23] (for domains of finite type). The reader should keep in mind that we are really proving two theorems here. One, the admissible version, holds for all C^2 domains, while the other only makes sense in domains of finite type. In this subclass of domains however, the second theorem is stronger than the one about admissible approach, since the approach regions considered in this context are, in general, larger than the admissible approach regions. We shall treat the two versions as one though, since their proofs depend only on certain relationships between the particular approach regions, and associated families of boundary balls and imbedded polydiscs—relationships which the reader can check are common to both types of approach. As noted before the introduction, Lempert has proved the admissible version in [21]; but we shall also derive a quantitative maximal function estimate which he avoided. The main idea in our proof will be to use the fact that $\log^+ |f|$ is not only subharmonic, but in fact plurisubharmonic when f is a holomorphic function in several variables. One then makes a second maximal function estimate, using the already proved non-tangential maximal function estimate. This second maximal function estimate is made with respect to the associated family of boundary balls, which are quite skewed in comparison to the Euclidean family considered in Section 3.

The reader is urged to consult [23] for a summary of the important interplay between the generalized approach regions and boundary balls, but we shall present here a brief summary of the definitions and results found there.

For Ω a smooth domain of finite type, let ρ be a C^∞ defining function for Ω, and let $R(z, w)$ be a *polarization* of ρ, i.e., $R(z, w)$ is a C^∞, complex-valued function on $\mathbf{C}^n \times \mathbf{C}^n$ such that

(4.4.1) $\quad R(z, z) = \rho(z)$,

(4.4.2) $\quad \bar{\partial}_z R(z, w)$ vanishes to infinite order on the diagonal $z = w$,

(4.4.3) $\quad R(z, w) - \overline{R(w, z)}$ vanishes to infinite order on the diagonal $z = w$.

For $w_0 \in \partial\Omega$ let $X_1, \ldots, X_{2n-2}$ be real vector fields defined in a small neighborhood $V = V_{w_0}$ of w_0 such that $L_j = X_j + iX_{n+j}$ $(1 \le j \le n-1)$ form a basis for the complex tangential holomorphic vector fields on $\partial\Omega$. Find also a transverse vector field T such that $X_1, \ldots, X_{2n-2}, T$ span the tangent space to $\partial\Omega$ at every point of $\partial\Omega$ contained in V.

We now define functions $\lambda_{i_1,\ldots,i_l}$ on $V \cap \partial\Omega$ by

$$(4.4.4) \quad [X_{i_l}, [X_{i_{l-1}}, \ldots [X_{i_2}, X_{i_1}] \ldots]] = \lambda_{i_1,\ldots,i_l} T \quad \mathrm{mod}(X_1, \ldots, X_{2n-2}).$$

We let $\mathcal{I}_k$ be the ideal in $C^\infty(V \cap \partial\Omega)$ generated by the $\lambda_{i_1,\ldots,i_l}$ with $l \le k$, and then define

$$(4.4.5) \qquad \Lambda_k(\zeta) = \left(\sum \lambda_{i_1,\ldots,i_l}(\zeta)^2\right)^{1/2}$$

where the sum is over the generators of $\mathcal{I}_k$. Because Ω is compact and of finite type, there will be some integer m such that $\Lambda_m(\zeta) > 0$ for all $\zeta \in \partial\Omega$. So for $\theta > 0$ put

$$(4.4.6) \qquad \Lambda^\theta(\zeta) = \sum_{k=2}^{m} \theta^k \Lambda_k(\zeta).$$

$R(z, w)$ can be used to define a pseudometric and a family of balls on $\partial\Omega$. For z, w in a coordinate neighborhood of $\partial\Omega$ (i.e. $V \cap \partial\Omega$) we say

$$(4.4.7) \qquad d(z, w) < \theta \Leftrightarrow |z - w| < \theta \quad \text{and} \quad |R(z, w)| < \Lambda^\theta(w).$$

Use this "distance" to define balls by

$$(4.4.8) \qquad B_2(z; \theta) = \{w \in V \cap \partial\Omega : \ d(z, w) < \theta\}.$$

For $z \in \overline{\Omega} \cap V$ set

$$(4.4.9) \qquad D(z) = \inf_{2 \le k \le m} (\delta(z)/\Lambda_k(\pi(z)))^{1/k}.$$

(Note that z is not restricted to $\partial\Omega$ here.)

The *generalized approach region* with "aperture" $\alpha > 0$ at $w \in \partial\Omega$ is defined by

$$(4.4.10) \qquad A_\alpha(w) = \{z \in \Omega \cap V : \pi(z) \in B_2(w; \alpha D(z))\}.$$

We embed polydiscs P_ε in these approach regions by defining for $z \in A_\alpha(w)$ and $\varepsilon > 0$ small

$$(4.4.11) \qquad P_\varepsilon(z; w) = \left\{\zeta \in \Omega : \begin{array}{l} |R(\zeta, w) - R(z, w)| < \varepsilon\delta(z) \text{ and} \\ |z_j - \zeta_j| < \varepsilon D(z) \text{ for } 1 \le j \le n-1 \end{array}\right\}$$

where the coordinate system is chosen in such a way that $(\partial R/\partial\zeta_n)(\zeta, w) \ne 0$ for all $\zeta \in V \cap \Omega$ $(V = V_w)$. When the boundary point w is understood we shall just write $P_\varepsilon(z)$.

The relationships (which we need to use) between these approach regions, and their associated boundary balls and embedded polydiscs are given by

PROPOSITION 4.5. *For each $\alpha > 0$, there are positive constants C_1 and C_2 so that for $z \in \Omega$*

$$(4.5.1)\qquad B_2(\pi(z); C_1 D(z)) \subset \{w \in \partial\Omega : z \in A_\alpha(w)\} \subset B_2(\pi(z); C_2 D(z)).$$

Also, for any pair $\mathrm{A} > \alpha > 0$, there exist $\varepsilon = \varepsilon(\alpha, \mathrm{A}) > 0$ such that

$$(4.5.2)\qquad z \in A_\alpha(w) \text{ implies } P_\varepsilon(z) \subset A_{\mathrm{A}}(w).$$

Remark 4.5.3. As a small point of interest, we remark that, for the purposes of the arguments in the rest of this section, 4.5.2 could actually be replaced by the weaker condition that, for any $\alpha > 0$, *there exist* $\mathrm{A} > \alpha$ *and* $\varepsilon > 0$ which satisfy 4.5.2.

At one point in our proof, we would like the pole set of f to be a closed set. This need not be true if f has indeterminate points. However, $\mathcal{P}(f) \cup \mathcal{I}(f)$ is always a closed set. (See 1.13.1–4 and the paragraph that follows these definitions.) In this section then, when we refer to a pole we really mean either a pole or an indeterminate point and $\mathcal{P}(f)$ will stand for all of $\mathcal{P}(f) \cup \mathcal{I}(f)$. The reader should notice that this assumption is only made for the sake of simplicity. Thanks to Theorem 1.1, the quantity $P(f)$ is not changed by this simplification, and all our estimates will hold without it.

Remark 4.5.4. Notice that Proposition 4.5 is easily translated to the situation of admissible approach. In this case we can simply use 4.3 in place of 4.4.10, and replace $D(z)$ by $\delta(z)^{1/2}$ in 4.4.11, the definition of $P_\varepsilon(z)$. The definition of the boundary balls corresponding to the $B_2(z;\theta)$ can be found in [34], p. 33. Thus, since the only non-trivial properties of the approach regions, balls, and polydiscs that we use in the proofs found in the rest of this section are the ones stated in 4.5, we will proceed without making explicit distinctions between the kinds of approach, by just referring to them both as "generalized" approach.

Following the notation of Nagel, Stein and Wainger we define $\mathcal{M}_\alpha$, the *generalized maximal function* by

$$(4.6)\qquad (\mathcal{M}_\alpha h)(\zeta) = \sup\{|h(z)| : z \in A_\alpha(\zeta)\}.$$

M^2 will denote the boundary maximal operator with respect to the generalized boundary balls $B_2(\zeta; r)$ (as opposed to the Euclidean versions M^1 and B_1 introduced in the previous section). $\mathcal{T}_\alpha(z)$ is the *generalized projection*

$$(4.7)\qquad \mathcal{T}_\alpha(z) = \{\zeta \in \partial\Omega : z \in A_\alpha(\zeta)\}.$$

We now fix, for the remainder of this section, an arbitrary $\alpha > 0$, and fix also the associated $\mathrm{A} > \alpha$ and $\varepsilon > 0$ as in 4.5. (If we are willing to take the stronger condition, 4.5.2, in Proposition 4.5, then we could simplify notation by replacing A by 2α everywhere in this section, as was done throughout Section 3.) For $\lambda > 0$ we define the analogies of 3.5.1–4:

$$\text{(4.8.1)} \quad \mathcal{F}_\alpha(\zeta) = (\mathcal{M}_\alpha \log^+ |f|)(\zeta) = \log^+((\mathcal{M}_\alpha f)(\zeta)) \quad (\zeta \in \partial\Omega),$$

$$\text{(4.8.2)} \quad \mathcal{S}_\alpha(\lambda) = \{\zeta \in \partial\Omega : \mathcal{F}_\alpha(\zeta) > \lambda\},$$

$$\text{(4.8.3)} \quad \mathcal{V}_\alpha = \mathcal{T}_{\mathrm{A}}(\mathcal{P}(f)) = \{\zeta \in \partial\Omega : \ \mathcal{A}_{\mathrm{A}}(\zeta) \cap \mathcal{P}(f) \neq \emptyset\},$$

$$\text{(4.8.4)} \quad \mathcal{W}_\alpha(\lambda) = \mathcal{S}_\alpha(\lambda) - \mathcal{V}_\alpha.$$

If one studies the proof of the generalized maximal function estimate in the case of holomorphic functions, it is apparent that the fact that f is globally holomorphic is never used. It is only important that f be holomorphic in an approach region with a "slightly larger" aperture. One then makes use of the fact that $\log|f|$ is plurisubharmonic by integrating the non-tangential estimate we proved in Section 3 over the embedded polydiscs, which results in a maximal function estimate (with respect to the B_2 balls) of the maximal function (with respect to the B_1 balls) of f. The only catch is that since M^1 is not bounded on L^1, $M^1 f$ need not be an integrable function, so we cannot apply a maximal operator to it directly. Instead we apply M^2 to the square root of $M^1 f$, which is integrable since M^1 is weak-type $(1,1)$. (See [3] for a good example of the method used.) Thus if we simply mimic the standard proof (using Proposition 3.7 in the place of the usual non-tangential maximal function estimate for holomorphic functions), we are led to the following:

PROPOSITION 4.9. *Suppose $f \in MN(\Omega)$. Then for all $\zeta \in \partial\Omega - \mathcal{V}_\alpha$*

$$\text{(4.9.1)} \qquad (\mathcal{M}_\alpha \log^+ |f|)(\zeta) \leq C_3[M^2((M^1(\mu+\nu))^{1/2})]^2(\zeta).$$

(Here μ and ν are as in Section 3. Notice that since $M^1(\mu+\nu)$ is a weak-type 1 function, $(M^1(\mu+\nu))^{1/2}$ is an $L^1(\partial\Omega)$ function. In particular, Theorem 1.12(2) guarantees that $M^2((M^1(\mu+\nu))^{1/2})$ is finite almost everywhere. Of course $C_3 = C_3(\alpha, \Omega)$.)

To prove that boundary convergence takes place in the generalized approach regions, our plan of attack is to derive an estimate analogous to Theorem 3.2 and then argue in a manner similar to the proof of Theorem 3.8. The obstacle we have is that we must control the projected area of the pole set $\sigma(\mathcal{V}_\alpha)$.

Recall that in the proof of Proposition 3.6 we were very fortunate to have the projected area of a point $z \in \Omega$ bounded both above and below by a constant times $\delta(z)^{2n-1}$. This is because Theorem 1.7 gives us the exponent $2n-2$, and the extra factor of $\delta(z)$ is provided by the factor of $\delta(z)$ in the definition of the quantity $P(f)$. In the generalized case though we have

$$\text{(4.10)} \qquad B_2(\pi(z); C_4 D(z)) \subset \mathcal{T}_\alpha(\zeta) \subset B_2(\pi(z); C_5 D(z))$$

where C_4 and C_5 depend only on α and Ω. (The "radius function", $D(z)$, in general depends in a rather complicated way on both $\delta(z)$ and the geometry of $\partial\Omega$ near $\pi(z)$, but in the case of the older admissible approach regions we have $D(z) = \delta(z)^{1/2}$.) The radii of the balls $B_2(\pi(z); C_i D(z))$ (in the Euclidean sense of distance on $\partial\Omega$) will be $\sim D(z)$ in the complex tangent (to $\partial\Omega$) directions, and $\sim \delta(z)$ in the remaining tangent direction. Thus we have

$$C_6\delta(z)D(z)^{2n-2} < \sigma(\mathcal{T}_\alpha(z)) < C_7\delta(z)D(z)^{2n-2} \tag{4.11}$$

where as usual C_6 and C_7 depend only on α and Ω.

This indicates that we will not be able to use Theorem 1.7, as we did in the proof of Proposition 3.6, to prove a generalized analog of that proposition, because $D(z)$ is much larger than $\delta(z)$. That is, $\lim_{\delta(z)\to 0} \delta(z)^{-1}D(z) = \infty$. The rest of this section will be dedicated to finding a way to tackle this problem.

The way in which we can embed polydiscs in the approach regions will be important to us. Recalling 4.5, if $\zeta \in \partial\Omega$ and $z \in A_\alpha(\zeta)$ then

$$P_\varepsilon(z) = P_\varepsilon(z;\zeta) \subset A_{\mathrm{A}}(\zeta). \tag{4.12}$$

These polydiscs should remind the reader of the balls $B(p_j; 2A_8\delta(p_j))$ introduced in the proof of Proposition 3.6. So, as a first attempt at a generalized version of this proposition, we might argue as follows. The collection of projected areas $\{\mathcal{T}_{\mathrm{A}}(p)\}_{p\in\mathcal{P}(f)}$ covers $\mathcal{V}_\alpha$. By 4.10 and Theorem 1.11, we can find a disjoint subcollection, $\{\mathcal{T}_{\mathrm{A}}(p_j)\}_{j=1}^{\infty}$, such that

$$\sigma(\mathcal{V}_\alpha) < C_8\sigma\Big(\bigcup_{j=1}^{\infty} \mathcal{T}_{\mathrm{A}}(p_j)\Big) = C_8\sum_{j=1}^{\infty}\sigma(\mathcal{T}_{\mathrm{A}}(p_j)). \tag{4.13}$$

It follows from 4.12 that there is an $\varepsilon > 0$ such that the polydiscs $P_\varepsilon(p_j;\pi(p_j))$ are pairwise disjoint. But here is where we get stuck. A direct application of Theorem 1.7 only tells us that

$$\mathrm{H}_{2n-2}(\mathcal{P}(f) \cap P_\varepsilon(p_j;\pi(p_j))) \geq C_9\delta(p_j)^{2n-2}. \tag{4.14}$$

(Embed a ball of radius $\varepsilon'\delta(p_j)$ in the polydisc P_ε.) So, if we estimate $P(f)$ as in the proof of Proposition 3.6 we get

$$P(f) > C_{10}\sum_{j=1}^{\infty}\delta(p_j)^{2n-1}. \tag{4.15}$$

But this is not good enough to bound $\sigma(\mathcal{V}_\alpha)$ from above, because 4.13 and 4.11 only give the upper bound

$$\sigma(\mathcal{V}_\alpha) < C_{11}\sum_{j=1}^{\infty}\delta(p_j)D(p_j)^{2n-2}. \tag{4.16}$$

It is true that one can iterate Theorem 1.7 to conclude

$$\mathrm{H}_{2n-2}(\mathcal{P}(f)\cap P_\epsilon(p_j;\pi(p_j))) \geq C_{12}\delta(p_j)^2 D(p_j)^{2n-4} \qquad (4.17)$$

but this only improves 4.15 to

$$P(f) > C_{13}\sum_{j=1}^{\infty}\delta(p_j)^3 D(p_j)^{2n-4} \qquad (4.18)$$

and comparing this with 4.16 one sees that it still is not good enough. However, 4.17 is really the best we can hope to do for a general pole of f, as is easily seen by observing the simple example $p = (0,0,\ldots,0)$, $P_\epsilon = D(0; D, D, \ldots, D, \delta)$ and $f(z_1,\ldots,z_n) = z_1^{-1}$.

The way out of this bind is to be more careful about how we pick our sequence of poles $\{p_j\}_{j=1}^{\infty}$. To this end we introduce a concept rather special for our purposes.

DEFINITION 4.19. A pole $p \in \mathcal{P}(f)$ will be called a *distant* pole of f if there is *no other* pole $p' \in \mathcal{P}(f)$ satisfying $\mathcal{T}_{\mathrm{A}}(p) \subset \mathcal{T}_{\mathrm{A}}(p')$ and $\delta(p') > \delta(p)$.

The term distant is perhaps not the best possible. It is not meant to imply that the pole p is necessarily far away from $\partial\Omega$, only that, in a certain sense, at p, the pole set of f does not travel very much in a direction which is orthogonal to $\partial\Omega$.

Let $\mathcal{P}_d(f)$ represent the set of distant poles of f. We claim that the collection $\{\mathcal{T}_{\mathrm{A}}(p)\}_{p\in\mathcal{P}_d(f)}$ still covers all of $\mathcal{V}_\alpha$. To see this, let p_0 be any pole of f. We need to show that there is a *distant* pole $\tilde{p}$ such that $\mathcal{T}_{\mathrm{A}}(p_0) \subset \mathcal{T}_{\mathrm{A}}(\tilde{p})$. Since the boundary sets $\mathcal{T}_{\mathrm{A}}(z)$ are open sets (in the boundary topology), which vary continuously with $z \in \Omega$, the set, $F(p_0)$, consisting of all points $z \in \Omega$ with $\mathcal{T}_{\mathrm{A}}(p_0) \subset \mathcal{T}_{\mathrm{A}}(z)$, is *closed* in Ω. Set $\kappa = \delta(p_0)$. Then, in particular, the set $\mathcal{P}(f)\cap\Omega_\kappa\cap F(p_0)$ is a closed Euclidean set. (Remember that, in this section, "the pole set", $\mathcal{P}(f)$, refers to all of $\mathcal{P}(f)\cup\mathcal{I}(f)$.) Being closed (and bounded, since Ω is bounded), there is a point $\tilde{p}$ that achieves a maximum distance from $\partial\Omega$. Clearly such a $\tilde{p}$ is distant.

Remark 4.19.1. The reader is urged to check that, actually, the fact that Ω is bounded is not absolutely crucial to the previous argument. This is because $P(f)$ is finite, and this automatically guarantees that no poles of f are "too far" from $\partial\Omega$.

Now that we have the above claim, we can select a disjoint subcollection of the $\mathcal{T}_{\mathrm{A}}(p)$ as p ranges over the set of *distant* poles, and thus may assume that all of the poles p_j in 4.13 are distant. To take advantage of this fact though, we need a new theorem, which is in some ways stronger than Theorem 1.7.

THEOREM 4.20. *Suppose that f is a function holomorphic in a neighborhood of the closure of the polydisc $P = D(0; r_1, r_2, \ldots, r_n) \subset \mathbf{C}^n$, and suppose that f has no zeros in the smaller polydisc $P' = D(0; r_1, r_2, \ldots, \beta_1 r_n)$ (where $0 < \beta_1 < 1$); i.e. we assume that*

$$\chi_f \cap P' = \emptyset . \tag{4.20.1}$$

Suppose also that

$$f(0, 0, \ldots, 0, \beta_2 r_n) = 0 \quad (\beta_1 < \beta_2 < 1). \tag{4.20.2}$$

Then

$$\mathrm{H}_{2n-2}(\chi_f \cap P) \geq \pi^{n-1} \frac{\log \beta_2}{\log \beta_1} \Big(\prod_{j=1}^{n-1} r_j^2 \Big) . \tag{4.20.3}$$

For us the important part of 4.20.3 is the factor $\prod_{j=1}^{n-1} r_j^2$. The $\pi^{n-1} \log \beta_2 / \log \beta_1$ term will just be treated as a constant.

Of course it is important to notice that, since polydiscs are convex, we also have

COROLLARY 4.21. *If P, P', β_1, and β_2 are the same as in the statement of Theorem 4.20, if f is now* meromorphic *in a neighborhood of the closure of P, and if we replace 4.20.1 and 4.20.2 respectively by*

$$\mathcal{P}(f) \cap P' = \emptyset , \tag{4.21.1}$$

$$(0, 0, \ldots, 0, \beta_2 r_n) \in \mathcal{P}(f) \quad (\beta_1 < \beta_2 < 1), \tag{4.21.2}$$

then

$$\mathrm{H}_{2n-2}(\mathcal{P}(f) \cap P) \geq \pi^{n-1} \frac{\log \beta_2}{\log \beta_1} \Big(\prod_{j=1}^{n-1} r_j^2 \Big) . \tag{4.21.3}$$

Before we prove Theorem 4.20 let's see what it does for us.

First we make a simplifying assumption about the distance of the pole set of f from $\partial\Omega$. Recall the definition of ε_0 from Section 1 (loosely, this is the thickness of a "smooth tube" around $\partial\Omega$), and set $E = \varepsilon_0/3$. We claim that it is fair to assume that

$$\mathcal{P}(f) \cap \Omega_E = \emptyset . \tag{4.22}$$

To see this, we reason as follows. Theorem 1.7 allows us to conclude that there is a small constant $\tau > 0$ such that *for any f* which is meromorphic on Ω and has a pole in Ω_E, $P(f) > \tau$. But we can make the conclusion of our main estimate (Theorem 4.29) trivial for all f with $P(f) > \tau$, simply by picking our constant, $C = C(\alpha, \Omega)$, so large that $C_\tau > \sigma(\partial\Omega)$.

Now recall from 4.12 that

$$P_\varepsilon(z_1; \pi(z_1)) \cap P_\varepsilon(z_2; \pi(z_2)) = \emptyset \quad \text{whenever} \quad \mathcal{T}_{\mathrm{A}}(z_1) \cap \mathcal{T}_{\mathrm{A}}(z_2) = \emptyset . \tag{4.23}$$

For each $z \in \Omega$ (near $\partial\Omega$) we define the new point $t(z) \in \Omega$ by

$$\pi(t(z)) = \pi(z) \quad \text{and} \quad \delta(t(z)) = (1+\varepsilon/2)\delta(z). \tag{4.24}$$

Now because each p_j is a distant pole of f, it follows from 4.10, the ball condition, 1.9.4, and 4.22, that we can find an ε' with $0 < \varepsilon' < \varepsilon/4$ such that

$$P_{\varepsilon'}(t(p_j); \pi(p_j)) \cap \mathcal{P}(f) = \emptyset \quad \forall j. \tag{4.25}$$

(Notice that this is easily verified directly for the admissible approach regions defined in 4.3.)

Choose coordinates so that $t(p_j) = (0, \dots, 0)$ and $\nu_{\pi(p_j)} = (0, \dots, 0, 1)$, and set $D = D(p_j)$ and $\delta = \delta(p_j)$. We can now apply Corollary 4.21 to the polydisc, $P_j = D(0; \varepsilon' D, \dots, \varepsilon' D, \varepsilon\delta)$, with $P_j' = D(0; \varepsilon' D, \dots, \varepsilon' D, \varepsilon'\delta)$. The conclusion is that

$$\mathrm{H}_{2n-2}(\mathcal{P}(f) \cap P_j) \geq C_{14} D(p_j)^{2n-2}. \tag{4.26}$$

But clearly $P_j \subset P_\varepsilon(t(p_j); \pi(p_j))$, and the latter are pairwise disjoint, so we get the improvement of 4.18 that we need, namely

$$P(f) > C_{15} \sum_{j=1}^{\infty} \delta(p_j) D(p_j)^{2n-2}. \tag{4.27}$$

Combining this with 4.16 gives us the following analog of Proposition 3.6, which is just what we were after.

PROPOSITION 4.28. $\sigma(\mathcal{V}_\alpha) < C_{16} P(f)$.

Combining this with Proposition 4.9 we get

THEOREM 4.29. *Let $\Omega \subset \mathbf{C}^n$ $(n \geq 1)$ be a bounded C^2 domain and fix $\alpha > 0$. There is a constant $C = C(\alpha, \Omega) > 0$ such that for all $f \in MN(\Omega)$ and all $\lambda > 0$*

$$\begin{aligned} \sigma(\{\zeta \in \partial\Omega : (\mathcal{M}_\alpha \log^+ |f|)(\zeta) > \lambda\}) \\ \leq C[P(f) + \lambda^{-1/2}(P(f) + \|f\|_0)^{1/2}]. \end{aligned} \tag{4.29.1}$$

Theorem 4.29 works nicely as a uniform estimate when we allow f to vary, while keeping Ω and α *fixed*; but, in spite of the fact that it looks very similar to Theorem 3.2 (which we used to prove the non-tangential limit theorem, 3.8), we cannot use it in the same way to prove a generalized limit theorem. This is because, in order to satisfy 4.22 (which was used in the proof of 4.29), we had to construct the constant C in 4.29 in such a way that it depends very strongly on the "thinness" of the domain Ω. Thus, we cannot apply 4.29 *uniformly* to the domains Ω_ε' as $\varepsilon \to 0$, which is what we need to do to get a generalized analog of Theorem 3.8. We need a theorem similar to 4.29, but one which works better in the context where we fix f and "vary the domain Ω". That is, in order to get an estimate with a single constant, which works for all the Ω'_ε, we will use the fact that f is defined in the interior of Ω, well outside of Ω'_ε. For that purpose, we state the following

THEOREM 4.30. *Suppose that $\Omega \subset \mathbf{C}^n$ and $\mathrm{A} > \alpha > 0$ are all as before. For $0 < \varepsilon < \varepsilon_0$, we set $D(\varepsilon) = \sup\{D(z) : z \in \Omega_\varepsilon\}$, and let $d > 0$ be any real number so small that $2D(d) + 2d < \varepsilon_0$. As shorthand, write $D = D(d)$. If f is any function in $MN(\Omega)$, we define*

$$\Sigma_d(f) = \mathcal{P}(f) \cap (\Omega - \Omega_d), \tag{4.30.1}$$

$$\Upsilon_d(f) = \mathcal{P}(f) \cap (\Omega - \Omega_{2D+2d}), \tag{4.30.2}$$

$$P_d(f) = -\int_{\Upsilon_d(f)} \delta(z)\gamma_f(z)\, d\mathrm{H}_{2n-2}(z). \tag{4.30.3}$$

Then there is a constant $\mathrm{K} = \mathrm{K}(\Omega, \alpha, \mathrm{A}) > 0$, independent *of d and f, such that*

$$\sigma(\mathcal{T}_{\mathrm{A}}(\Sigma_d(f))) < \mathrm{K} P_d(f). \tag{4.30.4}$$

Proof. For $z \in \Omega - \Omega_{\varepsilon_0}$, we define a family of polydiscs in Ω, and associated "balls" on $\partial\Omega$ parameterized by $b = (b_1, b_2)(0 < b_1 < 1/2, 0 < b_2)$ as follows: For $z \in \Omega - \Omega_d$, i.e. $\delta(z) \le d$,

$$P_b(z) = \left\{ \zeta \in \Omega : \begin{array}{l} |z_n - \zeta_n| < b_1\delta(z) \text{ and} \\ |z_j - \zeta_j| < b_2 D(z) \text{ for } 1 \le j \le n-1 \end{array} \right\}. \tag{4.30.5}$$

For $z \in \Omega_d$, i.e. $\delta(z) > d$,

$$P_b(z) = \left\{ \zeta \in \Omega : \begin{array}{l} |z_n - \zeta_n| < b_1\delta(z) \text{ and} \\ |z_j - \zeta_j| < b_2[D(\pi_d(z)) + \delta(z) - d] \\ \text{for } 1 \le j \le n-1 \end{array} \right\} \tag{4.30.6}$$

(where coordinates are chosen as in 4.4.11),

$$B_b(z) = \pi(P_b(z)). \tag{4.30.7}$$

(Recall that $\pi_d(z)$ is the orthogonal projection of z onto $\partial\Omega_d$.) In a way, these balls "interpolate" between the generalized balls (when z is near $\partial\Omega$) and the ordinary Euclidean balls (when z is far from $\partial\Omega$).

For an appropriate choice of $b = (b_1, b_2)$ (depending on Ω), we have

$$P_b(z) \subset \Omega \quad \forall z \in \Omega. \tag{4.30.8}$$

So fix this b for the rest of the proof.

Moreover, having fixed b, it follows from the definitions 4.4.8 and 4.4.9 of the generalized boundary balls B_2 and the function D that there are constants $\mathrm{K}_1, \mathrm{K}_2 > 0$ (depending on Ω, but not on d) such that

$$B_2(\pi(z); \mathrm{K}_1 D(z)) \subset B_b(z) \subset B_2(\pi(z); \mathrm{K}_2 D(z)) \tag{4.30.9}$$

whenever $z \in \Omega - \Omega_d$. In particular, it follows from Proposition 4.5 that there are constants K_3, $K_4 > 0$ such that

$$K_3\sigma(\mathcal{T}_A(z)) \leq \sigma(B_b(z)) \leq K_4\sigma(\mathcal{T}_A(z)) \quad \forall z \in \Omega - \Omega_d. \tag{4.30.10}$$

Let $S_d(f) \subset \partial\Omega$ be the union of the collection of balls $\{B_b(z) : z \in \Upsilon_d(f)\}$. We claim that to prove 4.30.4, it is enough to show that there is a constants $K' = K'(\Omega, \alpha, A) > 0$, independent of d and f, such that

$$\sigma(S_d(f)) < K' P_d(f). \tag{4.30.11}$$

The justification of this claim follows from the ball properties 1.9.1–5. Notice that from 4.30.9 we have

$$S_d(f) \supset \bigcup_{z \in \Sigma_d(f)} B_2(\pi(z); K_1 D(z)) \tag{4.30.12}$$

and from 4.5.1 we have

$$\bigcup_{z \in \Sigma_d(f)} B_2(\pi(z); C_2 D(z)) \supset \bigcup_{z \in \Sigma_d(f)} \mathcal{T}_A(z) \tag{4.30.13}$$

and neither of the constants K_1 nor C_2 depends on d.

Thus, if $K_1 \geq C_2$, our claim is proved. If not we simply use 1.9 to write

$$\begin{aligned} &\sigma\Big(\bigcup_{z \in \Sigma_d(f)} B_2(\pi(z); C_2 D(z))\Big) \\ &\leq K'' \sum_{i=1}^{\infty} \sigma(B_2(\pi(z_i); C_2 D(z))) \\ &\leq K'' \sum_{i=1}^{\infty} \sigma(B_2(\pi(z_i); K_1 D(z))) \leq K''\sigma\Big(\bigcup_{z \in \Sigma_d(f)} B_2(\pi(z); K_1 D(z))\Big). \end{aligned} \tag{4.30.14}$$

So we now focus on proving 4.30.11.

As before, we pick a disjoint subcollection, $\{B_b(z_j)\}_{j=1}^{\infty}$, but this time we do it carefully. We select z_j so that $B_b(z_j)$ is disjoint from all the $B_b(z_i)$, with $i < j$, *and* $\delta(z_j)$ is as large as possible.

Define $\xi(z)$ by

$$\pi(\xi(z)) = \pi(z) \quad \text{and} \quad \delta(\xi(z)) = (1 + b_1/2)\delta(z). \tag{4.30.15}$$

We call z *essentially distant* (resisting the temptation to call it nearly distant) if *either* $\delta(z) > d + D$, *or* $\delta(z) \leq d + D$ and the polydisc $P_b(\xi(z))$ satisfies condition 4.21.1 , with $\beta_1 = b_1/8$. (Notice that $P_b(\xi(z))$ always satisfies condition 4.21.2 for some $b_1/3 < \beta_2 < b_1/2$, because z itself is a pole of f.)

The polydiscs P_b have the ball property, that there is some large constant $K_5 > 2$, independent of d, such that whenever $P_b(z) \cap P_b(z_0) \neq \emptyset$ and $\delta(z) \geq \delta(z_0)$ then $P_b(z_0) \subset P_{K_5 b}(z)$. It is also not difficult to verify that there is a constant K_6, independent of d, such that whenever $\delta(z) \geq (1+b_1/4)\delta(z_0)$ and $P_{K_5 b}(z) \cap P_{K_5 b}(z_0) \neq \emptyset$ then

$$P_{K_6 b}(z_0) \subset P_{K_6 b}(z). \tag{4.30.16}$$

From this, 1.9.1–5, and the way that we selected the disjoint subcollection, $\{B_b(z_j)\}_{j=1}^{\infty}$, it follows that for each z_j in our subcollection, either z_j is essentially distant, or there is a z_i which is essentially distant, and also has the property that

$$P_{K_6 b}(z_j) \subset P_{K_6 b}(z_i). \tag{4.30.17}$$

To see this, notice that if z_j is not essentially distant, then there is a pole p_j with $\delta(p_j) \geq (1 + b_1/4)\delta(z_j)$ and $p_j \in P_{2b}(z_j)$. From the way that the subcollection was selected, there must be a z_i such that $\delta(z_i) \geq \delta(p_j)$ and $P_b(p_j) \cap P_b(z_i) \neq \emptyset$. As above, we must have $P_b(p_j) \subset P_{K_5 b}(z_i)$, so that

$$p_j \in P_{2b}(z_j) \cap P_{K_5 b}(z_i) \subset P_{K_5 b}(z_j) \cap P_{K_5 b}(z_i). \tag{4.30.18}$$

Hence 4.30.17 follows from 4.30.16.

Thus, if $ED = \{z_{j_1}, z_{j_2}, \ldots\}$ is the collection of z_j which are essentially distant, then

$$\sum_{j=1}^{\infty} \sigma(B_b(z_j)) \leq K_7 \sum_{z_j \in ED} \sigma(B_b(z_j)). \tag{4.30.19}$$

But we can split the sum on the right in 4.30.19 into the sum over those z_j with $\delta(z_j) \leq d + D$ and the sum over those z_j with $\delta(z_j) > d + D$. When $\delta(z_j) \leq d + D$, Corollary 4.21 applies. So if we set $\Lambda(z) = D(z)$ for $z \in \Omega - \Omega_d$, and $\Lambda(z) = D(\pi_d(z)) + \delta(z) - d$ for $z \in \Omega_d$, then

$$\begin{aligned} \sigma(B_b(z_j)) &\leq K_8 \delta(z_j) \Lambda(z_j)^{2n-2} \\ &\leq -K_9 \int_{P_b(z_j) \cap \mathcal{P}(f)} \delta(z) \gamma_f(z)\, dH_{2n-2}(z). \end{aligned} \tag{4.30.20}$$

And when $\delta(z_j) > d + D$, the balls $B_b(z_j)$ have "almost Euclidean" dimensions, so we can apply Corollary 1.8 to an imbedded Euclidean ball, obtaining

$$\begin{aligned} \sigma(B_b(z_j)) &\leq K_{10} \delta(z_j)^{2n-1} \\ &\leq -K_{11} \int_{P_b(z_j) \cap \mathcal{P}(f)} \delta(z) \gamma_f(z)\, dH_{2n-2}(z). \end{aligned} \tag{4.30.21}$$

Combining 4.30.19–21, and summing as usual, gives 4.30.11, which finishes the proof of Theorem 4.30. ■

It is now apparent how to use Theorem 4.30 to prove generalized convergence of f at almost every point of $\partial\Omega$. We do not have a uniform bound on

$\mathcal{T}_A(\mathcal{P}(f)\cap\Omega'_\varepsilon)$ purely in terms of $P(f)$ ($P(f)$ here means with respect to the thin domain Ω'_ε), but we do know that $D(d)\to 0$ as $d\to 0$. Thus, the fact that $P(f)$ is finite tells us that $P_d(f)$ also goes to 0 as d goes to 0. Thus, Theorem 4.30 finally allows us to conclude that the set of points $\zeta\in\partial\Omega$ which are generalized accumulation points of $\mathcal{P}(f)$ is of measure zero. Combining this with Proposition 4.9, and mimicking the proof of Theorem 3.8, we are led to

THEOREM 4.31. *If $f\in MN(\Omega)$ then f has a limit at almost every $\zeta\in\partial\Omega$ in the generalized approach region at ζ.*

The only loose end left to tie up is

4.32. Proof of Theorem 4.20. In order to make the proof at all legible we shall need to set up some notation. (This is only a small modification of standard notation from the theory of functions of one complex variable.) Thus we shall write for $z\in\mathbf{C}^n$, $w\in\mathbf{C}$, $r>0$, $j=1,\ldots,n$:

(4.32.1) $$f_j(w;z)=f(z_1,\ldots,z_{j-1},w,z_{j+1},\ldots,z_n),$$

(4.32.2) $$D_j(r;z)=\{\tilde z\in\mathbf{C}^n:|\tilde z_j|\le r,\ \tilde z_k=z_k \text{ if } k\ne j\},$$

(4.32.3) $$P_j(r_1,\ldots,r_n)=\{\tilde z\in\mathbf{C}^n:\tilde z_j=0,\ |\tilde z_k|\le r_k \text{ if } k\ne j\},$$

(4.32.4) $$\begin{aligned} dV^j_{2n-2}&=(i/2)^{n-1}dz_1d\bar z_1\ldots d\bar z_{j-1}dz_{j+1}\ldots dz_nd\bar z_n\\ &=dx_1dy_1\ldots dy_{j-1}dx_{j+1}\ldots dx_ndy_n,\end{aligned}$$

(4.32.5) $n^j_f(r;z)=$ number of zeros (including multiplicity) of the one variable function, $f(z_1,\ldots,z_{j-1},w,z_{j+1},\ldots,z_n)=f_j(w;z)$, in the (one-dimensional) complex disc $D_j(r;z)$,

(4.32.6) $$N^j_f(r;z)=\int_0^r\frac{n^j_f(s;z)-n^j_f(0;z)}{s}\,ds.$$

Jensen's formula in one variable says that if $n^j_f(0;z)=0$ then

(4.32.7) $$N^j_f(r;z)=\frac{1}{2\pi}\int_0^{2\pi}[\log|f_j(re^{i\theta};z)|-\log|f_j(0;z)|]\,d\theta.$$

Theorem 1.2 and Lemma 1.4 combine to give us

(4.32.8) $$\begin{aligned}\mathrm{H}_{2n-2}(\chi_f\cap P)&=\sum_{j=1}^n\int_{P_j(r_1,\ldots,r_n)}n^j_f(r_j;z)\,dV^j_{2n-2}(z)\\ &\ge\int_{P_n(r_1,\ldots,r_n)}n^n_f(r_n;z)\,dV^n_{2n-2}(z).\end{aligned}$$

The key point is that assumption 4.20.1 guarantees that for all $z = (z_1, \ldots, z_{n-1}, 0) \in P_n(r_1, \ldots, r_n)$

$$(4.32.9) \qquad N_f^n(r_n; z) = \int_{\beta_1 r_n}^{r_n} \frac{n_f^n(s; z)}{s} ds \le \int_{\beta_1 r_n}^{r_n} \frac{n_f^n(r_n; z)}{s} ds$$

$$= n_f^n(r_n; z) \int_{\beta_1 r_n}^{r_n} \frac{ds}{s} = (-\log \beta_1) n_f^n(r_n; z).$$

Combining 4.32.8 and 4.32.9 and invoking 4.32.7 we have

$$(4.32.10) \qquad \mathrm{H}_{2n-2}(\chi_f \cap P)$$

$$\ge (-\log \beta_1)^{-1} \int_{P_n(r_1, \ldots, r_n)} N_f^n(r_n; z)\, dV_{2n-2}^n$$

$$= (-\log \beta_1)^{-1} \int_{P_n(r_1, \ldots, r_n)} \frac{1}{2\pi} \int_0^{2\pi} [\log |f_n(r_n e^{i\theta}; z)| - \log |f_n(0; z)|]\, d\theta\, dV_{2n}^n$$

$$= \frac{(-\log \beta_1)^{-1}}{2\pi} \int_0^{2\pi} d\theta \int_{P_n(r_1, \ldots, r_n)} [\log |f_n(r_n e^{i\theta}; z)| - \log |f_n(0; z)|]\, dV_{2n}^n.$$

(The last equality comes, of course, from Fubini's theorem.)

Assumption 4.20.1 tells us something else, namely that for each fixed $\theta \in [0, 2\pi]$ the function $f_n(r_n e^{i\theta}; z)/f_n(0; z)$ is a holomorphic function of the $n-1$ variables $z_1, \ldots, z_{n-1}$ in the $(n-1)$-dimensional polydisc $P_n(r_1, \ldots, r_n)$. Thus the function

$$\log |f_n(r_n e^{i\theta}; z)/f_n(0; z)| = \log |f_n(r_n e^{i\theta}; z)| - \log |f_n(0; z)|$$

is *plurisubharmonic* in $P_n(r_1, \ldots, r_n)$. It thus follows from the last inequality that

$$(4.32.11) \qquad \mathrm{H}_{2n-2}(\chi_f \cap P)$$

$$\ge \frac{-(\log \beta_1)^{-1}}{2\pi} \int_0^{2\pi} \left(\prod_{j=1}^{n-1} \pi r_j^2\right) [\log |f_n(r_n e^{i\theta}; 0)| - \log |f(0)|]\, d\theta$$

$$= -\frac{\pi^{n-1}}{\log \beta_1} \left(\prod_{j=1}^{n-1} r_j^2\right) N_f^n(r_n; 0).$$

But assumption 4.20.2 says that $n_f^n(s; 0) \ge 1$ for all $s \ge \beta_2 r_n$, hence

$$(4.32.12) \qquad N_f^n(r_n; 0) \ge \int_{\beta_2 r_n}^{r_n} \frac{ds}{s} = -\log \beta_2.$$

This finishes the proof.

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