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LIMIT REDUCED POWERS

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Introduction. In this paper (1) we study a generalization of the notion of reduced power, which we shall call a limit reduced power. A limit reduced power of a relational structure is defined as a substructure of a reduced power consisting of equivalence classes of those functions from a direct power which are "almost constant" with respect to some filter over the cartesian square of the set of indexes. This notion was introduced by Keisler [2]. In Keisler's paper [2], mainly the limit ultrapowers are studied. Some his results can be generalized for limit reduced powers, obviously in a weaker form. In particular, instead of considering properties which are expressed by arbitrary formulas of the language, we shall examine the class of Horn formulas only. (Horn formulas are those which are preserved by reduced products, see also [3]).

1. Preliminaries. Let ϱ be an arbitrary ordinal number and let μ be a sequence of natural numbers with domain ϱ ($\mu \in \omega^{\varrho}$). A sequence $\mathfrak{A} = \langle A, R_\lambda \rangle_{\lambda<\varrho}$ is said to be a *relational structure of type* μ if A is a non-empty set and R_λ is a $\mu(\lambda)$-ary relation on A for each $\lambda < \varrho$.

Let $\varrho' < \varrho$ and $\mu' \in \omega^{\varrho'}$ be an initial segment of μ. By $\mathfrak{A}{\restriction}\mu'$ we denote the structure $\mathfrak{A}' = \langle A, R_\lambda \rangle_{\lambda<\varrho'}$ of type μ'. If $C \subseteq A$, then by $\mathfrak{A}{\restriction}C$ is denoted the only substructure of A with the universum C.

Let A and I be non-empty sets and let $\mathscr{D}$ be a filter over I and $\mathscr{G}$ a filter over $I \times I$. For any function $f \in A^I$ we write

$$eq(f) = \{\langle i, j\rangle \in I \times I : f(i) = f(j)\}.$$

Now we define a *limit reduced power of the set* A as

$$A^I_{\mathscr{D}} | \mathscr{G} = \{a \in A^I_{\mathscr{D}} : \textit{there is an } f \in a \textit{ with } eq(f) \in \mathscr{G}\},$$

where $A^I_{\mathscr{D}}$ is a reduced power of A and a is an equivalence class modulo $\mathscr{D}$.

By a *limit reduced power of structure* $\mathfrak{A}$ we mean the structure $\mathfrak{A}^I_{\mathscr{D}} | \mathscr{G} = \mathfrak{A}^I_{\mathscr{D}} {\restriction} (A^I_{\mathscr{D}} | \mathscr{G})$.

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(1) The results presented here were obtained under direction of Professor C. Ryll-Nardzewski when the authoress was a student at the Wrocław University.

The reduced power operation is a special case of that of the limit reduced power, namely it corresponds to the choice $\mathscr{G} = 2^{I\times I}$.

A structure $\mathfrak{A}$ is isomorphic with the limit reduced power of itself for $\mathscr{G} = \{I\times I\}$, and $\mathfrak{A}$ as a diagonal can be embedded into $\mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$, for proper $\mathscr{D}$ and arbitrary $\mathscr{G}$:

$$\mathfrak{A} \cong \mathfrak{A}^I_{\mathscr{D}}|\{I\times I\} \subseteq \mathfrak{A}^I_{\mathscr{D}}|\mathscr{G} \subseteq \mathfrak{A}^I_{\mathscr{D}}.$$

The following characterization of limit reduced powers was given by Keisler in [2]:

THEOREM 1.1. *A subset $B \subseteq A^I_{\mathscr{D}}$ satisfies $B = A^I_{\mathscr{D}}|\mathscr{G}$ for some filter $\mathscr{G}$ if and only if $B \neq 0$ and for every $f|_{\mathscr{D}}, g|_{\mathscr{D}} \in B$ and every $h \in A^I$ such that $eq(f) \cap eq(g) \subseteq eq(h)$, we have $h|_{\mathscr{D}} \in B$.*

We examine the first-order language $\mathscr{L}(\mu)$ with countably many individual variables $v_1, v_2, \ldots$ and with $\mu(\lambda)$-placed predicate symbols P_λ for all $\lambda < \varrho$. Atomic formulas are all those of the forms: $v_n = v_m$ and $P\ (v_1, \ldots, v_n)$. Basic Horn formulas are atomic formulas and implications of them. We obtain the set $\mathscr{H}$ of all Horn formulas by forming conjunctions and quantifications of basic Horn formulas. The notion of validity of formulas is defined in the usual way.

In our considerations, we shall use the following theorems of Keisler [3]:

THEOREM 1.2. *Let K be an elementary class of similar structures ($K \in \mathbf{EC}_\Delta$). The class K is closed under operation of reduced power if and only if K may be defined using finite disjunctions of Horn sentences.*

THEOREM 1.3. *The two following conditions are equivalent:*

(i) *For any Horn sentence Φ, if Φ is satisfied in structure $\mathfrak{A}$, then Φ is satisfied in $\mathfrak{B}$.*

(ii) *Certain elementary extension of $\mathfrak{B}$ is isomorphic with some reduced power of $\mathfrak{A}$.*

In [3], Keisler proved Theorem 1.3 assuming the Generalized Continuum Hypothesis, but by the result of Galvin [1] this assumption is unnecessary.

2. Properties of limit reduced powers. Let $\mathfrak{A}^I_{\mathscr{D}}|\mathscr{G}$ be a limit reduced power of the structure $\mathfrak{A}$ and let $f_1/_{\mathscr{D}}, f_2/_{\mathscr{D}}, \ldots \in A^I_{\mathscr{D}}|\mathscr{G}$. Let $\boldsymbol{f}$ denote the sequence $\boldsymbol{f} = \langle f_1, f_2, \ldots\rangle$. Let Φ be any Horn formula. We denote by $J_\Phi(\boldsymbol{f})$ the set

$$J_\Phi(\boldsymbol{f}) = \{i \in I : \mathfrak{A} \models \Phi[f_1(i), f_2(i), \ldots]\}.$$

THEOREM 2.1. *If $J_\Phi(\boldsymbol{f}) \in \mathscr{D}$, then*

$$\mathfrak{A}^I_{\mathscr{D}}|\mathscr{G} \models \Phi[f_1/_{\mathscr{D}}, f_2/_{\mathscr{D}}, \ldots]$$

Proof. We argue by induction on the length of the formula Φ. The theorem is true for atomic formulas, and the converse is also, by the definition of limit reduced power. Let $v_1, \ldots, v_n$ be all free variables of Φ. Suppose that Φ is of the form $\Phi_1 \to \Phi_2$, where Φ_1 and Φ_2 are atomic, and let $J_\Phi(f) \in \mathscr{D}$. If $\mathfrak{A}^I_\mathscr{D} | \mathscr{G} \models \Phi_1[f_1/_\mathscr{D}, \ldots, f_n/_\mathscr{D}]$, then $J_{\Phi_1}(f) \in \mathscr{D}$ and $J_\Phi(f) \cap J_{\Phi_1}(f) \in \mathscr{D}$ by the definition of a filter. Because $J_\Phi(f) \cap J_{\Phi_1}(f) \subseteq J_{\Phi_2}(f)$, we have also $J_{\Phi_2}(f) \in \mathscr{D}$ and $\mathfrak{A}^I_\mathscr{D} \mathscr{G} \models \Phi_2[f_1/_\mathscr{D}, \ldots, f_n/_\mathscr{D}]$. Hence the elements $f_1/_\mathscr{D}, \ldots, f_n/_\mathscr{D}$ satisfy Φ in $\mathfrak{A}^I_\mathscr{D} | \mathscr{G}$.

Let $\Phi = \Phi_1 \wedge \Phi_2$ and suppose that for Φ_1 and Φ_2 our theorem holds and that $J_\Phi(f) \in \mathscr{D}$. Obviously we have

$$J_\Phi(f) = \{i \in I : \mathfrak{A} \models \Phi_1[f_1(i), \ldots, f_n(i)] \textit{ and } \mathfrak{A} \models \Phi_2[f_1(i), \ldots, f_n(i)]\},$$

whence $J_\Phi(f) = J_{\Phi_1}(f) \cap J_{\Phi_1}(f)$ and $J_{\Phi_1}(f), J_{\Phi_2}(f) \in \mathscr{D}$. By the inductive assumption, $\mathfrak{A}^I_\mathscr{D} | \mathscr{G} \models \Phi_1[f_1/_\mathscr{D}, \ldots, f_n/_\mathscr{D}]$ and $\mathfrak{A}^I_\mathscr{D} | \mathscr{G} \models \Phi_2[f_1/_\mathscr{D}, \ldots, f_n/_\mathscr{D}]$, that is $\mathfrak{A}^I_\mathscr{D} | \mathscr{G} \models \Phi[f_1/_\mathscr{D}, \ldots, f_n/_\mathscr{D}]$.

Now, let $\Phi = \forall v_m \Psi$, where $1 \leqslant m \leqslant n$ and Ψ satisfies our theorem and $J_\Phi(f) \in \mathscr{D}$. For an arbitrary function f'_m with $f'_m/_\mathscr{D} \in A^I_\mathscr{D} | \mathscr{G}$ we have

$$J_\Phi(f) \subseteq \{i \colon \mathfrak{A} \models \Psi[f_1(i), \ldots, f_m(i), \ldots, f_n(i)]\} = J_\Psi(f') \in \mathscr{D}.$$

Hence, for arbitrary $f'_m/_\mathscr{D} \in A^I_\mathscr{D} | \mathscr{G}$, we have

$$\mathfrak{A}^I_\mathscr{D} | \mathscr{G} \models \Psi[f_1/_\mathscr{D}, \ldots, f'_m/_\mathscr{D}, \ldots, f_n/_\mathscr{D}],$$

and so $\mathfrak{A}^I_\mathscr{D} | \mathscr{G} \models \Phi[f_1/_\mathscr{D}, \ldots\ f_n/_\mathscr{D}]$.

Suppose that $\Phi = \exists v_m \Psi$, where Ψ satisfies our theorem and $J_\Phi(f) \in \mathscr{D}$. It is obvious that

$J_\Phi(f) = \{i$: *there is an element* $a_m(i) \in A$ *such that* $\mathfrak{A} \models \Psi[f_1(i), \ldots$ $\ldots, a_m(i), \ldots, f_n(i)]\}$.

Let $E = eq(f_1) \cap \ldots \cap eq(f_m) \cap \ldots \cap eq(f_n) \in \mathscr{G}$. We construct the function $g \in A^I$ as follows: for every pair $\langle i, j\rangle \in E$ it is possible to set $a_m(i) = a_m(j)$. Then we define $g(i) = a_m(i) = g(j)$. Hence we have $E \subset eq(g)$ and, consequently, $eq(g) \in \mathscr{G}$ and $g_\mathscr{D} \in A^I_\mathscr{D} | \mathscr{G}$. By the definition of g it follows that

$$J_\Phi(f) = \{i : \mathfrak{A} \models \Psi[f_1, (i), \ldots, g(i), \ldots, f_n(i)]\} = J_\Psi(f_g) \in \mathscr{D}.$$

Hence $\mathfrak{A}^I_\mathscr{D} | \mathscr{G} \models \Psi[f_1/_\mathscr{D}, \ldots, g/_\mathscr{D}, \ldots, f_n/_\mathscr{D}]$ and, consequently,

$$\mathfrak{A}^I_\mathscr{D} | \mathscr{G} \models \Phi[f_1/_\mathscr{D}, \ldots, f_n/_\mathscr{D}].$$

This completes the proof.

From Theorem 2.1 follows at once

COROLLARY 2.2. *If Θ is a sentence of the form $\Phi_1 \vee \ldots \vee \Phi_n$, where $\Phi_1, \ldots, \Phi_n$ are Horn sentences, then for any structure $\mathfrak{A}$ we have*

$$\mathfrak{A} \models \Theta \textit{ implies } \mathfrak{A}^I_\mathscr{D} | \mathscr{G} \models \Theta.$$

By Corollary 2.2 and Theorem 1.2 we have

COROLLARY 2.3. *Let $K \in \mathbf{EC}_\Delta$ be a class of similar structures. K is closed under limit reduced powers if and only if K may be defined using finite disjunctions of Horn sentences.*

In particular, if K is defined by one sentence Θ and K is closed under limit reduced powers, then Θ is equivalent to a finite disjunction of Horn sentences. So, if the implication $\mathfrak{A} \models \Theta \Rightarrow \mathfrak{A}^I_{\mathscr{D}} | \mathscr{G} \models \Theta$ is true for every limit reduced power $\mathfrak{A}^I_{\mathscr{D}} | \mathscr{G}$, then $\Theta = \Phi_1 \vee \ldots \vee \Phi_n$, where $\Phi_1, \ldots, \Phi_n$ are Horn sentences. This is the converse of Corollary 2.2.

Definition. Let Φ be a formula with free variables $v_1, \ldots, v_n$. We say that Φ *is satisfiable* in the structure $\mathfrak{A}$, if $\mathfrak{A} \models \exists v_1, \ldots, v_n \Phi$.

Let Σ be a set of formulas. The structure $\mathfrak{A}$ is *Σ-pure in* $\mathfrak{B}$ if every formula $\Phi \in \Sigma$ satisfiable in $\mathfrak{A}$ is satisfiable in $\mathfrak{B}$ (see [5]). We shall examine this notion for $\Sigma = \mathscr{H}$ and we shall use the notation $A \prec B$ for the Horn purity of $\mathfrak{A}$ in $\mathfrak{B}$.

THEOREM 2.4 *For an arbitrary filter $\mathscr{G}$ over $I \times I$ we have* $\mathfrak{A} \prec \mathfrak{A}^I_{\mathscr{D}} | \mathscr{G}$ (2).

Proof. Suppose that a Horn formula Φ with free variables $v_1, \ldots, v_n$ is satisfiable in $\mathfrak{A}$, i.e. there are elements $a_1, \ldots, a_n \in A$ such that $\mathfrak{A} \models \Phi[a_1, \ldots, a_n]$. Let f_k, for $k = 1, \ldots, n$, be a function $f_k\colon I \to A$ such that $f_k(i) = a_k$ for every $i \in I$. Then $f_k/_{\mathscr{D}} \in A^I_{\mathscr{D}} | \{I \times I\} \subseteq A^I_{\mathscr{D}} | \mathscr{G}$ and $J_\Phi(f) = \{i \in I\colon \mathfrak{A} \models \Phi[a_1, \ldots, a_n]\} = I \in \mathscr{D}$. Hence, by Theorem 2.1, we have $\mathfrak{A}^I_{\mathscr{D}} | \mathscr{G} \models \Phi[f_1/_{\mathscr{D}}, \ldots, f_n/_{\mathscr{D}}]$, and thus Φ is satisfiable in $\mathfrak{A}^I_{\mathscr{D}} | \mathscr{G}$, q.e.d.

THEOREM 2.5 *The structure $\mathfrak{A}$ is $\mathscr{H}$-pure in $\mathfrak{B}$ if and only if $\mathfrak{B}$ is isomorphic to an elementary substructure of a reduced power of $\mathfrak{A}$.*

Proof. Let us observe that $\mathfrak{A} \prec \mathfrak{B}$ if and only if every Horn sentence which is true in $\mathfrak{A}$ is also true in $\mathfrak{B}$. By Theorem 1.3, $\mathfrak{A} \prec \mathfrak{B}$ implies that an elementary extension of $\mathfrak{B}$ is isomorphic to a reduced power of $\mathfrak{A}$, hence $\mathfrak{B}$ is isomoprhic to an elementary substructure of a reduced power of $\mathfrak{A}$.

Conversely, if $\mathfrak{B}$ is elementarily embeddable into a reduced power of $\mathfrak{A}$, then, by Theorem 2.4 and properties of elementary extensions, a Horn sentence true in A is also true in B, whence $\mathfrak{A} \prec \mathfrak{B}$.

Definition. A structure $\mathfrak{A}$ is said to be *complete* if for every $n < \omega$ and every $R \subseteq A^n$ there exists a $\lambda < \varrho$ such that $R = R_\lambda$.

We shall write $\mathfrak{A} \prec\!\!\prec \mathfrak{B}$ if there is a complete structure $\mathfrak{A}'$ and a structure $\mathfrak{B}'$, both of type μ', such that $\mathfrak{A}' \upharpoonright \mu = \mathfrak{A}, \mathfrak{B}' \upharpoonright \mu = \mathfrak{B}$ and $\mathfrak{A}' \prec \mathfrak{B}'$.

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(2) But for arbitrary filters $\mathscr{F} \subseteq \mathscr{G}$ over $I \times I$ the relation $\mathfrak{A}^I_{\mathscr{D}} | \mathscr{F} \prec \mathfrak{A}^I_{\mathscr{D}} | \mathscr{G}$ is not necessarily true. An easy counter-example follows from a result of J. Waszkiewicz and B. Węglorz [4].

LEMMA 2.6. (i) $\mathfrak{A} \prec\!\!\prec \mathfrak{A}$;

(ii) *if* $\mathfrak{A} \prec\!\!\prec \mathfrak{B}$ *and* $\mathfrak{B} \prec\!\!\prec \mathfrak{C}$, *then* $\mathfrak{A} \prec\!\!\prec \mathfrak{C}$;

(iii) *if* $\mathfrak{A} \prec\!\!\prec \mathfrak{B}$, *then* $\mathfrak{A} \prec \mathfrak{B}$;

(iv) $\mathfrak{A} \prec\!\!\prec \mathfrak{A}^I_{\mathscr{D}} | \mathscr{G}$;

(v) *if* $\mathfrak{A}$ *is complete and* $\mathfrak{A} \prec \mathfrak{B}$, *then* $\mathfrak{A} \prec\!\!\prec \mathfrak{B}$.

Proof. The conditions (i)-(iii) and (v) follow at once from the definition and (iv) follows from Theorem 2.4 and from the fact that $(\mathfrak{A}'^I_{\mathscr{D}} | \mathscr{G}) \upharpoonright \mu = (\mathfrak{A}' \upharpoonright \mu)^I_{\mathscr{D}} | \mathscr{G}$.

THEOREM 2.7 *A structure* $\mathfrak{B}$ *is isomorphic to a limit reduced power of* $\mathfrak{A}$ *if and only if* $\mathfrak{A} \prec\!\!\prec \mathfrak{B}$.

Proof. If $\mathfrak{B} \cong \mathfrak{A}^I_{\mathscr{D}} | \mathscr{G}$, then by 2.6 (iv) we have $\mathfrak{A} \prec\!\!\prec \mathfrak{B}$.

Suppose now that $\mathfrak{A} \prec\!\!\prec \mathfrak{B}$. Then there are similar structures $\mathfrak{A}' \prec \mathfrak{B}'$ such that $\mathfrak{A}' \upharpoonright \mu = \mathfrak{A}, \mathfrak{B}' \upharpoonright \mu = \mathfrak{B}$, and $\mathfrak{A}'$ is complete. Because of $\mathfrak{A}' \prec \mathfrak{B}'$ and of Theorem 2.4 there exists an isomorphism $\varkappa$ from $\mathfrak{B}$ into $\mathfrak{A}^I_{\mathscr{D}}$. Let $C = \varkappa(B) \subseteq A^I_{\mathscr{D}}$ and let $f/_{\mathscr{D}}, g/_{\mathscr{D}} \in C, h \in A^I$. By Theorem 1.1 it is sufficient to show that $eq(f) \cap eq(g) \subseteq eq(h)$ implies $h/_{\mathscr{D}} \in C$.

Let $eq(f) \cap eq(g) \subseteq eq(h)$. Then we can find a function $k: A \times A \to A$ such that $k(a, b) = c$, whenever $f(i) = a, g(i) = b$ and $h(i) = c$ for some $i \in I$. Let us consider the relation $R \subseteq A^3$ defined as $R(a, b, c) \leftrightarrow$ $\leftrightarrow k(a, b) = c$. By the completeness of $\mathfrak{A}$, $R = R_\lambda$ for some $\lambda < \varrho'$. The Horn sentence

$$(*) \qquad \forall v_1, v_2 \exists v_3 \forall v_4 [P_\lambda(v_1, v_2, v_4) \leftrightarrow v_3 = v_4]$$

holds in $\mathfrak{A}'$, and therefore holds in $\mathfrak{A}'^I_{\mathscr{D}}, \mathfrak{B}'$, and in $\mathfrak{A}'^I_{\mathscr{D}} \upharpoonright C$ as an isomorphic image of $\mathfrak{B}'$. Since

$$\{i \in I : \mathfrak{A}' \models R_\lambda[f(i), g(i), h(i)]\} = I \in \mathscr{D},$$

we have $\mathfrak{A}'^I_{\mathscr{D}} \models R_\lambda[f/_{\mathscr{D}}, g/_{\mathscr{D}}, h/_{\mathscr{D}}]$, and so $h/_{\mathscr{D}} \in C$ by (*). Hence, by Theorem 1.1, $C = A'^I_{\mathscr{D}} | \mathscr{G}$ for some filter $\mathscr{G}$. But then $\mathfrak{B}' \cong \mathfrak{A}'^I_{\mathscr{D}} | \mathscr{G}$ and, consequently, $\mathfrak{B} \cong \mathfrak{A}^I_{\mathscr{D}} | \mathscr{G}$. The proof is thus complete.

COROLLARY 2.8. *For any complete structure* $\mathfrak{A}$, *the only extensions in which* $\mathfrak{A}$ *is* $\mathscr{H}$*-pure are limit reduced powers of* $\mathfrak{A}$.

Proof. It follows at once from Theorem 2.7 and Lemma 2.6 (v).

THEOREM 2.9. *Let* $\{\mathfrak{A}_n\}_{n<\omega}$ *be a sequence of relational structures such that* $\mathfrak{A}_0 = \mathfrak{A}, \mathfrak{A}_{n+1} = \mathfrak{A}^{I_n}_{n\mathscr{D}_n}$ *with* $I_n \neq 0$ *and* $\mathscr{D}_n$ *being a filter over* I_n. *Then the structure* $\mathfrak{B} = \bigcup_{n<\omega} \mathfrak{A}_n$ *is isomorphic to some limit reduced power* $\mathfrak{A}^I_{\mathscr{D}} | \mathscr{G}$.

Proof. By Theorem 2.7 it is sufficient to prove that $\mathfrak{A} \prec\!\!\prec \mathfrak{B}$. Let $\mathfrak{A}'_0$ be a complete structure for which $\mathfrak{A}'_0 \upharpoonright \mu = \mathfrak{A}_0$. For every $n < \omega$ we define $\mathfrak{A}'_{n+1} = \mathfrak{A}'^{I_n}_{n\mathscr{D}_n}$ and $\mathfrak{B}' = \bigcup_{n<\omega} \mathfrak{A}'_n$. Then obviously $\mathfrak{B}' \upharpoonright \mu = \mathfrak{B}$. By Theorem 2.4 we have $\mathfrak{A}'_n \prec \mathfrak{A}'_{n+1}$ for each $n < \omega$. Hence, by the result in [5], we have $\mathfrak{A}'_0 \prec \mathfrak{B}'$, whence $\mathfrak{A} \prec\!\!\prec \mathfrak{B}$, q.e.d.

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