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SUPERIOR ESTIMATION OF CARATHÉODORY'S DIMENSION FOR n-CELL CONVEXITY

BY

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1. Carathéodory's dimension of multiplicative families, c-independence. Let $\mathscr{C}$ be a multiplicative family of subsets of a given set X. Obviously, X belongs to $\mathscr{C}$ as the intersection of the empty family of subsets of X. Sets of the family $\mathscr{C}$ will be called *$\mathscr{C}$-convex*.

For an arbitrary $A \subset X$ we define *$\mathscr{C}$-hull* of A (denoted by $\mathscr{C}\text{-conv}\,A$) as the intersection of all $\mathscr{C}$-convex sets containing the set A. Obviously, $\mathscr{C}\text{-conv}\,A$ is the smallest $\mathscr{C}$-convex set containing A.

By *Carathéodory's dimension* $\operatorname{cim}\mathscr{C}$ of the family $\mathscr{C}$ we mean the smallest integer $k \geqslant -1$ such that, for any $A \subset X$ and $a \in \mathscr{C}\text{-conv}\,A$, there exists a subset $B \subset A$ with $|B| \leqslant k+1$ such that $a \in \mathscr{C}\text{-conv}\,B$. If such an integer k does not exist, then we put $\operatorname{cim}\mathscr{C} = \infty$ (cf. [2], p. 160).

A set $T \subset X$ is called *c-independent* [4] if

$$T = \emptyset \quad \text{or} \quad \mathscr{C}\text{-conv}\,T \neq \bigcup_{a\in T} \mathscr{C}\text{-conv}(T\setminus\{a\}).$$

LEMMA. *Carathéodory's dimension* $\operatorname{cim}\mathscr{C}$ *is equal to the greatest integer k for which there exists a $(k+1)$-element c-independent subset of X; if such an integer k does not exist, then* $\operatorname{cim}\mathscr{C} = \infty$ (see [4]).

2. n-cell convexity. Let X be an arbitrary set and let $\mathscr{X}_n$ be the family of all subsets of X containing at most n elements. Let $C_n\colon \mathscr{X}_n \to 2^X$ be a function. The sets of the form $C_n(F)$, where $|F| \leqslant n$, will be called *n-cells*.

A set $A \subset X$ is said to be *n-cell convex* if $C_n(F) \subset A$ for any $F \subset A$, $|F| \leqslant n$. Let $\mathscr{C}_n$ denote the family of n-cell convex sets. Obviously, the family $\mathscr{C}_n$ is multiplicative.

Examples. Let M be an arbitrary metric space with metric d.

1. The *family of d-convex sets* (the bibliography and properties of d-convexity are presented in [1]) is the family of 2-cell convex sets where

$$C_2(\{a, c\}) = \{b \in M\colon d(a, b)+d(b, c) = d(a, c)\} \quad \text{and} \quad C_2(\emptyset) = \emptyset.$$

2. We define *B_n-convexity* (comp. with B-convexity [5]) as the family of n-cell convex sets, where $C_n(F)$ is the intersection of all balls of the space M containing $F \subset M$, $|F| \leqslant n$.

Among various kinds of generalized convexity (a survey of some of them is given in Section 9 of [2]) there are further examples of n-cell convexity. In many examples, C_n fulfils some natural conditions such as $C_n(G) \subset C_n(H)$ for $G \subset H$, $C_n(G) = \emptyset$ for $|G| < n$, $G \subset C_n(G)$. In this paper those extra conditions are not essential.

For an arbitrary subset G of X we put

$$L_n(G) = G \cup \bigcup\{C_n(F)\colon\ F \subset G \text{ and } |F| \leqslant n\}.$$

Now, we define recurrently a sequence of sets

$$L_n^0(G) = G, \quad L_n^{i+1}(G) = L_n\big(L_n^i(G)\big),\ i = 0, 1, \ldots$$

The sets have the following properties:

(1) $$G = L_n^0(G) \subset L_n^1(G) \subset \ldots,$$

(2) $$L_n^i(G) \subset L_n^i(H) \quad \text{for } G \subset H,\ i = 0, 1, \ldots,$$

(3) $$A = L_n^0(A) = L_n^1(A) = \ldots \quad \text{for } A \in \mathscr{C}_n,$$

(4) $$\mathscr{C}_n\text{-conv}G = \bigcup_{i=0}^{\infty} L_n^i(G).$$

Properties (1)-(3) are obvious.

We show (4). Since $G \subset \mathscr{C}_n$-conv$G$, by (2) and (3) we have

$$L_n^i(G) \subset L_n^i(\mathscr{C}_n\text{-conv}G) = \mathscr{C}_n\text{-conv}G, \quad i = 0, 1, \ldots$$

Hence

$$\bigcup_{i=0}^{\infty} L_n^i(G) \subset \mathscr{C}_n\text{-conv}G.$$

Now, it is sufficient to show that $\bigcup_{i=0}^{\infty} L_n^i(G) \in \mathscr{C}_n$. Let

$$F \subset \bigcup_{i=0}^{\infty} L_n^i(G) \quad \text{and} \quad |F| \leqslant n.$$

By the finiteness of F and by (1) there exists a number j such that $F \subset L_n^j(G)$. Hence

$$C_n(F) \subset L_n(F) \subset L_n\big(L_n^j(G)\big) = L_n^{j+1}(G) \subset \bigcup_{i=0}^{\infty} L_n^i(G).$$

Consequently, the set $\bigcup_{i=0}^{\infty} L_n^i(G)$ is $\mathscr{C}_n$-convex. Therefore (4) holds.

From (4) we infer immediately that for any $G \subset X$ and $x \in \mathscr{C}_n$-conv$G$ the smallest integer $r_G(x)$ exists such that $x \in L_n^{r_G(x)}(G)$. We call $r_G(x)$ the *rank* of x in G. Obviously, $r_G(x) = 0$ iff $x \in G$.

The existence of the finite rank implies that the $\mathscr{C}_n$-hull of any set $G \subset X$ is the union of $\mathscr{C}_n$-hulls of finite subsets of G. Consequently, from generalizations in [4], p. 66, it follows that for n-cell convexity any c-independent set is finite.

3. Inferior estimation of the number of elements in n-cell hulls of c-independent sets.

THEOREM 1. *If T is a c-independent set with respect to a family $\mathscr{C}_n$ of n-cell convex sets and $n \geqslant 2$, then*

$$\frac{n|T|-1}{n-1} \leqslant |\mathscr{C}_n\text{-conv}\,T|.$$

Proof. If $|T| = 0$ or $|T| = 1$, then the theorem is obvious. The case $|T| = \infty$ is impossible as was seen at the end of Section 2.

Let $|T| = t \geqslant 2$. Since T is c-independent, there exists $y_0 \in \mathscr{C}_n\text{-conv}\,T$ such that $y_0 \notin \mathscr{C}_n\text{-conv}(T \setminus \{x\})$ for any $x \in T$. Hence $y_0 \notin T$ and, consequently, $r_T(y_0) \geqslant 1$.

We shall recurrently define a finite number of pairs of disjoint sets W_i and V_i.

Put $W_0 = \emptyset$ and $V_0 = \{y_0\}$. Obviously, W_0 and V_0 are disjoint.

Let the disjoint sets W_j and V_j be defined for $j \geqslant 0$. Among all elements of V_j choose an element y_j of the highest rank $r_T(y_j)$ in T. If $r_T(y_j) \geqslant 1$, then there exists a set $T_j \subset L_n^{r_T(y_j)-1}(T)$, $|T_j| \leqslant n$, such that $y_j \in L_n(T_j)$. Put $W_{j+1} = W_j \cup \{y_j\}$ and $V_{j+1} = (V_j \setminus \{y_j\}) \cup T_j$. Since the rank of any element of T_j is smaller than $r_T(y_j)$ and the rank of any element of W_j is not smaller than $r_T(y_j)$, we have $T_j \cap (W_j \cup \{y_j\}) = \emptyset$. Hence

$$\begin{aligned} W_{j+1} \cap V_{j+1} &= (W_j \cup \{y_j\}) \cap [(V_j \setminus \{y_j\}) \cup T_j] \\ &= [(W_j \cup \{y_j\}) \cap (V_j \setminus \{y_j\})] \cup [(W_j \cup \{y_j\}) \cap T_j] = \emptyset. \end{aligned}$$

Since $y_0 \in \mathscr{C}_n\text{-conv}\,T$ and $y_0 \notin \mathscr{C}_n\text{-conv}(T \setminus \{x\})$ for any $x \in T$, after a finite number of steps s we get $V_s = T$. Therefore $r_T(y_s) = 0$ for any $y_s \in V_s$. Hence the sets W_s and V_s are the last which we define.

Obviously, $|W_i| = i$ for $i = 0, \ldots, s$.

We show recurrently that $|V_i| \leqslant (n-1)i+1$ for $i = 0, \ldots, s$.

For $i = 0$ the inequality holds.

Assume $|V_j| \leqslant (n-1)j+1$ for $0 \leqslant j < s$. Since $y_j \in V_j$, we have

$$|V_{j+1}| \leqslant |V_j \setminus \{y_j\}| + |T_j| \leqslant (n-1)j+n = (n-1)(j+1)+1.$$

Consequently, $|V_i| \leqslant (n-1)i+1$ for $i = 0, \ldots, s$.

In particular, $|V_s| \leqslant (n-1)s+1$. Moreover, $T = V_s$ and $W_s \cap V_s = \emptyset$. From the inequality $|T| \leqslant (n-1)s+1$ we get $(|T|-1)/(n-1) \leqslant s$. Finally,

$$|\mathscr{C}_n\text{-conv}\,T| \geqslant |W_s \cup V_s| = |W_s| + |V_s| = s + |T| \geqslant \frac{|T|-1}{n-1} + |T| = \frac{n|T|-1}{n-1}.$$

Thus the proof is complete.

Since $|\mathscr{C}_n\text{-conv}\,T|$ is an integer or ∞, the estimation in Theorem 1 can be written in the form

$$-\left[-\frac{n|T|-1}{n-1}\right] \leqslant |\mathscr{C}_n\text{-conv}\,T|,$$

where the symbol $[\cdot]$ denotes the integer part. The construction below in the proof of Theorem 2 shows that the last inequality is essential for every natural number n and for every number of elements of T.

4. Superior estimation of $\operatorname{cim}\mathscr{C}_n$**.** In connection with the supposition [3] that Carathéodory's dimension of the family of d-convex sets of an arbitrary metric space M is not greater than $[(|M|-1)/2]$ we give a more general theorem.

THEOREM 2. *Carathéodory's dimension of the family $\mathscr{C}_n$ of n-cell convex subsets of a finite set X is not greater than*

$$\left[\frac{(n-1)(|X|-1)}{n}\right].$$

The estimation cannot be improved in a general case (for any n and $|X|$).

Proof. If $n=1$, then the estimation is obvious.

Let $n \geqslant 2$. From Theorem 1 it follows that if there exists a c-independent set $T \subset X$, then

$$\frac{n|T|-1}{n-1} \leqslant |\mathscr{C}_n\text{-conv}\,T| \leqslant |X|.$$

Therefore

$$|T| \leqslant \frac{(n-1)|X|+1}{n}.$$

Now, from the Lemma we get

$$\operatorname{cim}\mathscr{C}_n \leqslant \frac{(n-1)|X|+1}{n} - 1 = \frac{(n-1)(|X|-1)}{n}.$$

Since Carathéodory's dimension is an integer, we have

$$\operatorname{cim}\mathscr{C}_n \leqslant \left[\frac{(n-1)(|X|-1)}{n}\right].$$

We show that the estimation cannot be improved in a general case.

In the case $|X| \leqslant 1$ or $n=1$ a proper example of an n-cell convexity is easy to construct.

Assume that $n \geqslant 2$ and $2 \leqslant |X| < \infty$. There exist integers $t \geqslant 1$ and r $(0 \leqslant r \leqslant n-1)$ such that $|X| = 2+(t-1)n+r$. Let

$$X = \{a_0^0, a_0^1, \ldots, a_{n-1}^1, \ldots, a_0^{t-1}, \ldots, a_{n-1}^{t-1}, a_0^t, \ldots, a_r^t\} \quad \text{for } t > 1$$

and

$$X = \{a_0^0, a_0^1, \ldots, a_r^1\} \quad \text{for } t = 1.$$

Let

$$C_n(\{a_0^{j-1}, a_1^j, \ldots, a_{n-1}^j\}) = \{a_0^{j-1}, a_0^j, \ldots, a_{n-1}^j\}) \quad \text{for } 0 < j < t$$

and, moreover, if $r > 0$, let

$$C_n(\{a_0^{t-1}, a_1^t, \ldots, a_r^t\}) = \{a_0^{t-1}, a_0^t, \ldots, a_r^t\}.$$

Let $C_n(F) = F$ for any farther $F \subset X$ with $|F| \leqslant n$.

From the above construction it follows that each of the sets

$$\{a_0^0, a_1^1, \ldots, a_{n-1}^1, \ldots, a_1^{t-1}, \ldots, a_{n-1}^{t-1}, a_1^t, \ldots, a_r^t\} \quad \text{for } t > 1$$

and

$$\{a_0^0, a_1^1, \ldots, a^1\} \quad \text{for } t = 1$$

is c-independent. Since we have $(n-1)(t-1)+r+1$ elements in a c-independent set, we infer from the Lemma that Carathéodory's dimension of the constructed n-cell convexity is not greater than $(n-1)(t-1)+r$. From the equality

$$\left[\frac{(n-1)(|X|-1)}{n}\right] = \left[\frac{(n-1)\{1+(t-1)n+r\}}{n}\right] = \left[(t-1)(n-1)+r+\frac{n-r-1}{n}\right] = (t-1)(n-1)+r$$

we infer that Carathéodory's dimension in the example equals

$$\left[\frac{(n-1)(|X|-1)}{n}\right].$$

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