COLLOQUIUM MATHEMATICUM

VOL. XXII 1971 FASC. 2

*ON CONDITIONS UNDER WHICH LOCAL ISOMETRIES ARE MOTIONS**

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A *G-space*, in the terminology of Busemann [1], p. 37, is a metric space which is finitely compact, metrically convex, and which possesses unique local prolongation. A mapping φ of a G-space R on itself is called *locally isometric* if for each point $p \in R$ there is a number $\eta_p > 0$ such that φ maps the spherical neighborhood $S(p, \eta_p)$ (centered at p with radius η_p) isometrically on the spherical neighborhood $S(\varphi(p), \eta_p)$. Recent attention has been given the problem of determining conditions on R which imply that every such local isometry must be an isometry. (Such conditions may be found in Busemann [1], [2], Kirk [4], [6], and Szenthe [7], [8].)

In [4] it is proved that if the local isometry φ has a fixed point $p \in R$, then φ is an isometry (from which it follows that if the motions of R form a transitive group, then every local isometry is an isometry). This approach suggests the seeking of conditions on φ, rather than on R, which imply that φ is an isometry. Thus in [5] we generalized the above theorem by proving that if φ is a local isometry mapping R on itself which has the property that for some point $p \in R$ the sequence $\{\varphi^n(p)\}$ is bounded, then φ must be an isometry. Our purpose in this paper is to note a further generalization of this theorem.

THEOREM. *Let φ be a locally isometric mapping of a G-space R on itself. If for some point $p \in R$ it is the case that $\{\varphi^n(p)\}$ has a subsequence which is bounded, then φ is an isometry.*

Before proving this theorem we list some basic properties of local isometries which will be needed in the proof.

Let R be a given G-space and φ a locally isometric mapping of R on itself. Let $\varrho(p)$ denote the supremum of those numbers ϱ such that if x, y are in the spherical neighborhood $S(p, \varrho)$ and $x \neq y$, then there exists a point $z \neq y$ such that $xy + yz = xz$. Local prolongation in R

* Research supported by National Science Foundation Grant GP 8367.

implies $\varrho(p) > 0$. If $\varrho(p) = \infty$ for some $p \in R$, then R is a "straight space" for which it is known the only local isometries are motions [1]. *Thus we may assume* $0 < \varrho(p) < \infty$ *for all* $p \in R$.

The properties we shall need are the following:

(1) *If φ is one-to-one, then φ is an isometry.*

(2) *If $\varphi(p_1) = \varphi(p_2) = p$, $p_1 \neq p_2$, then $p_1p_2 \geqslant 2\varrho(p)$.*

(3) *The number of points of R which lie over a given point of R is the same for different points of R.*

(4) *If $a, b \in R$ are given, and if $\varphi(\bar{a}) = a$, then there is a point $\bar{b} \in R$ such that $\varphi(\bar{b}) = b$ and $\bar{a}\bar{b} = ab$.*

(5) *$\varrho(p)$ is a continuous function of $p \in R$.*

(6) *For each $a, b \in R$, $\varphi(a)\varphi(b) \leqslant ab$.*

Properties (1)-(3), (5), and (6) are explicitly stated in Busemann [1] while (4) follows from (27.4) and (27.11) of [1]. In this note, as in our previous ones, we make strong use of property (4). Also, we will use the following:

(7) *If some subsequence of $\{\varphi^n(x)\}$ converges to $t \in R$, then some subsequence of $\{\varphi^n(t)\}$ converges to t.*

Proof. Let $\varepsilon > 0$. Select $i > j$ so that $\varphi^i(x)t < \varepsilon/2$, $\varphi^j(x)t < \varepsilon/2$. Then, using (6) and the fact that φ^{i-j} is also a local isometry,

$$\varphi^{i-j}(t)t \leqslant \varphi^i(x)\varphi^{i-j}(t) + \varphi^i(x)t \leqslant \varphi^{i-j}\big(\varphi^j(x)\big)\varphi^{i-j}(t) + \varepsilon/2 \leqslant \varphi^j(x)t + \varepsilon/2 < \varepsilon.$$

Proof of the Theorem. Because R is finitely compact, some subsequence of $\{\varphi^n(p)\}$ converges, so by (7) we may assume p has been chosen so that

$$\lim_{k\to\infty}\varphi^{n_k}(p) = p.$$

Continuity of $\varrho(x)$ in R implies

$$\delta = \inf\{\varrho\big(\varphi^{n_k}(p)\big): k = 1, 2, \ldots\} > 0.$$

For each $q \in R$ let

$$W(q) = \{x \in R: \varphi(x) = \varphi(q), x \neq q\}.$$

Now assume φ is not an isometry. Then, by (1), φ is not one-to-one, so $W(q) \neq \emptyset$, $q \in R$. Define the sequence $\{p_n\}$ as follows: for each integer $n = 1, 2, \ldots$, let p_n be a point of $W\big(\varphi^n(p)\big)$ for which

$$\varphi^n(p)p_n = \inf\{\varphi^n(p)x: x \in W\big(\varphi^n(p)\big)\}.$$

Thus of all the points which are mapped by φ into $\varphi^{n+1}(p)$ and are different from $\varphi^n(p)$, none is nearer to $\varphi^n(p)$ than p_n. Take p_0 to be a point of $W(p)$ nearest p. (The points p_n exist as defined because of finite compactness of R.)

Using (4) let $\overline{p}_n$ be a point of R such that

$$\varphi^n(\overline{p}_n) = p_n \text{ and } p\overline{p}_n = \varphi^n(p)p_n \qquad (n = 1, 2, \ldots),$$

and take $\overline{p}_0 = p_0$.

Let $\varepsilon > 0$, $\varepsilon < pp_0/2$, and choose j so that $p\varphi^j(p) < \varepsilon$. Again using (4) there is a point $z \in R$ such that

$$\varphi(z) = \varphi^{j+1}(p) \quad \text{ and } \quad p_0 z = \varphi(p)\varphi^{j+1}(p).$$

Furthermore,

$$pz \geqslant pp_0 - p_0 z > 2\varepsilon - \varphi(p)\varphi^{j+1}(p) \geqslant 2\varepsilon - p\varphi^j(p) > \varepsilon,$$

so $z \neq \varphi^j(p)$. It follows that z is a candidate for p_j, so

$$\varphi^j(p)p_j \leqslant \varphi^j(p)z \leqslant \varphi^j(p)p + pp_0 + p_0 z < \varepsilon + pp_0 + \varphi(p)\varphi^{j+1}(p) < pp_0 + 2\varepsilon.$$

Since $p\overline{p}_j = \varphi^j(p)p_j$ by definition, we have proved that if $p\varphi^j(p) < \varepsilon$, then $p\overline{p}_j < pp_0 + 2\varepsilon$. In particular, since $p\varphi^{n_i}(p) < \varepsilon$ for i sufficiently large, it follows that the sequence $\{\overline{p}_{n_i}\}_{i=1}^{\infty}$ is bounded. Finite compactness of R implies that there exist distinct integers u, v, say $u = n_i$, $v = n_j$, such that $\overline{p}_u\overline{p}_v < \delta$. Suppose $u > v$; then

$$\varphi^{u+1}(\overline{p}_v) = \varphi^{u-v}\big(\varphi^{v+1}(\overline{p}_v)\big) = \varphi^{u-v}\big(\varphi^{v+1}(p)\big) = \varphi^{u+1}(p),$$

and

$$\varphi^{u+1}(\overline{p}_u) = \varphi^{u+1}(p).$$

Also $\varphi^u(\overline{p}_v) = \varphi^u(p) \neq p_u = \varphi^u(\overline{p}_u)$, so $\overline{p}_v \neq \overline{p}_u$. If $t > u$, then both of these points are mapped by the local isometry φ^t into the point $\varphi^t(p)$. For such t, (2) implies

$$\overline{p}_u\overline{p}_v \geqslant 2\varrho\big(\varphi^t(p)\big).$$

But when $t = n_k$, then $\varrho\big(\varphi^t(p)\big) \geqslant \delta$ and we have a contradiction, completing the proof.

The above theorem gives rise to a pertinent and seemingly difficult question.

QUESTION. If φ is an isometry of a G-space R on itself and if some subsequence of $\{\varphi^n(p)\}$, $p \in R$, is bounded, then is the sequence $\{\varphi^n(p)\}$ bounded? (**P 706**).

An affirmative answer to this question would of course indirectly enable one to conclude that the hypothesis of the theorem just proved implies that of the theorem generalized. In spaces which are not G-spaces, boundedness of such a subsequence does not necessarily imply boundedness of the sequence, and in fact it is possible for the subsequence to converge while the sequence of iterates remains unbounded. (For an example of such an isometry in Hilbert space, see Edelstein [3], Theorem 2.1.)

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Reçu par la Rédaction le 18. 5. 1969