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P^λ-SPACES AND L^λ-SPACES

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In the study of the extension problem in the categories of normed linear spaces and bounded linear operators and of metric spaces and Lipschitz maps, two types of spaces, i.e., a P^λ-space and an L^λ-space, were introduced and studied by Nachbin [11], Goodner [6], Kelley [9], Pełczyński [12], Sobczyk [14], Aronszajn and Panitchpakdi [1], Cipszer and Geher [4], Grunbaum [7], [8].

Definition 1. Let $\lambda \in [1, \infty)$. A normed space Y is said to be a *P^λ-space* if, whenever X is a normed space and A is a closed subspace of X, bounded linear operator f: $A \to Y$ can be extended to a bounded linear operator F: $X \to Y$ such that $\|F\| \leqslant \lambda \|f\|$.

Definition 2. Let $\lambda \in [1, \infty)$. A metric space Y is called an *L^λ-space* if, whenever A and X are metric spaces such that A is a closed subset of X and f: $A \to Y$ is a Lipschitz map, there exists a Lipschitz map F: $X \to Y$ such that $F|A = f$ and $\|F\| \leqslant \lambda \|f\|$. Here $\|\cdot\|$ denotes the *Lipschitz norm*, i.e.,

$$\|g\| = \inf\{K: d_Y(g(x), g(y)) \leqslant K d_X(x, y) \text{ for all } x, y \in X\}.$$

for any Lipschitz map g: $(X, d_X) \to (Y, d_Y)$.

In this note we prove that every P^λ-space is an L^λ-space. At the same time we show that the Banach space c_0 is an L^λ-space for $\lambda = 74$, but it is not a P^λ-space for any $\lambda \geqslant 1$.

Our main result is the following

THEOREM 1. *Let Y be a normed P^λ-space. Then $(Y, \|\cdot\|)$ is an L^λ-space.*

Proof. Let f: $A \to Y$ be a Lipschitz map from a closed subset A of a metric space (X, d). Without loss of generality we may assume that $\|f\| = 1$. We consider Y as a normed linear subspace of a space $m(D)$ of all bounded functions on a set D (with the supremum norm).

Since $m(D) \in L^1$ (see [1], [3] or [4]), there exists a Lipschitz map $g: X \to m(D)$ such that

$$g|A = f \quad \text{and} \quad \|g(x) - g(y)\| \leqslant d(x, y) \text{ for all } x, y \in X.$$

Let $L(X)$ and $L(A)$ denote the linear spaces with the linear bases X and A, respectively. Then $L(A)$ is the linear subspace of $L(X)$ spanned by A.

Given

$$t = \sum_{i=1}^{n} \lambda_i x_i \in L(X),$$

put

$$s(t) = \max\left\{\left\|\sum_{i=1}^{n} \lambda_i g(x_i)\right\|, \sup_{\varphi\in\Phi}\left|\sum_{i=1}^{n} \lambda_i \varphi(x_i)\right|\right\},$$

where

$$\Phi = \{\varphi \in C(X) : \varphi|A = 0 \text{ and } |\varphi(x) - \varphi(y)| \leqslant d(x, y) \text{ for all } x, y \in X\}.$$

Finally, let $E = L(X)/s^{-1}(0)$ and $F = L(A)/s^{-1}(0)$, E being considered in the quotient norm $\tilde{s}$ induced by s. Then

(a) The linear operator $h: F \to Y$ defined by

$$h\left(\left[\sum_{i=1}^{n}\lambda_i a_i\right]\right) = \sum_{i=1}^{n}\lambda_i f(a_i)$$

satisfies $\|h(z)\| \leqslant \tilde{s}(z)$ for every $z \in F$.

(b) F is closed in E.

In fact, if $z \in E\setminus F$, then

$$z = \left[\sum_{i=1}^{n}\lambda_i x_i\right],$$

where $x_1, x_2, \ldots, x_n \in X$ are mutually distinct, $\lambda_1, \lambda_2, \ldots, \lambda_n$ are different from zero and, say, $x_1 \notin A$. For every $y \in F$,

$$y = \left[\sum_{i=1}^{k}\mu_i a_i\right], \quad \text{where } a_1, a_2, \ldots, a_n \in A.$$

We then get

$$\tilde{s}(z-y) = s\left(\sum_{i=1}^{n}\lambda_i x_i - \sum_{i=1}^{k}\mu_i a_i\right) \geqslant \left|\sum_{i=1}^{n}\lambda_i\varphi(x_i) - \sum_{i=1}^{k}\mu_i\varphi(a_i)\right|,$$

where $\varphi(x) = d(x, A\cup\{x_2, x_3, \ldots, x_n\}) \in \Phi$.

Hence

$$\tilde{s}(z-y) \geqslant |\lambda_1| d(x_1, A\cup\{x_2, x_3, \ldots, x_n\}) > 0,$$

which shows that $z \notin \overline{F}$ and completes the proof of (b).

Since Y is a P^λ-space, we infer from (a) and (b) that there exists a bounded linear operator $H\colon E \to Y$ such that $H|A = h$ and

$$\|H(z)\| \leqslant \lambda s(z) \quad \text{for every } z \in E.$$

For every $x \in X$ put $\tilde{F}(x) = H([x])$. We then get for $x, y \in X$

$$\|\tilde{F}(x) - \tilde{F}(y)\| = \|H([x]) - H([y])\| \leqslant \lambda s(x-y)$$
$$= \lambda \max\{\|g(x)-g(y)\|, \sup_{\varphi\in\Phi} |\varphi(x)-\varphi(y)|\}.$$

Since $\|g(x)-g(y)\| \leqslant d(x,y)$ and $|\varphi(x)-\varphi(y)| \leqslant d(x,y)$ for every $\varphi \in \Phi$, we infer that

$$\|\tilde{F}(x) - \tilde{F}(y)\| \leqslant \lambda d(x,y).$$

Thus the theorem is proved.

From the results of Nachbin [11] and Aronszajn and Panitchpakdi [1] we infer that a Banach space Y is a P^1-space if and only if Y is an L^1-space. The situation, however, is different in the case where $\lambda > 1$. We shall see that $c_0 \in L^{74}$ but it is not a P^λ-space for any $\lambda \geqslant 1$.

Lemma 1. *If a Banach space $Y \in L^\lambda$, then $Y^{**} \in P^\lambda$.*

Proof. Let Z be a Banach space containing Y^{**} as a closed subspace. Since $Y \in L^\lambda$, there exists a bounded linear projection p^{**} from Z^{**} onto Y^{**} of norm less than or equal to λ (see [10]). Putting $p = p^{**}|Z$ we get a bounded linear projection from Z onto Y^{**} with norm less than or equal to λ.

A Banach space Y is called *conjugate* if $Y = X^*$ for some Banach space X. It is known (see [5]) that if Y is a conjugate Banach space, then there exists a bounded linear projection p from Y^{**} onto Y of norm 1. Combining this with Theorem 1 and Lemma 1 we get

Proposition 1. *A conjugate Banach space Y is a P^λ-space if and only if Y is an L^λ-space.*

We now show that there exist normed spaces which are in $L^\lambda \setminus P^\lambda$. We need the following definition:

Definition 3. A normed space Y (respectively, a metric space Y) is called a *$P^\lambda(s)$-space* (respectively, an *$L^\lambda(s)$-space*) if it satisfies the requirements of Definition 1 (respectively, of Definition 2) for every separable normed space X (respectively, for every separable metric space X).

The same argument as in the proof of Theorem 1 shows the following

Proposition 2. *If a normed space Y is a $P^\lambda(s)$-space, then $(Y, \|\cdot\|)$ is an $L^\lambda(s)$-space.*

Now let us prove the following

Proposition 3. *Let Y be a separable $L^\lambda(s)$-space. Then Y is an $L^{37\lambda}$-space.*

Proof. Let X be an arbitrary metric space and let $f\colon A \to Y$ be a Lipschitz map from a closed subset A of X into Y. Without loss of generality we may assume that $\|f\| = 1$. By a theorem of Banach and Mazur [2] we may assume that Y is a subset of the space $E = C_{[0,1]}$. A theorem of Lindenstrauss [10] shows that there is a Lipschitz map $g\colon X \to E$ such that $g|A = f$ and $\|g\| \leqslant 37\|f\| = 37$.

Let $F = E \times R^1$ with the max-norm and write

$$h(x) = \big(g(x), d(x, A)\big).$$

Finally, put $Z = h(X) \cup Y \times \{0\} \subset F$. Identifying Y with $Y \times \{0\}$ we easily see that Y is closed in Z. Moreover, since Z is separable, there exists a Lipschitz retraction $R\colon Z \to Y$ of norm λ. Then $\tilde{F} = R \cdot h$ is an extension of f and

$$\|\tilde{F}(x) - \tilde{F}(y)\| \leqslant \lambda \max\{\|g(x) - g(y)\|, |d(x, A) - d(y, A)|\} \leqslant 37 d(x, y)$$

for all $x, y \in X$. This completes the proof.

Remark. The corresponding statement for P^λ-spaces and $P^\lambda(s)$-spaces does not hold. Indeed, if m denotes the Banach space of all bounded scalar-valued sequences with the supremum norm and c_0 is its closed linear subspace of all sequences convergent to zero, then, by Sobczyk [13],

(1) c_0 is a $P^2(s)$-space;

(2) there is no bounded linear projection from m onto c_0.

Combining Sobczyk's results with Propositions 2 and 3 we get

PROPOSITION 4. *c_0 is an L^{74}-space; however, it is not a P^λ-space for any $\lambda \geqslant 1$.*

Remark. Lemma 1 and Proposition 4 show that c_0^{**} is a P^λ-space for some $\lambda \geqslant 1$; however, c_0 is not a P^λ-space for any $\lambda \geqslant 1$. On the other hand, it is well known (see also Theorem 1 and Lemma 1) that if a Banach space Y is a P^λ-space, then so is Y^{**}.

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