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SOME PROBLEMS AND REMARKS ON RELATIVE MULTIPLIERS

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0. Let G be a compact abelian group and Γ its dual. There is a fairly developed theory of multipliers on various function and measure spaces on G. Under a *multiplier* on the space X on G we understand in this paper a function φ on Γ such that $\varphi\hat{x}$ is the Fourier transform of a member of X provided that $x \in X$. (We suppose every time that any member $x \in X$ has a well-defined Fourier transform $\hat{x}$.) In a few classical cases the space $\mathcal{M}(X)$ of multipliers on X is equal to $\hat{M}(\Gamma)$ – the space of Fourier–Stieltjes transforms of finite Borel measures on G. Thus $\mathcal{M}(X) = \hat{M}(\Gamma)$ if $X = L^1(G)$ or $C(G)$ or $L^\infty(G)$ or $M(G)$. For L^p $(1 < p < \infty)$ the multiplier theory is much more complicated but much is known (see [1], Chapter 16, [2], and [7], for example). We are concentrated on *relative multipliers*. By those we mean multipliers on subspaces obtained from X by restricting the carrier of the Fourier transform (the spectrum) of its elements. Thus L_E^1 means integrable functions with spectrum in $E \subset \Gamma$, and L_E^p, C_E, M_E have analogous meaning. To this kind of problems less attention was paid. Interesting results of Meyer [8] about $\mathcal{M}(L_E^1)$ concern mainly the case Γ not discrete.

1. The aim of this paper is to state some simple relations between multiplier spaces and to raise some problems. In the sequel we admit the usual notation $B(E)$ for the quotient space of Fourier–Stieltjes transforms restricted to E. We begin with

THEOREM 1. *The space $\mathcal{M}(M_E)$ equipped with the multiplier norm is the closure of the linear space of functions on E with finite support under pointwise and norm bounded convergence.*

Proof. Let (h_α) be a net of members of $\mathcal{M}(M_E)$ pointwise convergent to some h and let $\mu \in M_E$. If $\|h_\alpha\| < K$, then for every α there is a $\nu_\alpha \in M_E$ such that $\hat{\mu}h_\alpha = \hat{\nu}_\alpha$ and $\|\nu_\alpha\| \leqslant K\|\mu\|$. Hence (ν_α) converges * weak to a measure whose transform is $h\hat{\mu}$. Thus $h \in \mathcal{M}(M_E)$. Conversely, let $h \in \mathcal{M}(M_E)$. Suppose that k_α is an approximative unit in $L^1(G)$ such that k_α's have finite supports. Then $h\hat{k}_\alpha|E \to h$ pointwise and, for every $\mu \in M_E$, $\hat{\mu}\hat{k}_\alpha$ is the Fourier transform of a measure of norm $\leqslant \|\mu\|$. Thus $h\hat{k}_\alpha|E\cdot\hat{\mu} = h\cdot\hat{k}_\alpha\hat{\mu}$ is the Fourier transform of a measure of norm $\leqslant \|\mu\|\cdot\|h\|$. Hence the operator norms $\|h\hat{k}_\alpha|E\|$ are bounded (by $\|h\|$).

THEOREM 2. *Let E be any subset of the discrete abelian group Γ and*

$$J_E = \{f\in L^\infty\colon \hat{f}|E \equiv 0\}.$$

Then

$$\mathcal{M}(L_E^1) = \mathcal{M}(L^\infty/J_E) = \mathcal{M}(C/J_E\cap C) = \mathcal{M}(M_E) \supset B(E). \tag{1}$$

The operator norms in all multiplier spaces occurring in (1) *are equal.*

We prove the theorem by steps.

1° $\mathcal{M}(L_E^1) \subset \mathcal{M}(L^\infty/J_E)$. Let $h\in\mathcal{M}(L_E^1)$, $\|h\|$ the norm of h as operator on L_E^1, $f\in L_E^1$, $g\in L^\infty$, and let $[g]$ be the equivalence class of $g \pmod{J_E}$ equipped with quotient norm. We have

$$\langle\hat{f}h, \hat{g}|E\rangle = \langle\hat{f}, \bar{h}\hat{g}|E\rangle \quad\text{and}\quad |\langle\hat{f}h, \hat{g}|E\rangle| \leqslant \|h\|\cdot\|f\|_1\cdot\|[g]\|.$$

Hence $\bar{h}\hat{g}$ determines a linear functional on L_E^1 with norm $\leqslant \|h\|\cdot\|[g]\|$. Since $L^\infty/J_E = L_E^{1*}$, it follows that $\bar{h}\hat{g}|E = \hat{k}|E$ for some $k\in L^\infty$, so $h\in\mathcal{M}(L^\infty/J_E)$ and the operator norm of h in $\mathcal{M}(L^\infty/J_E)$ does not exceed $\|h\|$.

2° $\mathcal{M}(C/J_E\cap C) \subset \mathcal{M}(M_E)$. This is proved like step 1° because $M_E = (C/J_E\cap C)^*$.

3° $\mathcal{M}(L^\infty/J_E) \subset \mathcal{M}(M_E)$. Let $h\in\mathcal{M}(L^\infty/J_E)$ with norm $\|h\|$, $\varepsilon>0$, and $f\in C(G)$. Then there exists a $g\in L^\infty$ such that $\hat{g}|E = h\hat{f}|E$ and

$$\|g\|_\infty - \varepsilon \leqslant \|[g]\|_{L^\infty/J_E} \leqslant \|h\|\cdot\|[f]\|_{L^\infty/J_E} \leqslant \|h\|\cdot\|[f]\|_{C/J_E\cap C}. \tag{2}$$

Let again (k_α) be an approximative unit in $L^1(G)$. Then $g*k_\alpha\in C(G)$, and since $h\hat{k}_\alpha\hat{f}|E = \hat{k}_\alpha\hat{g}|E$, we have $h\hat{k}_\alpha\in(C/J_E\cap C)$. As $\|g*k_\alpha\|_\infty \leqslant \|g\|_\infty$, by (2) we obtain

$$\|[g*k_\alpha]\|_{C/J_E\cap C} \leqslant \|g\|_\infty \leqslant \|[g]\|_{L^\infty/J_E}+\varepsilon \leqslant \|h\|\cdot\|[f]\|_{C/J_E\cap C}+\varepsilon.$$

This proves that the norms of $h\hat{k}_\alpha$ in $\mathcal{M}(C/J_E\cap C)$ are bounded: $\|hk_\alpha\| \leqslant \|h\|$. By step 2° we may write $h\hat{k}_\alpha\in\mathcal{M}(M_E)$ without increasing the norm. Hence taking into account that $k_\alpha\to 1$ pointwise we infer from Theorem 1 that $h\in\mathcal{M}(M_E)$.

4°(1) $\mathcal{M}(M_E) \subset \mathcal{M}(C/J_E\cap C)$. Let $h\in\mathcal{M}(M_E)$ with norm $\|h\|$ and $f\in C(G)$. We denote by PT_E the linear space of trigonometric polynomials in C_E. Members of PT_E may be viewed as elements of $C/J_E\cap C$. If $\mu\in M_E$, there exists a measure $\nu\in M_E$ such that $\bar{h}\hat{\mu} = \hat{\nu}$. Since $M_E = (C/J_E\cap C)^*$, for any $w\in PT_E$ we have

$$|\langle h\hat{w}, \hat{\mu}\rangle| = |\langle\hat{w}, \hat{\nu}\rangle| \leqslant \|h\|\cdot\|\mu\|\cdot\|w\|_{C/J_E\cap C}.$$

(1) The author is indebted to M. Bożejko for this step.

Let v be the polynomial having $h\hat{w}$ as its Fourier transform. Then

$$(3) \qquad \|v\|_{C/J_E \cap C} = \sup_{\substack{\mu \in M_E \\ \|\mu\| = 1}} |\langle \hat{v}, \hat{\mu} \rangle| \leqslant \|h\| \cdot \|w\|_{C/J_E \cap C}.$$

Let (w_n) be a sequence in PT_E such that $\|w_n - [f]\|_{C/J_E \cap C} \to 0$. Such sequences exist because PT_E is dense in $C/J_E \cap C$, since PT is dense in $C(G)$. If v_n is the polynomial having $h\hat{w}_n$ as its Fourier transform, then, by (3),

$$\|v_n - v_m\|_{C/J_E \cap C} \leqslant \|h\| \cdot \|w_n - w_m\|_{C/J_E \cap C}.$$

Hence the sequence (v_n) is norm convergent to some $[\varphi] \in C/J_E \cap C$ such that $\hat{\varphi}|E = h\hat{f}|E$; thus $h \in \mathcal{M}(C/J_E \cap C)$ and by (3) the operator norm of h in this space does not exceed $\|h\|$.

5° $\mathcal{M}(M_E) \subset \mathcal{M}(L^1_E)$. Let $h \in (M_E)$ with operator norm $\|h\|$ and $f \in L^1_E$. There exists a sequence (w_n) of trigonometric polynomials with spectrum in E norm convergent to f. Let v_n denote the polynomial whose Fourier transform is $h\hat{w}_n$. We have

$$\|v_n\|_1 \leqslant \|h\| \cdot \|w_n\|_1 \leqslant \|h\| (\|f\|_1 + \varepsilon)$$

for n large, whence (v_n) is also convergent in L^1 (via Cauchy condition) to $g \in L^1_E$, say. Of course, $\hat{g}|E = h\hat{f}|E$. Thus $h \in \mathcal{M}(L^1_E)$ and the operator norm of h in this space is $\leqslant \|h\|$.

6° If in (1) we set $E = \Gamma$, the classical result $\mathcal{M}(L^1) = \mathcal{M}(L^\infty) = \mathcal{M}(C) = \mathcal{M}(M) = B(\Gamma)$ appears. The last equality follows trivially from the fact that $B(\Gamma) \simeq M(G)$ has a unit element (the function $1 = \delta_0$). From $\mathcal{M}(M) = B(\Gamma)$ it is now obvious that $\mathcal{M}(M_E) \supset B(E)$. This completes the proof of Theorem 2.

The following chain of equalities, analogous to (1), is well known:

$$(4) \qquad \mathcal{M}(L^1/I_E \cap L^1) = \mathcal{M}(L^\infty_E) = \mathcal{M}(C_E) = \mathcal{M}(M/I_E) = B(E),$$

where I_E denotes the ideal in $M(G)$ consisting of measures in $M(G)$ such that $\hat{\mu} = 0$ on E. The last equality in (4) holds because $M/I_E \simeq B(E)$ has a unit element. $B(E) \subset \mathcal{M}(L^1/I_E \cap L^1)$ follows from the fact that $\mu * f \in L^1$ for any $f \in L^1$ and any μ. Further, we obtain $\mathcal{M}(L^1/I_E \cap L^1) \subset \mathcal{M}(L^\infty_E)$ like step 1° in the proof of Theorem 2, since $L^\infty_E = (L^1/I_E \cap L_1)^*$. In the same way we infer that $\mathcal{M}(C_E) \subset \mathcal{M}(M/I_E)$. Finally, we obtain $\mathcal{M}(L^\infty_E) \subset \mathcal{M}(C_E)$ by setting $f = \operatorname{lim\,unif} w_n$ for any $f \in C_E$, where $w_n \in PT_E$. All these inclusions do not increase the norm.

Thus the multipliers for spaces occurring in (4) are known as far as the knowledge of $B(E)$ reaches. Yet the multipliers in (1) are to a great deal a mystery. We intend to shed some light thereupon.

2. Let us begin with the nearly obvious remark that, for any $h \in \mathcal{M}(L^1_E)$, $\|h\|_{l^\infty(E)} \leqslant \|h\|$ and, for any $h \in B(E)$,

$$(5) \qquad \|h\| \leqslant \|h\|_{B(E)}.$$

If $h \in B(E)$, we call h a *tame multiplier* for L^1_E. Otherwise, it will be called a *wild multiplier*. To obtain examples of wild multipliers we may set $G = \boldsymbol{T}$ and $E = \boldsymbol{Z}^+$. Then $L^1_E = H^1$. Let the infinite set S be a finite union of Hadamard sequences in $\boldsymbol{Z}^+$. Then by the Paley inequality we have, for any $f \in H^1$,

$$\sum_{n \in S} |\hat{f}(n)|^2 \leqslant A \|f\|_1^2.$$

Thus the function 1_S defined in $\boldsymbol{Z}^+$ is a multiplier of H^1. To see that $1_S \notin B(\boldsymbol{Z}^+)$ the shortest argument is to base on the result of Host and Parreau [5] which reads as follows:

The set $K \subset \Gamma$ is said to be of M_0 *type* if

$$\lim_{x \to \infty} \hat{\mu}(x) = 0$$

holds for every measure in M_K. If K is of M_0 type, then every 0-1 function in $B(\Gamma \setminus K)$ is the restriction of an idempotent of $B(\Gamma)$.

This combined with Cohen's idempotent theorem and the F. and M. Riesz theorem stating that every measure in $M_{\boldsymbol{Z}^-}$ is in $L^1(\boldsymbol{T})$ (in other words, $\boldsymbol{Z}^-$ is a "Riesz set") yields the desired result.

A larger class of wild multipliers on H^1 can be obtained by means of recent interesting results of Peller [11].

Instead of $\boldsymbol{Z}^+$ we may set $E = \boldsymbol{Z}^+ \setminus F$, where $F \subset \boldsymbol{Z}^+$ is closed in $\boldsymbol{Z}$ in Bohr topology. We then take for S a subset of E having the same property as above. According to a theorem of Meyer [9], $F \cup \boldsymbol{Z}^-$ is a Riesz set and we can repeat the above argument, thus coming to the result that the function 1_S defined in E is a wild multiplier in L^1_E. As an example of a set in $\boldsymbol{Z}^+$ closed in Bohr topology we can take any Hadamard sequence but also larger sets like that of primes in the progression $8k+3$.

We may also assume that $F \subset \boldsymbol{Z}^+$ is such that $\boldsymbol{Z}^- \cup F$ is a *Rajchman set*, i.e., such that if $\lim \hat{\mu} = 0$ on $\boldsymbol{Z} \setminus (\boldsymbol{Z}^- \cup F)$, then $\lim \hat{\mu} = 0$ on $\boldsymbol{Z}$. Logically, this is a stronger condition than to be an M_0 set. It is not known whether it is essentially stronger but it can be characterized in arithmetic terms as follows [6]:

$\Lambda \subseteq \boldsymbol{Z}$ is a Rajchman set if and only if there does not exist any infinite set $\Theta \subset \boldsymbol{Z}$ such that

$$\Lambda \supset \{\alpha + \sum_{i \in I} \pm \gamma_i \colon I \text{ finite}, \ \gamma_i \in \Theta, \ \gamma_i \text{ different}\}.$$

If we want to have an example of a wild multiplier in L^1_E for a set E such that $|E \cap \boldsymbol{Z}^+| = |E \cap \boldsymbol{Z}^-| = \omega$, we may set

$$E = 2\boldsymbol{Z}^+ \cup (\boldsymbol{Z}^- \setminus 2\boldsymbol{Z}^-) \quad \text{and} \quad h = 1_S,$$

where 1_S is the characteristic function defined on E of an Hadamard set $S \subset 2\boldsymbol{Z}^+$. Then h is a multiplier in L^1_E by the Paley inequality on account of the fact that the sets $2Z$ and $Z\setminus 2Z$ are harmonically separated (this means there exists a μ with $\hat{\mu}|2Z = 1$ and $\hat{\mu}|(Z\setminus 2Z) = 0$). Owing to the same separation argument the set $Z\setminus E$ is a Riesz set. Now, the wildness of h follows from the theorem of Host and Parreau and that of Cohen.

Let E be an element of the "coset ring" of an ordered discrete abelian group Γ. (The *coset ring* is the complementative ring generated by all subgroups of Γ and their cosets.) Then $1_E \in B(\Gamma) = \mathcal{M}(M_\Gamma)$ so that $B(E) = \mathcal{M}(M_E)$. On the other hand, this equality holds trivially for Sidon sets because then $B(E) = l^\infty(E)$. Thus in both cases (i.e., for the "big" infinite sets in the coset ring and for the "small" Sidon sets) M_E has no wild multipliers. Now the problem arises whether there exist other sets having this property.

It is obvious that in all cases just considered we would obtain again a wild multiplier in M_E if we take for h any bounded function tending to 0 at infinity on $E\setminus S$ and not tending to 0 at infinity on S. This remark suggests the following question: Suppose that M_E has wild multipliers. Must it then have wild idempotent multipliers? **(P 1332)**

3. Let us call a set $E \subset Z$ an *L^1C set* if the Fourier series of any function in L^1_E is norm convergent in L^1. By the Banach–Steinhaus theorem this is equivalent to the uniform norm-boundedness of all operators $f \mapsto S_N f$ ($f \in L^1_E$), where $S_N f$ denotes the N-th partial Fourier sum. A set $E \subset Z$ is called a *UC set* [10] if the Fourier series of any function in C_E is norm convergent in $C(G)$. Again, by the Banach–Steinhaus theorem this is equivalent to the uniform norm-boundedness of all operators $f \mapsto S_N f$ ($f \in C_E$). From (1), (4) and (5) we infer that $\mathrm{UC} \Rightarrow L^1\mathrm{C}$. The converse is false as proved by Fournier [3]. Owing to (1) we can characterize L^1C sets in terms of continuous functions. In fact, E is L^1C if and only if for any $f \in C_E$ the series $\sum \hat{f}(n) e^{inx}$ is convergent in the quotient norm $\|\cdot\|_{C/J_E \cap C}$. The last means that for every $\varepsilon > 0$ there exist N and a continuous function g such that $\operatorname{supp}\hat{g} \subset Z\setminus[-N, N]$, $\|g\|_\infty < \varepsilon$ and $\hat{g} = \hat{f}$ on $E\setminus[-N, N]$. Let us add that an L^1C set is an M_0 set. In fact, by (1) the finite sum operators $f \mapsto S_N f$ are norm bounded also if considered as 0-1 multipliers on M_E. This means that for every $\mu \in M_E$ its Fourier series has bounded partial sums, whence $\hat{\mu} \to 0$ owing to [4].

Going a step further we meet subsets of a discrete abelian group Γ such that for a suitable constant C and any finite set $F \subset E$ (whose characteristic function on E will be denoted by 1_F) the projection S_F of L^1_E onto L^1_F defined by

$$S_F f(\cdot) = \sum 1_F \hat{f}(\gamma) \langle \gamma, \cdot \rangle$$

is norm bounded by C. That is to say that E is a Λ_2 set; in other words, that $L^1_E \subset L^2_E$ or $\|f\|_2 \leqslant \mathrm{const}\,\|f\|_1$ for $f \in L^1_E$. This equivalence is known but let us sketch its proof for completeness. We have to do this only in one direction. We may suppose E is countable. Let us fix a well-ordering (γ_n) of E and let r_n denote the n-th Rademacher function. Let us observe that the condition $\|S_F\| \leqslant C$ for all finite sets $F \subset E$ implies unconditional convergence of the Fourier series of any $f \in L_E^1$ and is actually equivalent to it by the Banach–Steinhaus theorem.

We infer from the condition $\|S_F f\|_1 \leqslant C\|f\|_1$ for all finite sets $F \subset E$ and any $f \in L_E^1$ that for every $t \in [0, 1]$ and $N \in \mathbf{Z}^+$

$$\int_G \Big|\sum_1^N r_n(t)\hat{f}(\gamma_n)\langle \gamma_n, x\rangle\Big|\,dx \leqslant 2C\|f\|_1,$$

whence

$$\int_0^1\int_G \Big|\sum_1^N r_n(t)\hat{f}(\gamma_n)\langle \gamma_n, x\rangle\Big|\,dx\,dt \leqslant 2C\|f\|_1$$

and, using Khintchine's inequality,

$$2C\|f\|_1 \geqslant \int_G dx \int_0^1 \Big|\sum_1^N r_n(t)\hat{f}(\gamma_n)\langle \gamma_n, x\rangle\Big|\,dt \geqslant K\Big(\sum_1^N |\hat{f}(\gamma_n)|^2\Big)^{1/2},$$

where K is the Khintchine constant. Thus $f \in L^2$.

There are also other ways to characterize Λ_2 sets. Let us prove

THEOREM 3. *A set $E = (\gamma_n)$ in Γ is a Λ_2 set if and only if one of the following equivalent conditions holds*:

(i) $l^\infty(E) \subset \mathcal{M}(L_E^1)$;

(ii) $\overline{B(E)} \subset \mathcal{M}(L_E^1)$;

(iii) $c_0(E) \subset \mathcal{M}(L_E^1)$;

(iv) *for any $g \in C(G)$ the series $\sum 1_E(\gamma)\hat{g}(\gamma)\langle \gamma, \cdot\rangle$ is unconditionally convergent in the norm of $C(G)/J_E \cap C$*;

(iv′) *for any $g \in L^\infty(G)$ the series in* (iv) *is unconditionally convergent in the norm of $L^\infty(G)/J_E$*.

Since, as was stated just before, unconditional norm convergence of Fourier series of all f in L_E^1 means that E is a Λ_2 set, the necessity and sufficiency of (iv) and (iv′) follow from (1). Condition (i) is obviously necessary. Since (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), we must prove that (iii) is sufficient. To this aim we observe that $c_0(E) \cap \mathcal{M}(L_E^1)$ is closed in $\mathcal{M}(L_E^1)$. Hence, assuming (iii), $c_0(E)$ is closed in $\mathcal{M}(L_E^1)$. Thus the uniform norm $\|\cdot\|_\infty$ and the multiplier norm $\|\cdot\|$ are equivalent for $c_0(E)$. Now, on account of Theorem 1, every $h \in \mathcal{M}(L_E^1)$ is the pointwise limit of a sequence of members of $c_0(E)$ bounded in multiplier norm. But this time this means uniform boundedness. Thus $l^\infty(E) \subset \mathcal{M}(L_E^1)$, whence $\mathcal{M}(L_E^1) = l^\infty(E)$ and $\|h\| \simeq \|h\|_\infty$ in this space. This implies uniform boundedness of all finite sum operators S_F for $f \in L_E^1$ (by taking 0-1 multipliers), which means the Λ_2 condition.

Remark. It is well known that Λ_2 sets can be characterized by the identity

$$L_E^2 = C(G)/J_E \cap C \quad \text{or} \quad L_E^2 = L^\infty(G)/J_E$$

(see [1], pp. 230–231). Thus (iv) and (iv′) mean that unconditional convergence involved there implies these identities. The converse is obvious.

THEOREM 4. *If $c_0 \cap \mathcal{M}(L_E^1) \subset B(E)$, then $\mathcal{M}(L_E^1) = B(E)$.*

Proof. The space $c_0(E) \cap \mathcal{M}(L_E^1)$ is closed in $B(E)$ by (5). Thus the multiplier norm and the $B(E)$ norm are equivalent for $c_0(E) \cap \mathcal{M}(L_E^1)$. In particular, they are equivalent for multipliers with finite support. Hence the result follows from Theorem 1.

Theorem 4 means that if there exist wild multipliers on L_E^1, then there exist such multipliers also in $c_0(E)$.

Let us observe that examples of wild multipliers we gave in Section 2 belong to $\overline{B(E)}$. In fact, their supports lie in sets (called S) being finite unions of Hadamard sets, thus Sidon sets, and by the known theorem of Drury the characteristic function of a Sidon set is in $\overline{B(\Gamma)}$. It would be interesting to have some characterization of those sets E for which $\mathcal{M}(L_E^1)$ has some (wild) multipliers beyond $\overline{B(E)}$.

4. In this last section we prove a general theorem characterizing in some way sets like Sidon sets (where all 0-1 functions are in $\mathcal{M}(C_E)$), Λ_2 sets (where all 0-1 functions are in $\mathcal{M}(L_E^1)$), etc.

THEOREM 5. *Let X be a Banach space and T a countable set whose members x_n fulfil the following conditions:*

(i) *There exist linear forms a_n such that $a_m(x_n) = \delta_{n,m}$ and every $x \in X$ is uniquely determined by the sequence $(a_n(x))_1^\infty$ or, in other words, by the formal series*

$$S(x) \stackrel{\text{def}}{=} \sum_1^\infty a_n(x)\, x_n$$

(we then write $x \sim \sum a_n(x)\, x_n$).

(ii) *For every subset A of T there is a projection*

$$P_A\colon\ x \mapsto \sum_{x_n \in A} a_n(x)\, x_n \sim P_A x \in X.$$

Then all P_A are uniformly bounded in the operator norm. If in addition $\{x_n\}$ is linearly dense in X, then (x_n) is an unconditional basis of X.

Proof. Writing $\varphi_A(x_n) = 1$ if $x_n \in A$ and $\varphi_A(x_n) = 0$ otherwise we have defined a 1-1 correspondence between the class of subsets of x_n and the Cantor group $D = C_2^\omega$. Thus we may identify φ_A with a point in D. By the closed graph theorem all projections referred to in (2) are bounded. For any $C > 0$ let $\Phi_C = \{\varphi_A \colon \|P_A\| \leqslant C\}$. Then

$$D = \bigcup_{m=1}^{\infty} \Phi_m.$$

Let us write $A \dot{-} B$ for the symmetric difference $(A \setminus B) \cup (B \setminus A)$. Then $\varphi_A \dot{-} \varphi_B$ is the group operation in D. The crucial point is that, for any m and n, there exists r such that $\Phi_m \dot{-} \Phi_n \subset \Phi_r$. In fact, let $A \in \Phi_m$ and $B \in \Phi_n$. We observe first that for any $E \in \Phi_m$ we have $\|P_{E^c}\| \leqslant m+1$, which follows from the identity $x = P_E x + P_{E^c} x$. Hence

$$\|P_{A \setminus B}\| = \|P_{B^c} P_A\| \leqslant m(n+1).$$

Equally, $\|P_{B \setminus A}\| \leqslant n(m+1)$. This proves the claim with $r = 2mn+m+n$.

It is easy to prove that Φ_m are measurable sets. Thus the Haar measure $|\Phi_m|$ is positive for $m \geqslant m_0$. Consequently, $\Phi_{m_0} \dot{-} \Phi_{m_0}$ contains an open set in D and is contained in some Φ_r. By compactness argument there is a finite set of points $t_j \in D$; in other words, a finite number of sets $A_j \subset T$ ($\varphi_{A_j} = t_j$), such that

$$D = \bigcup_j [(\Phi_{m_0} \dot{-} \Phi_{m_0}) \dot{-} \{t_j\}].$$

Hence the first part of Theorem 5 is proved taking into account that

$$\sup_{A \in 2^T} \|P_A\| \leqslant 2rs+r+s, \qquad \text{where } s = \max \|P_{A_j}\|.$$

Since, in particular, the norms $\|P_A\|$ are commonly bounded for all finite sets $A \subset T$, the partial sum operators on $S(x)$ $(x \in X)$ are commonly bounded for any fixed ordering of T. Hence the second part of Theorem 5 follows from the first one.

Remark 1. It is obvious that if the norms $\|P_A\|$ are commonly bounded for finite A's, they are commonly bounded for all A's. Hence, on account of the Banach–Steinhaus theorem, we obtain the following converse of Theorem 5:

If $T = (x_n)_1^\infty$ is an unconditional basis for a Banach space X, then the projections P_A into the subspaces of X generated by A are commonly bounded for all $A \in 2^T$.

Remark 2. If we consider the Haar measure on D as probability, we at once deduce from the proof of Theorem 5 that if (i) and (ii) are fulfilled, then either the projection P_A exists for every $A \subset T$ or for almost none.

We do not think that Theorem 5 has any application. It rather expresses a general and very plausible regularity. We can illustrate it, for example, on function spaces with no natural ordering: if Γ is a discrete countable non-ordered abelian group, we may take $X = L_E^p$ $(= L_E^p(\hat{\Gamma}), p > 1, E \subset \Gamma)$. Then we obtain a characterization of "sets of unconditional convergence" (this means that the Fourier series of any $f \in L^p_E$ converges unconditionally) as those for which the projections $P_A f$ ($f \in L^p_E$) exist for all subsets A of E.

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