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*A FIRST COUNTABLE SUPERCOMPACT HAUSDORFF SPACE WITH A CLOSED G_δ NON-SUPERCOMPACT SUBSPACE**

BY

MURRAY G. BELL (WINNIPEG, MANITOBA)

I. INTRODUCTION

A topological space is called ***supercompact*** [9] if it has a closed subbase such that every linked subcollection has a non-empty intersection (every two members have a non-empty intersection). Each supercompact space is compact. A major problem in this area is to arrive at a more intuitive geometric characterization of this concept. Several algebraic characterizations are known, e.g., by graphs [10] and by interval structures [5].

Among the supercompact spaces there are included compact metric spaces ([15] and [6]) and compact tree-like spaces ([5] and [11]). Supercompactness is productive and each Tychonoff space can be embedded in many supercompact extensions [16]. Supercompact spaces are particular examples of spaces with finite compactness number [4].

Among the compact non-supercompact spaces there are included: βX for X non-pseudocompact ([2] and [7]), infinite compact spaces with no non-trivial convergent sequences [7], and any compact space X which contains a dense subspace D of weight less than the cellularity of the growth $X-D$ [3]. The simplest known example in this category, due to van Douwen, is the Alexandroff one-point compactification of the complete Cantor tree (cf. [14]). There is also a consistent example of a countable space no compactification of which is supercompact [12]. All of the foregoing spaces are not even continuous images of neighbourhood retracts of supercompact spaces.

It is unknown whether dyadic spaces (Hausdorff continuous images of some power of 2) are supercompact (**P 1179**). Hausdorff continuous

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images of supercompact Hausdorff spaces are natural generalizations of dyadic spaces. The large body of knowledge on dyadic spaces suggests many similar questions about supercompact spaces. A good discussion of this is contained in [7]. The authors raise the question of whether a closed G_δ-subspace of a supercompact Hausdorff space is supercompact or at least the continuous image of one. Closed G_δ-subspaces of dyadic spaces are known to be dyadic [8]. In this paper we answer this question in the negative, moreover, our counterexample is also first countable.

II. BASIC DEFINITIONS AND NOTATION

Let $\omega = \{0, 1, 2, \ldots\}$ and $n = \{0, 1, \ldots, n-1\}$. If $h: X \to Y$, then $\operatorname{dom} h = X$ and, for a subset A of X, $h \upharpoonright A$ is the restriction of the mapping h to A, ${}^{\omega}2 = \{f: f: \omega \to 2 \text{ and } f \neq \emptyset\}$ and ${}^{\omega}2 = \{f \upharpoonright k: f \in {}^{\omega}2 \text{ and } 1 \leqslant k < \omega\}$.

${}^{\omega}2$ has a natural lexicographic total order defined on it; namely, $f < g$ iff for the least number n such that $f(n) \neq g(n)$ we have $f(n) < g(n)$. The symbol $<$ is also used for the usual total order on ω; however, no confusion should arise since we use f and g for mappings and n and m for natural numbers.

By definition, a collection $\mathscr{S}$ of closed subsets of a topological space X is a *closed subbase* iff for each $x \in X$ and for each closed subset C of X with $x \notin C$ there exists a finite subcollection $\mathscr{F}$ of $\mathscr{S}$ such that $x \notin \bigcup \mathscr{F}$ and $C \subseteq \bigcup \mathscr{F}$. A collection of sets is *binary* if each linked subcollection has a non-empty intersection. Therefore, a space X is *supercompact* if X has a binary closed subbase.

III. A FIRST COUNTABLE SUPERCOMPACT HAUSDORFF SPACE X WITH A CLOSED G_δ NON-SUPERCOMPACT SUBSPACE

To begin, we first product a space K with certain crucial properties.

PROPOSITION. *There exists a first countable supercompact Hausdorff space K with binary closed subbase $\mathscr{S}$ satisfying the following properties: For each $f \in {}^{\omega}2$, there exists $\{S_f, L_f, R_f\} \subseteq \mathscr{S}$ such that*

K0. *S_f is both open and closed in K and both L_f and R_f are closed in K.*

K1. *$S_f \cup L_f \cup R_f = K$ and $S_f \cap (L_f \cup R_f) = \emptyset$.*

K2. *$\{S_f: f \in {}^{\omega}2\}$ is linked. Consequently, since $\mathscr{S}$ is binary, we have $\bigcap \{S_f: f \in {}^{\omega}2\} \neq \emptyset$.*

K3. *$f \leqslant g$ implies $R_f \cap L_g \neq \emptyset$.*

K4. *$f < g$ implies* (a) *$L_f \subseteq L_g$ and $R_g \subseteq R_f$;* (b) *$S_f \cap L_g \neq \emptyset$; and* (c) *$R_f \cap S_g \neq \emptyset$.*

Proof. Let $M = {}^{\omega}2\times\{0,1\}$. For each $f\in{}^{\omega}2$, write

$$A_f = \{(g,0): g\leqslant f\}\cup\{(g,1): g<f\}$$

and

$$B_f = \{(g,0): g>f\}\cup\{(g,1): g\geqslant f\}.$$

Generate a topology on M by using $\{A_f: f\in{}^{\omega}2\}\cup\{B_f: f\in{}^{\omega}2\}$ as a closed subbase. For the original construction of such a space, see [1]. M is a first countable supercompact Hausdorff space with binary closed subbase $\{A_f: f\in{}^{\omega}2\}\cup\{B_f: f\in{}^{\omega}2\}$. Set $K = M\times M$ and give K the product topology. For each $f\in{}^{\omega}2$, put

$$S_f = A_f\times B_f,\qquad L_f = M\times A_f,\qquad\text{and}\qquad R_f = B_f\times M.$$

Write

$$\mathscr{S} = \{S_f: f\in{}^{\omega}2\}\cup\{L_f: f\in{}^{\omega}2\}\cup\{R_f: f\in{}^{\omega}2\}\cup\{M\times B_f: f\in{}^{\omega}2\}\cup$$
$$\cup\{A_f\times M: f\in{}^{\omega}2\}.$$

Then K and $\mathscr{S}$ are as required and the S_f's, L_f's, and R_f's satisfy K0-K4.

Now, take any such space K along with a binary closed subbase $\mathscr{S}$ which contains S_f's, L_f's and R_f's satisfying K0-K4. Assume that $K\in\mathscr{S}$. Write

$$X = [{}^{\omega}2]\cup[{}^{\omega}2\times\{0,1\}]\cup[{}^{\omega}2\times K].$$

1. Definition of the topology on X. Points of ${}^{\omega}2$ are isolated. For each $p\in{}^{\omega}2$, $\{p\}\times K$ is open and closed in X and has the topology of K. Basic open neighbourhoods of $(f,0)$ are of the form

$$U_f^n = \{(f,0)\}\cup\{f\restriction k: k\geqslant n\}\cup\bigcup_{k\geqslant n}(\{f\restriction k\}\times S_f).$$

Basic open neighbourhoods of $(f,1)$ are of the form

$$V_f^n = \big[\{(g,i): g\restriction n = f\restriction n \text{ and } i\in\{0,1\}\}\cup\{p\in{}^{\omega}2: p\restriction n = f\restriction n\}\cup$$
$$\cup\bigcup\{\{p\}\times K: p\restriction n = f\restriction n\}\big] - U_f^1.$$

This is a valid neighbourhood assignment which endows X with a first countable Hausdorff topology. Moreover, as S_f is clopen in K, so also is each U_f^n clopen in X. Furthermore, this topology is compact. To see this, let $\mathcal{O}$ be an open cover of X by basic open sets. Since ${}^{\omega}2\times\{0,1\}$, as a subspace of X, is the Alexandroff double of the Cantor discontinuum, finitely many members of $\mathcal{O}$ suffice to cover it. All that remains to be covered is finitely many points of ${}^{\omega}2$ and finitely many $\{p\}\times K$'s. Since K is compact, we are done.

Let $Y = [{}^{\omega}2]\cup[{}^{\omega}2\times\{0,1\}]$. Y is a closed G_δ-subspace of X. It is known that Y is not a continuous image of a supercompact space [7]. We proceed to show that X is supercompact.

2. Construction of a closed subbase $\mathscr{G}$ for X. For each $p \in {}^{<\omega}2$, put

$$D_p = \{q \in {}^{<\omega}2 \colon q{\restriction}\operatorname{dom}p = p\} \cup \{(f, i)\colon f{\restriction}\operatorname{dom}p = p \text{ and } i \in \{0, 1\}\} \cup \\ \cup \bigcup \{\{q\} \times K\colon q \in {}^{<\omega}2 \text{ and } q{\restriction}\operatorname{dom}p = p\}.$$

CLAIM 1. *For each $p \in {}^{<\omega}2$, D_p is closed in X.*

We have

$$X - D_p = \{q \in {}^{<\omega}2\colon q{\restriction}\operatorname{dom}p \neq p\} \cup \bigcup\{U_f^{\operatorname{dom}p}\colon f{\restriction}\operatorname{dom}p \neq p\} \cup \\ \cup \bigcup\{V_f^{\operatorname{dom}p}\colon f{\restriction}\operatorname{dom}p \neq p\} \cup \bigcup\{\{q\} \times K\colon q \in {}^{<\omega}2 \text{ and } q{\restriction}\operatorname{dom}p \neq p\},$$

which is a union of open sets.

For each $f \in {}^{\omega}2$, put

$$A_f^0 = \{g{\restriction}k\colon g < f,\ 1 \leqslant k < \omega \text{ and } g{\restriction}k \neq f{\restriction}k\} \cup \{(g, 0)\colon g < f\} \cup \\ \cup \{(g, 1)\colon g \leqslant f\} \cup \bigcup\{\{g{\restriction}k\} \times K\colon g < f,\ 1 \leqslant k < \omega \text{ and } g{\restriction}k \neq f{\restriction}k\} \cup \\ \cup \bigcup\{\{f{\restriction}k\} \times L_f\colon 1 \leqslant k < \omega\}$$

and

$$A_f^1 = \{g{\restriction}k\colon g > f,\ 1 \leqslant k < \omega \text{ and } g{\restriction}k \neq f{\restriction}k\} \cup \{(g, 0)\colon g > f\} \cup \\ \cup \{(g, 1)\colon g \geqslant f\} \cup \bigcup\{\{g{\restriction}k\} \times K\colon g > f,\ 1 \leqslant k < \omega \text{ and } g{\restriction}k \neq f{\restriction}k\} \cup \\ \cup \bigcup\{\{f{\restriction}k\} \times R_f\colon 1 \leqslant k < \omega\}.$$

CLAIM 2. *For each $f \in {}^{\omega}2$, both A_f^0 and A_f^1 are closed in X.*

We prove that A_f^0 is closed in X. For each $g > f$, choose $n_g < \omega$ such that $g{\restriction}n_g \neq f{\restriction}n_g$. Then

$$X - A_f^0 = \{g{\restriction}k\colon g \geqslant f \text{ and } 1 \leqslant k < \omega\} \cup \bigcup_{g>f} U_g^{n_g} \cup U_f^1 \cup \\ \cup \bigcup_{g>f} V_g^{n_g} \cup \bigcup\{\{g{\restriction}k\} \times K\colon g > f,\ 1 \leqslant k < \omega \text{ and } g{\restriction}k \neq f{\restriction}k\} \cup \\ \cup \bigcup\{\{f{\restriction}k\} \times (K - L_f)\colon 1 \leqslant k < \omega\},$$

which is open. Using K1 we get $S_f \cap L_f = \emptyset$, and, by K0, L_f is closed in K.

Similarly, to prove that A_f^1 is closed, we observe that $S_f \cap R_f = \emptyset$ and that R_f is closed in K.

CLAIM 3. *For each $f \in {}^{\omega}2$, $A_f^0 \cup A_f^1 = X - U_f^1$.*

Indeed, by K1, we get $S_f \cup (L_f \cup R_f) = K$.

CLAIM 4. *$f \leqslant g$ implies $A_f^0 \subseteq A_g^0$ and $A_g^1 \subseteq A_f^1$.*

Here we use K4(a): $f < g$ implies $L_f \subseteq L_g$ and $R_g \subseteq R_f$.

Now, set

$$\mathscr{A} = \{A_f^0\colon f \in {}^{\omega}2\}, \qquad \mathscr{B} = \{A_f^1\colon f \in {}^{\omega}2\},$$

$$\mathscr{C} = \{U_f^n\colon f \in {}^{\omega}2 \text{ and } 1 \leqslant n < \omega\}, \qquad \mathscr{D} = \{D_p\colon p \in {}^{<\omega}2\},$$

$$\mathscr{E} = \{\{p\} \times S\colon p \in {}^{<\omega}2 \text{ and } S \in \mathscr{S}\} \quad \text{and} \quad \mathscr{F} = \{\{p\}\colon p \in {}^{<\omega}2\}.$$

Finally, let

$$\mathscr{G} = \mathscr{A} \cup \mathscr{B} \cup \mathscr{C} \cup \mathscr{D} \cup \mathscr{E} \cup \mathscr{F}.$$

To show that $\mathscr{G}$ is a closed subbase for X we consider several cases. To begin, let $x \in X$ and let C be a closed subspace of X with $x \notin C$. For each case, we must find a finite subset $\mathscr{G}'$ of $\mathscr{G}$ with $x \notin \bigcup \mathscr{G}'$ and $C \subseteq \bigcup \mathscr{G}'$.

Case (i). $x \in {}^{\omega}2$.

Choose $f \in {}^{\omega}2$ such that $f \restriction \operatorname{dom} x = x$. Then

$$C \subseteq X - \{x\} = A_f^0 \cup A_f^1 \cup U_f^{\operatorname{dom} x+1} \cup \{f \restriction k : k < \operatorname{dom} x\} \cup \\ \cup \bigcup \{f \restriction k \times S_f : k \leqslant \operatorname{dom} x\}.$$

Case (ii). $x \in {}^{\omega}2 \times \{0\}$.

Since $x \notin C$, there exists a neighbourhood U_x^n of x such that $U_x^n \cap C = \emptyset$. Therefore,

$$C \subseteq X - U_x^n = A_x^0 \cup A_x^1 \cup \{x \restriction k : k < n\} \cup \bigcup \{\{x \restriction k\} \times S_x : k < n\}.$$

Case (iii). $x \in {}^{\omega}2 \times \{1\}$.

Since $x \notin C$, there exists a neighbourhood V_x^n of x such that $V_x^n \cap C = \emptyset$. Therefore,

$$C \subseteq X - V_x^n = \bigcup \{D_p : \operatorname{dom} p = n \text{ and } p \neq x \restriction n\} \cup \{p : \operatorname{dom} p < n\} \cup \\ \cup \bigcup \{\{p\} \times K : \operatorname{dom} p < n\} \cup U_x^1.$$

Case (iv). $x \in {}^{\omega}2 \times K$.

Let $x = (p, k)$. Since $C \cap (\{p\} \times K)$ is a closed subspace of $\{p\} \times K$ and $\mathscr{S}$ is a closed subbase for K, there exists a finite $\mathscr{S}' \subseteq \mathscr{S}$ such that

$$C \cap (\{p\} \times K) \subseteq \bigcup \{\{p\} \times S : S \in \mathscr{S}'\} \quad \text{and} \quad x \notin \bigcup \{\{p\} \times S : S \in \mathscr{S}'\}.$$

Therefore

$$\begin{aligned} C &\subseteq [C \cap (\{p\} \times K)] \cup [X - (\{p\} \times K)] \\ &\subseteq \bigcup \{\{p\} \times S : S \in \mathscr{S}'\} \cup \bigcup \{D_q : \operatorname{dom} q = \operatorname{dom} p + 1\} \cup \\ &\quad \cup \bigcup \{\{q\} \times K : \operatorname{dom} q \leqslant \operatorname{dom} p \text{ and } q \neq p\} \cup \{q \in {}^{\omega}2 : \operatorname{dom} q \leqslant \operatorname{dom} p\}. \end{aligned}$$

Note that x is not an element of the right-hand side of this inclusion.

3. Linkage conditions. At this point we list conditions that are implied by pairs of members of $\mathscr{G}$ being linked. These conditions follow directly from the definitions. The reader is encouraged to check each one as this is the backbone of our argument.

LC1. $\{A_f^0, A_g^0\}$ is always linked. Moreover, if $f \leqslant g$, then $A_f^0 \subseteq A_g^0$.

LC2. $\{A_f^0, A_g^1\}$ linked implies $g \leqslant f$.

LC3. $\{A_f^0, U_g^n\}$ linked implies $g < f$.

LC4. $\{A^0_f, D_p\}$ linked implies there exists a $g \leqslant f$ such that $g \upharpoonright \mathrm{dom}\, p = p$.

LC5. $\{A^0_f, \{p\} \times S\}$ linked implies either $f \upharpoonright \mathrm{dom}\, p = p$, in which case $S \cap L_f \neq \emptyset$, or $f \upharpoonright \mathrm{dom}\, p \neq p$, in which case there exists a $g < f$ such that $g \upharpoonright \mathrm{dom}\, p = p$. In the latter case, $\{p\} \times K \subseteq A^0_f$.

LC6. $\{A^1_f, A^1_g\}$ is always linked. Moreover, if $f \leqslant g$, then $A^1_g \subseteq A^1_f$.

LC7. $\{A^1_f, U^n_g\}$ linked implies $g > f$.

LC8. $\{A^1_f, D_p\}$ linked implies there exists a $g \geqslant f$ such that $g \upharpoonright \mathrm{dom}\, p = p$.

LC9. $\{A^1_f, \{p\} \times S\}$ linked implies either $f \upharpoonright \mathrm{dom}\, p = p$, in which case $S \cap R_f \neq \emptyset$, or $f \upharpoonright \mathrm{dom}\, p \neq p$, in which case there exists a $g > f$ such that $g \upharpoonright \mathrm{dom}\, p = p$. In the latter case, $\{p\} \times K \subseteq A^1_f$.

LC10. $\{U^n_f, U^m_g\}$ linked implies $f \upharpoonright \max\{n, m\} = g \upharpoonright \max\{n, m\}$.

LC11. $\{U^n_f, D_p\}$ linked implies $f \upharpoonright \mathrm{dom}\, p = p$.

LC12. $\{U^n_f, \{p\} \times S\}$ linked implies $\mathrm{dom}\, p \geqslant n$, $f \upharpoonright \mathrm{dom}\, p = p$, and $S \cap S_f \neq \emptyset$.

LC13. $\{D_p, D_q\}$ linked implies either $p \upharpoonright \mathrm{dom}\, q = q$, in which case $D_p \subseteq D_q$, or $q \upharpoonright \mathrm{dom}\, p = p$, in which case $D_q \subseteq D_p$.

LC14. $\{D_p, \{q\} \times S\}$ linked implies $q \upharpoonright \mathrm{dom}\, p = p$, in which case $\{q\} \times S \subseteq D_p$.

LC15. $\{\{p\} \times S, \{q\} \times T\}$ linked implies $p = q$ and $S \cap T \neq \emptyset$.

4. $\mathscr{G}$ is binary. Since X is compact, it suffices to show that each non-empty finite linked subcollection $\mathscr{G}'$ of $\mathscr{G}$ has a non-empty intersection. Let $\mathscr{G}'$ be such a collection. If $\mathscr{G}' \cap \mathscr{F} \neq \emptyset$, then $\bigcap \mathscr{G}'$ is clearly non-empty. So, assume that $\mathscr{G}' \cap \mathscr{F} = \emptyset$.

Case 1. $\mathscr{G}' \cap \mathscr{E} \neq \emptyset$.

By LC15, we conclude that there exist $p \in {}^{\omega}2$ and a non-empty finite linked subcollection $\mathscr{S}' \subseteq \mathscr{S}$ such that

$$\mathscr{G}' \cap \mathscr{E} = \{\{p\} \times S : S \in \mathscr{S}'\}.$$

Since we are trying to show that $\bigcap \mathscr{G}' \neq \emptyset$, we may assume that no one member of $\mathscr{G}'$ is contained in another member. This implies, by LC14, LC1 and LC6, that $\mathscr{G}' \cap \mathscr{D} = \emptyset$ and that $\mathscr{G}'$ contains at most one A^0_g and at most one A^1_h. It also implies, by LC5 and LC9, that $g \upharpoonright \mathrm{dom}\, p = p$ and $\mathscr{S}' \cup \{L_g\}$ is linked, and $h \upharpoonright \mathrm{dom}\, p = p$ and $\mathscr{S}' \cup \{R_h\}$ is linked. If both A^0_g and A^1_h are present, then LC2 implies $h \leqslant g$. In this case, K3 implies $L_g \cap R_h \neq \emptyset$. Hence $\mathscr{S}' \cup \{R_h, L_g\}$ is linked. Furthermore, there exist a finite (possibly empty) $F \subseteq {}^{\omega}2$ and, for each $f \in F$, an $n_f < \omega$ such that

$$\mathscr{G}' \cap \mathscr{C} = \{U^{n_f}_f : f \in F\}.$$

From LC12 and K2 it follows that $\mathscr{S}'\cup\{S_f: f\in F\}$ is linked, $\operatorname{dom} p \geqslant \max\{n_f: f\in F\}$ and, for each $f\in F$, $f\upharpoonright \operatorname{dom} p = p$. LC3 implies $\max\{f: f\in F\} < g$, whence from K4(b) we conclude that $\{S_f: f\in F\}\cup\{L_g\}$ is linked. LC7 implies that $h<\min\{f: f\in F\}$, whence from K4 (c) we conclude that $\{S_f: f\in F\}\cup\{R_h\}$ is linked. Hence $\mathscr{S}'\cup\{L_g, R_h\}\cup\{S_f: f\in F\}$ is linked. Since $\mathscr{S}$ is binary, choose

$$x\in\bigcap\mathscr{S}'\cap L_g\cap R_h\cap\bigcap_{f\in F} S_f.$$

Then

$$(p,x)\in\bigcap\{\{p\}\times S: S\in\mathscr{S}'\}\cap A^0_g\cap A^1_h\cap\bigcap_{f\in F} U^{n_f}_f.$$

Case 2. $\mathscr{G}'\cap(\mathscr{E}\cup\mathscr{F})=\emptyset$ and $\mathscr{G}'\cap\mathscr{C}\neq\emptyset$.

By LC10, we conclude that there exist a non-empty finite subset $F\subseteq{}^{\omega}2$ and, for each $f\in F$, an $n_f<\omega$ such that $\mathscr{G}'\cap\mathscr{C}=\{U^{n_f}_f: f\in F\}$ and, for each $f_1\neq f_2$ in F,

$$f_1\upharpoonright\max\{n_f: f\in F\} = f_2\upharpoonright\max\{n_f: f\in F\}.$$

Again, we may assume, by LC1, LC6 and LC13, that $\mathscr{G}'$ contains at most one A^0_g, at most one A^1_h and at most one D_p. LC7 and LC3 imply that

$$h<\min\{f: f\in F\}\leqslant\max\{f: f\in F\}<g.$$

K2, K4(b), K4(c), and K3 imply that $\{S_f: f\in F\}\cup\{L_g, R_h\}$ is linked. Since $\mathscr{S}$ is binary, choose

$$x\in\bigcap_{f\in F} S_f\cap L_g\cap R_h.$$

Let $m=\max(\{n_f: f\in F\}\cup\{\operatorname{dom} p\})$. By LC11, we infer that, for each $f\in F$, $f\upharpoonright\operatorname{dom} p = p$. Thus, for each $f_1\neq f_2$ in F, $f_1\upharpoonright m = f_2\upharpoonright m$. Let q be this common value. Then $q\upharpoonright\operatorname{dom} p = p$.

Subcase 2 (i). $g\upharpoonright m = q$ and $h\upharpoonright m = q$.

Then

$$\begin{aligned}(q,x)&\in\bigcap_{f\in F}[\{f\upharpoonright m\}\times S_f]\cap[\{g\upharpoonright m\}\times L_g]\cap[\{h\upharpoonright m\}\times R_h]\cap[\{q\}\times K]\\ &\subseteq\bigcap_{f\in F} U^{n_f}_f\cap A^0_g\cap A^1_h\cap D_p.\end{aligned}$$

Subcase 2 (ii). $g\upharpoonright m = q$ and $h\upharpoonright m\neq q$.

Since $h<\min\{f: f\in F\}$, we have $\{q\}\times K\subseteq A^1_h$. Then

$$(q,x)\in\bigcap_{f\in F}[\{f\upharpoonright m\}\times S_f]\cap[\{g\upharpoonright m\}\times L_g]\cap[\{q\}\times K]\subseteq\bigcap_{f\in F} U^{n_f}_f\cap A^0_g\cap A^1_h\cap D_p.$$

Subcase 2 (iii). $g\upharpoonright m\neq q$ and $h\upharpoonright m = q$.

Since $\max\{f: f\in F\}<g$, we have $\{q\}\times K\subseteq A^0_g$. Then

$$\begin{aligned}(q,x)&\in\bigcap_{f\in F}[\{f\upharpoonright m\}\times S_f]\cap[\{q\}\times K]\cap[\{h\upharpoonright m\}\times R_h]\cap[\{q\}\times K]\\ &\subseteq\bigcap_{f\in F} U^{n_f}_f\cap A^0_g\cap A^1_h\cap D_p.\end{aligned}$$

Subcase 2 (iv). $g{\restriction}m \neq q$ and $h{\restriction}m \neq q$.

Since $h < \min\{f: f \in F\} \leqslant \max\{f: f \in F\} < g$, we have $\{q\} \times K \subseteq A_g^0$ and $\{q\} \times K \subseteq A_h^1$. Then

$$\begin{aligned}(q, x) &\in \bigcap_{f \in F} [\{f{\restriction}m\} \times S_f] \cap [\{q\} \times K] \cap [\{q\} \times K] \cap [\{q\} \times K] \\ &\subseteq \bigcap_{f \in F} U_f^{n_f} \cap A_g^0 \cap A_h^1 \cap D_p.\end{aligned}$$

Case 3. $\mathscr{G}' \cap (\mathscr{C} \cup \mathscr{E} \cup \mathscr{F}) = \emptyset$ and $\mathscr{G}' \cap \mathscr{D} \neq \emptyset$.

Again, we may assume that $\mathscr{G}'$ contains exactly one D_p, at most one A_g^0, and at most one A_h^1. By LC4, LC8 and LC2, we conclude that there exists an $l \leqslant g$ such that $l{\restriction}\mathrm{dom}\, p = p$, there exists a $k \geqslant h$ such that $k{\restriction}\mathrm{dom}\, p = p$ and $h \leqslant g$. If $g{\restriction}\mathrm{dom}\, p = p$, then

$$(g, 1) \in D_p \cap A_g^0 \cap A_h^1.$$

If $h{\restriction}\mathrm{dom}\, p = p$, then

$$(h, 1) \in D_p \cap A_g^0 \cap A_h^1.$$

If $g{\restriction}\mathrm{dom}\, p \neq p$ and $h{\restriction}\mathrm{dom}\, p \neq p$, then $l < g$ and $k > h$. Hence

$$\{p\} \times K \subseteq D_p \cap A_g^0 \cap A_h^1.$$

Case 4. $\mathscr{G}' \cap (\mathscr{C} \cup \mathscr{D} \cup \mathscr{E} \cup \mathscr{F}) = \emptyset$ and $\mathscr{G}' \cap (\mathscr{A} \cup \mathscr{B}) \neq \emptyset$.

Again, we may assume that $\mathscr{G}'$ contains at most one A_g^0 and at most one A_h^1 and the fact that they are linked does the trick.

Exhausting all possible cases, we conclude that $\mathscr{G}$ is binary.

IV. DISCUSSION

In the example constructed here, the subspace ${}^{\omega}2 \times \{1\}$ is a closed non-G_δ-subspace of X. This leads to the question of whether a closed subspace of a perfectly normal supercompact space is supercompact. In general, is a compact perfectly normal space supercompact? **(P 1180)**

The difficulty in working with supercompact spaces is perhaps illustrated by the fact that it is unknown whether the union of two supercompact subspaces of a Hausdorff space is supercompact. **(P 1181)**

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UNIVERSITY OF MANITOBA
WINNIPEG, MANITOBA

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