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METRIC CHARACTERIZATIONS OF HYPERBOLIC AND EUCLIDEAN SPACES

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1. Introduction. Since 1922, when Banach, Hahn, and Wiener (independently) axiomatized and investigated complete normed linear spaces, much effort has been devoted to the study of these spaces. In 1928 Menger [9] laid the foundation for characterizing spaces metrically, when he gave necessary and sufficient conditions for a metric space to be congruent to an n-dimensional euclidean space. By appealing to some of Menger's theorems, Wilson [11] was able to characterize (metrically) generalized euclidean space among a certain class of metric spaces. Blumenthal [2] observed that Wilson's property could be weakened considerably and still obtain the same result.

Numerous papers have appeared which characterize inner products spaces among the class of normed linear spaces (over the reals). One of the most memorable of these is the condition of Jordan - von Neumann [8], which has long been a standard, showing it is implied by newly postulated conditions. Blumenthal [3] metrized the Jordan - von Neumann condition and by the use of this metrization was able to characterize real inner-product spaces among the class of complete, convex, externally convex metric spaces. He obtained generalizations of both the Jordan - von Neumann condition and his own [2]. Many of the properties which characterize real inner-product spaces have analogous properties which characterize non-euclidean spaces, for example see [5] and [10]. Busemann [6] used a somewhat different approach to characterize euclidean and hyperbolic spaces.

Young [12] introduced three conditions, which he used to distinguish between euclidean, hyperbolic and elliptic geometries in spaces satisfying the axioms of Hilbert's groups I, II, III and V, namely, the axioms of connection, order, congruence, and continuity. In 1935 two short articles by Aronszajn appeared [1], in which a metric characterization of normed linear spaces was stated. The characterization, which appeared without proof, made use of a metric restatement and generalization of Young's condition for euclidean geometry. Andalafte and Blumenthal [4] gave the following complete metrization of Aronszajn's assumption:

THE YOUNG POSTULATE. *If p, q and r are points of a metric space M, and if q' and r' are the midpoints of p and q, and p and r, respectively, then* $q'r' = \frac{1}{2}qr$.

Ficken [7] proved that an inner-product can be defined in a complete, real, normed linear space if and only if, for elements p and q of the space, the equality of the norms of p and q, $\|p\| = \|q\|$, implies

$$\|\lambda \cdot p + \mu \cdot q\| = \|\mu \cdot p + \lambda \cdot q\|$$

for all real numbers λ and μ.

Andalafte and Blumenthal [4] metrized this property in the following way:

THE FICKEN POSTULATE. *If f is the foot of a point p on a line L (p not on L) and if q and r are points of L with $fq = fr$, then $sq = sr$ for each point s of a line $L(p, f)$, joining p and f.*

Assuming the Young Postulate, they obtained a characterization of Banach spaces. Certain results were obtained using only the Ficken Postulate. Then, combining these results with the ones obtained from the Young Postulate, they gave a characterization of euclidean space.

In this paper we introduce a stronger form of the Ficken Postulate and show that it immediately implies the space is euclidean or hyperbolic. To distinguish between the spaces, we adjoin a weak form of the Young Postulate for the respective spaces.

In the process of obtaining this characterization, the authors observed that if a congruence between two triples of non-collinear points could be extended to a congruence between one of points and the line of the other two points and the corresponding point and the line of the remaining two points, then the space in question is hyperbolic or euclidean. This observation establishes a conjucture of Busemann [6], since it is a weakening of his assumption.

Unless otherwise noted all terms used in this paper have the same meanings as those given in [2].

2. Metric characterizations. We now suppose M is a metric space which is (1) finitely compact, (2) convex, (3) externally convex, and (4) has the two triple property. Assuming M also satisfies either the Congruence Extension Postulate or the Strong Ficken Postulate, we will show that M also satisfies the criteria of Busemann [6] and is, consequently, euclidean or hyperbolic. The Congruence Extension and the Strong Ficken Postulates are as follows:

(C.E.) THE CONGRUENCE EXTENSION POSTULATE. *If p, q, r and p', q', r' are any two triples of non-collinear points such that $p, q, r \approx p',$*

q', r', then this congruence can be extended to a congruence between $\{p\}\cup \cup L(q, r)$ and $\{p'\}\cup L(q', r')$.

(S.F.) STRONG FICKEN POSTULATE. *Suppose p and p' are points of M, L and L' are lines of M, and suppose f and f' are the feet of p and p' on L and L', respectively. If $pf = p'f'$, then for each pair of points q and r on $L(p,f)$ and L, respectively, and for each pair of points q' and r' on $L(p',f')$ and L', respectively, if $fq = f'q'$ and $fr = f'r'$, then $qr = q'r'$.*

In order to make use of the results of Busemann mentioned above, it is necessary to consider the equidistant locus of two points s and s' of M.

Definition 2.1. The *equidistant locus* of two distinct points s and s' of M, denoted by $m(s, s')$, is the set of points p such that $ps = ps'$.

THEOREM 2.1. *If M satisfies the Congruence Extension Postulate, then the equidistant locus of two distinct points of M is linear.*

Proof. Suppose s, s' are distinct points of M and p, q are points in $m(s, s')$. If x is any point on $L(p, q,)$, then since $p, q, s \approx p, q, s'$ (by C. E.), this congruence may be extended to $\{s\}\cup L(p, q) \approx \{s'\}\cup L(p, q)$, and since the line joining p and q is unique (by the two-triple property), it follows that $sx = s'x$. Therefore $L(p, q)$ is contained in $m(s, s')$.

It now follows that M is euclidean or hyperbolic, since Busemann [6] has shown that a finitely compact metric space with unique straight lines is hyperbolic or euclidean if and only if the equidistant locus is linear. We thus have the following characterization theorem:

THEOREM 2.2. *A finitely compact, convex, externally convex metric space which has the two-triple property is hyperbolic or euclidean if and only if it satisfies the Congruence Extension Postulate.*

THEOREM 2.3. *If M satisfies the Strong Ficken Postulate, then M satisfies the Ficken Postulate.*

Proof. This is clear, since p, p' and L, L' need not be distinct, in which case the Strong Ficken Postulate is just the Ficken Postulate.

THEOREM 2.4. *If M satisfies the Strong Ficken Postulate, then M satisfies the Congruence Extension Postulate.*

Proof. Suppose p, q, r and p', q', r' are two triples of non-collinear points of M with $p, q, r \approx p', q', r'$. Let f and f' be the feet of p and p' on $L(q, r)$ and $L(q', r')$, respectively. That the foot of a point on a line is unique follows as in [4]. If $pf = p'f'$, an easy application of the Strong Ficken Postulate gives the result. Suppose then that $pf \neq p'f'$, and assume the labeling such that $pf < p'f'$. It follows that a point t of $L(p,f)$ exists such that $tf = p'f'$ and tpf holds. By the monotone property [4], there are points x, y on $L(q, r)$ such that $tx = p'q'$, $ty = p'r'$, and x, f, y satisfy the same betweeness relations as q', f', r'. From the Strong Ficken

Postulate it is easily seen that f is the foot of q on $L(p, f)$. Thus x and y must both be between q and r, for if it is assumed that xqf holds, then, by the monotone property, it would follow that $xt > qt$. But the monotone property also implies $qt > qp$ contrary to the fact that $xt = qp$. Thus qxf holds and, similarly, fyr holds. However, it follows from the Strong Ficken Postulate that $xy = q'r'$, but this contradicts $xy < qr = q'r'$, which completes the proof.

Applying Theorem 2.2, we obtain the following characterization:

THEOREM 2.5. *A finitely compact, convex, externally convex metric space which has the two-triple property is euclidean or hyperbolic if and only if it satisfies the Strong Ficken Postulate.*

Since for any three points p, q, r of euclidean space, if q' and r' are the midpoints of p and q and p and r, respectively, it follows that $q'r' = \frac{1}{2}qr$ and, for any three non-collinear points p, q, r of hyperbolic space with q', r' as before, $q'r' < \frac{1}{2}qr$, we introduce the following weak forms of the Young Postulate:

WEAK YOUNG POSTULATE (EUCLIDEAN). *There exist three non-collinear points p, q, r of a metric space such that if q' and r' are the midpoints of p and q and p and r, respectively, then $q'r' = \frac{1}{2}qr$.*

WEAK YOUNG POSTULATE (HYPERBOLIC). *There exist three non-collinear points p, q, r of a metric space such that if q' and r' are the midpoints of p and q and p and r, respectively, then $q'r' < \frac{1}{2}qr$.*

Since no three non-collinear points of euclidean space satisfy the hyperbolic Weak Young Postulate, and no three non-collinear points of hyperbolic space satisfy the euclidean Weak Young Postulate the following four theorems are immediate:

THEOREM 2.6. *A finitely compact, convex, externally convex metric space which has the two-triple property is congruent to a euclidean space of finite dimension if and only if it satisfies the Strong Ficken Postulate and the Weak Young Postulate (Euclidean).*

THEOREM 2.7. *A finitely compact, convex, externally convex metric space which has the two-triple property is congruent to a hyperbolic space of finite dimension if and only if it satisfies the Strong Ficken Postulate and the Weak Young Postulate (Hyperbolic).*

THEOREM 2.8. *A finitely compact, convex, externally convex metric space which has the two-triple property is congruent to a euclidean space of finite dimension if and only if it satisfies the Congruence Extension Postulate and the Weak Young Postulate (Euclidean).*

THEOREM 2.9. *A finitely compact, convex, externally convex metric space which has the two-triple property is congruent to a hyperbolic space* *of finite dimension if and only if it satisfies the Congruence Extension Postulate and the Weak Young Postulate (Hyperbolic).*

3. Conclusions and questions. It follows from Section 2 that the Strong Ficken Postulate and the Congruence Extension Postulate are equivalent in finitely compact, convex, externally convex metric spaces which has the two-triple property, and the Strong Ficken Postulate implies the Ficken Postulate in such a setting. If M is a finitely compact normed linear space over the reals, then the Strong Ficken Postulate and the Congruence Extension Postulate may be stated in terms of norms as follows:

STRONG FICKEN POSTULATE. *For each quadruple of vectors p, q, p', q', if*

$$|p+q| = |p-q| = |p'-q'| = |p'+q'|,$$

then

$$|\lambda\cdot p+\mu\cdot q| = |\lambda\cdot p-\mu\cdot q| = |\lambda\cdot p'+\mu\cdot q'| = |\lambda\cdot p'-\mu\cdot q'|$$

for all real numbers λ and μ.

CONGRUENCE EXTENSION POSTULATE. *The equality of norms $|p| = |p'|$, $|q| = |q'|$ and $|p-q| = |p'-q'|$ implies $|p-\lambda\cdot q| = |p'-\lambda\cdot q'|$ for each real number λ.*

We then immediately obtain the following characterizations of real inner product spaces:

THEOREM 3.1. *A finitely compact normed linear space over the reals which has the two-triple property is an inner product space if and only if it satisfies the Strong Ficken Postulate.*

THEOREM 3.2. *A finitely compact normed linear space over the reals which has the two-triple property is an inner product space if and only if it satisfies the Congruence Extension Postulate.*

It follows that in a finitely compact normed linear space with unique straight lines the Strong Ficken Postulate, the Ficken Postulate, and the Congruence Extension Postulate are all equivalent. It would be interesting to know if the Ficken Postulate is equivalent to the other two postulates in a finitely compact, convex, externally convex metric space which has the two-triple property (**P 798**). Of course, if it is equivalent to one of them in such a setting, then it is equivalent to both.

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