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w^*-CLOSED SUBALGEBRAS OF $L^\infty(G)$

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1. Let X be a locally compact space equipped with a positive Radon measure μ. Let $L^\infty(X, \mu)$ denote the algebra of (equivalence classes of) complex-valued, essentially bounded, measurable functions on X with pointwise multiplication. First we prove a lemma which can also be used to simplify some arguments given in [5], Lemma 3, and [6], 5.9. As a further application, in the second part of this paper we will use this lemma to prove our main theorem.

LEMMA. *If A is an algebra of complex-valued, bounded, continuous functions on X, which is selfadjoint, contains the constant functions, and separates the points of X, then A is w^*-dense in $L^\infty(X, \mu)$.*

Proof. We may assume that A is closed with respect to the supremum norm on X. Assume that $f \in L^1(X, \mu)$, $\|f\|_1 = 1$ satisfies

$$\int_X f(t)g(t)\,d\mu(t) = 0 \quad \text{for all } g \in A.$$

There exists a continuous function h with $\|h\|_\infty = 1$ such that

$$\left|\int_X f(t)h(t)\,d\mu(t)\right| > 3/4.$$

By regularity, we get a compact subset C of X such that

$$\int_C |f(t)|\,d\mu(t) > 3/4.$$

The subalgebra A_0 of real functions in A forms a real norm-closed algebra. Consequently, A_0 is a lattice (this follows easily from the first part of the usual proof of the Stone-Weierstrass theorem; see, e.g., [2], p. 131 ff.). If $g \in A$ is any function which satisfies $|g(t)| \leqslant 1$ for $t \in C$, then the same inequality holds for its real and imaginary parts $\mathrm{Re}\,g$ and $\mathrm{Im}\,g$. Put

$$g_1 = \max\big(-1, \min(1, \mathrm{Re}\,g)\big), \quad g_2 = \max\big(-1, \min(1, \mathrm{Im}\,g)\big).$$

Since A_0 contains the constant 1, $g_1, g_2 \in A$, and $g_3 = g_1 + ig_2$ satisfies $g_3 = g$ on C and $\|g_3\|_\infty \leqslant 2$. Therefore, the restrictions of functions in A to C form a norm-closed algebra. Applying the Stone-Weierstrass theorem, we find some $g \in A$ such that $h = g$ on C and $\|g\|_\infty \leqslant 2$. But now

$$\Big|\int_X f(t)g(t)d\mu(t)\Big| \geqslant \Big|\int_X f(t)h(t)d\mu(t)\Big| - 1/4 - 2/4 > 0,$$

which contradicts our assumption on f.

2. Let $X = G$ be a locally compact group and μ a left Haar measure on G. If H is a closed subgroup of G, we write G/H for the family of left cosets equipped with the quotient topology. L_x (R_x) denotes the left (right) translation by x on $L^\infty(G)$.

THEOREM. *Let $A \neq (0)$ be a w^*-closed, selfadjoint, left invariant subalgebra of $L^\infty(G)$. Then $A = L^\infty(G/H)$ for some closed subgroup H of G.*

Proof. Since A is w^*-closed, $f*g \in A$ for $f \in L^1(G)$, $g \in A$. Taking an appropriate approximate identity in $L^1(G)$, we see that the subalgebra A_1 of all continuous functions in A is w^*-dense in A. Put

$$H = \{x \in G\colon R_x f = f \text{ for all } f \in A\} = \{x \in G\colon R_x f = f \text{ for all } f \in A_1\}.$$

Every member of A_1 may be regarded as a continuous function on G/H. Since $L^\infty(G/H)$ is w^*-closed, we have $A \subset L^\infty(G/H)$. The algebra A contains the constant 1 (by [3], chapitre 1, § 3, lemme 6), and so A_1 fulfills all conditions of the previous lemma.

COROLLARY. *Let A be a selfadjoint, left invariant subalgebra of $L^\infty(G)$ with the property that for any $x \in G$ there exists an element $f \in A$ such that $R_x f \neq f$. Then A is w^*-dense in $L^\infty(G)$.*

Proof. It follows again from [3], chapitre 1, § 3, lemme 6, that the w^*-closure of A fulfills the same conditions as A.

Remarks. 1. If one considers right coset spaces, then an analogous theorem holds for right invariant subalgebras of $L^\infty(G)$.

2. The condition $R_x f \neq f$ of the Corollary cannot be replaced by $L_x f \neq f$ (if one considers left invariant subalgebras). Take, e.g., for G a finite group, for H a cyclic non-normal subgroup of prime order, and $A = L^\infty(G/H)$.

3. The result can also be used to give a short proof of the following theorem of [4]: *If G is a maximally almost periodic group, then $L^1(G)$ is strongly semisimple.* (This observation is due to T. Pytlik.)

4. In [1] Cigler has proved similar results for the case of locally compact abelian groups. He used methods of spectral synthesis.

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