COLLOQUIUM MATHEMATICUM

VOL. XXVIII 1973 FASC. 2

A VARIATIONAL METHOD FOR A CLASS OF ODD FUNCTIONS IN AN ANNULUS

BY

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1. INTRODUCTION

Let R denote the annulus $\{z: r_0 < |z| < 1\}$. The class F, introduced by Gaier [2], consists of functions that are holomorphic and schlicht in R and that satisfy the three conditions:

(1) $$|f(z)| < 1 \quad (z \in R), \qquad |f(z)| = 1 \quad (|z| = 1),$$

(2) $$f(z) \neq 0 \quad (z \in R),$$

(3) $$f(1) = 1.$$

The class F and related classes have been considered in many papers. Duren and Schiffer [1] considered the classes F_0 (functions in F_0 satisfy all conditions for functions in F, except the normalization (3)) and F_1 (functions in F_1 need not satisfy conditions (2) and (3)). Gehring and Hällström [3] obtained distortion theorems for the class F and for the subclass of functions whose image is symmetric in the origin. Recently, a subclass F_s of F has been introduced [5] consisting of functions whose image is symmetric with respect to the real axis. Some extremal problems that are inaccessible in F could be solved in the smaller class F_s.

In this paper, we consider two subclasses of F_0 consisting of functions with a rotational symmetry, namely

$$f(-z) = -f(z).$$

These classes are

$$F_2 = \{f \in F_0 \colon f(-z) = -f(z)\} \quad \text{and} \quad F_3 = F_2 \cap F.$$

Clearly, both F_2 and F_3 are compact. Note that with each $f \in F_2$, each rotation $e^{i\theta} f$ (θ real) also belongs to F_2, whereas F_3 does not contain any non-trivial rotations of its members.

In Section 2, we derive variational formulas for F_2 and F_3, and in the following sections, we apply these variational formulas to various extremal problems. Known extremal functions for the class F_0 do not have the rotational symmetry of functions in F_2. The solution of extremal problems in F_2 or F_3 will therefore shed more light on the behavior of conformal maps of doubly connected domains.

2. VARIATIONAL FORMULAS

Each $f \in F_0$ maps R onto the unit disk minus some continuum Γ_f that contains the origin. If f belongs to F_2 or F_3, then Γ_f is symmetric with respect to the origin.

Suppose f belongs to F_2. Fix $w_0 \in \Gamma_f$, say $w_0 \neq 0$. Let $D_\varrho(w_0)$, $\varrho > 0$, denote the domain consisting of all points either exterior to Γ_f or exterior to the disk $|w - w_0| \leqslant \varrho$. There exist [1] functions of the form

$$F(w) = w + \frac{a\varrho^2 w}{(w - w_0)w_0} + O(\varrho^3)$$

that are analytic and univalent in $D_\varrho(w_0)$ and leave the origin fixed. Here the constant a depends on ϱ ($|a| = |a(\varrho)| \leqslant 1$), and the error term $O(\varrho^3)$ can be estimated uniformly in each closed subdomain of $D_\varrho(w_0)$.

Either $\operatorname{Re} w_0 \neq 0$ or $\operatorname{Im} w_0 \neq 0$; say $\operatorname{Im} w_0 > 0$. Choose ϱ so small that the circle $|w - w_0| = \varrho$ does not intersect the real axis. Then there exists a function of the form $h(w) = w/(w + w_0) + O(\varrho)$ such that h is defined and satisfies a Lipschitz condition in the half-plane $\operatorname{Im} w > -\frac{1}{2}\operatorname{Im} w_0$ (hence the function $w - a\varrho^2 h(w)/w_0$ is univalent in this half-plane for $\varrho < \varrho_0$) and such that the composite function

$$\begin{aligned} H(w) &= F(w) - \frac{a\varrho^2}{w_0} h\big(F(w)\big) \\ &= w + \frac{2a\varrho^2 w}{w^2 - w_0^2} + O(\varrho^3) \qquad (\operatorname{Im} w \geqslant 0,\ w \in D_\varrho(w_0)) \end{aligned}$$

satisfies the condition $H(-w) = -H(w)$ for real values of w. Clearly, H is univalent in $D_\varrho(w_0) \cap \{w : \operatorname{Im} w \geqslant 0\}$ and preserves the origin. Now extend H to a univalent function in $D_\varrho(w_0) \cap D_\varrho(-w_0)$ by setting $H(-w) = -H(w)$.

We may now proceed as in [1]. The function $w^3(1 - \bar{w}_0^2 w^2)^{-1}$ satisfies a Lipschitz condition in $|w| \leqslant 1$, so that the function

$$\varphi(w) = w - 2\bar{a}\varrho^2 w^3 (1 - \bar{w}_0^2 w^2)^{-1}$$

is univalent for sufficiently small values of ϱ. But then the composite function

$$\varphi(H(w)) = w+\frac{2a\varrho^2 w}{w^2-w_0^2}-\frac{2\bar{a}\varrho^2 w^3}{1-\bar{w}_0^2 w^2}+O(\varrho^3)$$

is univalent in $D_\varrho(w_0)\cap D_\varrho(-w_0)$. Moreover, it is easy to check that $\varphi(H(w))$ preserves the unit circle up to order ϱ^3, and we obtain the following variational formula for the class F_2:

$$V_\varrho^{(2)}(w) = w\left(1+\frac{a\varrho^2}{w^2-w_0^2}-\frac{\bar{a}\varrho^2 w^2}{1-\bar{w}_0^2 w^2}\right)+O(\varrho^3).$$

The variational formula for F_3 is given by the relation

$$V_\varrho^{(3)}(w) = \left(V_\varrho^{(2)}(1)\right)^{-1} V_\varrho^{(2)}(w),$$

and we summarize these results as follows:

THEOREM. *If $f(z)$ belongs to F_2, then $V_\varrho^{(2)}(f(z))$ $(\varrho\leqslant\varrho_1)$ also belongs to F_2, where*

$$V_\varrho^{(2)}(w) = w\left(1+\frac{a\varrho^2}{w^2-w_0^2}-\frac{\bar{a}\varrho^2 w^2}{1-\bar{w}_0^2 w^2}\right)+O(\varrho^3).$$

If $f(z)$ belongs to F_3, then $V_\varrho^{(3)}(f(z))$ $(\varrho\leqslant\varrho_1)$ also belongs to F_3, where

$$V_\varrho^{(3)}(w) = w\left(1+\frac{a\varrho^2(1-w^2)}{(w^2-w_0^2)(1-w_0^2)}+\frac{\bar{a}\varrho^2(1-w^2)}{(1-\bar{w}_0^2 w^2)(1-\bar{w}_0^2)}\right)+O(\varrho^3).$$

3. MAXIMUM AND MINIMUM VALUE OF $|f(z)|$

Suppose that f is an extremal function for the problem

$$\max_{g\in F_2}|g(z)|.$$

Then $|V_\varrho^{(2)}(f(z))|\leqslant|f(z)|$. This inequality leads to the relation

$$\operatorname{Re}\left[a\varrho^2\left(\frac{1}{a^2-w_0^2}-\frac{\bar{a}^2}{1-\bar{a}^2 w_0^2}\right)+O(\varrho^3)\right]\leqslant 0,$$

where $\alpha = f(z)$. By Schiffer's lemma (see [6] and [1]), the omitted continuum Γ_f satisfies the differential condition $w'(t)^2 s(w(t))>0$, where

$$s(w) = \frac{1}{a^2-w^2}-\frac{\bar{a}^2}{1-\bar{a}^2 w^2}.$$

Without loss of generality we may assume that $a>0$, so that

$$s(w) = (1-a^4)(a^2-w^2)^{-1}(1-a^2w^2)^{-1}.$$

The differential condition thus becomes

$$\frac{w'(t)^2}{\big(a^2-w(t)^2\big)\big(1-a^2w(t)^2\big)} > 0.$$

The solution through the origin is $w(t) = at$ $(-1 < t < 1)$, and Γ_f is an interval of the real axis, symmetric about the origin.

For the minimum problem, the differential condition for Γ_f becomes

$$w'(t)^2(a^2-w^2)^{-1}(1-a^2w^2)^{-1} < 0,$$

and Γ_f is an interval of the imaginary axis, symmetric about the origin.

(The results of this section should be compared with those of Gehring and Hällström [3] who use as an important tool a lemma of H. Grötzsch; see also Teichmüller [7], p. 631-635.)

4. *DISTORTION*

Suppose f is an extremal function for the problem

$$\max_{g\in F_2} |g'(z)|.$$

Then $\big|V_\varrho^{(2)}(f(z))'\big| \leqslant |f'(z)|$, which leads to the relation

$$\operatorname{Re}\left[a\varrho^2\left(\frac{a^2+w_0^2}{(a^2-w_0^2)^2}+\frac{\bar{a}^2(3-\bar{a}^2w_0^2)}{(1-\bar{a}^2w_0^2)^2}\right)+O(\varrho^3)\right] \geqslant 0,$$

where $a = f(z)$. Again we may assume that $a > 0$. Thus Γ_f satisfies the differential condition $w'(t)^2 s\big(w(t)\big) < 0$, where

$$s(w) = \frac{a^2+3a^6+(1-8a^4-a^8)w^2+(a^2+3a^6)w^4}{(a^2-w^2)^2(1-a^2w^2)^2}. \tag{4}$$

If $a > \sqrt{\sqrt{5}-2} = 0.4858\ldots$, then Γ_f is a segment of the imaginary axis, symmetric about the origin. If $a \leqslant \sqrt{\sqrt{5}-2}$, then the polynomial $1+cw^2+w^4$ $(c = (1-8a^4-a^8)/(a^2+3a^6))$ has a root $w = iw_1$ $(a < w_1 < 1)$. The solution curve through the origin of the differential condition consists of the segment $w(t) = it$ $(-w_1 < t < w_1)$, and at $-iw_1$ and iw_1 the curve sprouts a fork. The omitted continuum is a symmetric (about 0) piece of this curve.

If f is an extremal function for the minimum problem $\min\limits_{g\in F_2} |g'(z)|$, then Γ_f satisfies the differential condition $w'(t)^2 s\big(w(t)\big) > 0$, where $s(w)$ is given by (4). Since the numerator of $s(w)$ is positive for $-a < w < a$, the omitted continuum Γ_f is a segment $[-t_0, t_0]$ of the real axis.

5. A ROTATION PROBLEM

In the smaller class F_3, the problem

$$\max_{g\in F_3} \operatorname{Im}\log\frac{g(z)}{z}$$

is meaningful (select the branch of the logarithm for which $\operatorname{Im}\log g(z)/z = 0$ at $z = 1$). Suppose f is an extremal function, and set $a = f(z)$. The inequality

$$\operatorname{Im}\log\frac{V_\varrho^{(3)}(f(z))}{z} \leqslant \operatorname{Im}\log\frac{f(z)}{z}$$

leads to the relation

$$\operatorname{Re}\left[a\varrho^2\left(\frac{i(1-\bar{a}^2)}{(1-\bar{a}^2w_0^2)(1-w_0^2)} - \frac{i(1-a^2)}{(a^2-w_0^2)(1-w_0^2)}\right) + O(\varrho^3)\right] \leqslant 0.$$

By Schiffer's lemma, the continuum Γ_f satisfies the differential condition $w'(t)^2 \times s(w(t)) > 0$, where

$$s(w) = i\frac{2a^2-1-|a|^4+(2\bar{a}^2-1-|a|^4)w^2}{(1-w^2)(1-\bar{a}^2w^2)(a^2-w^2)}.$$

The local behavior of this differential condition is easy to describe (see [4], Chapter III). We have computed the direction field for $a^2 = 1/2$, and the results are shown in Fig. 1.

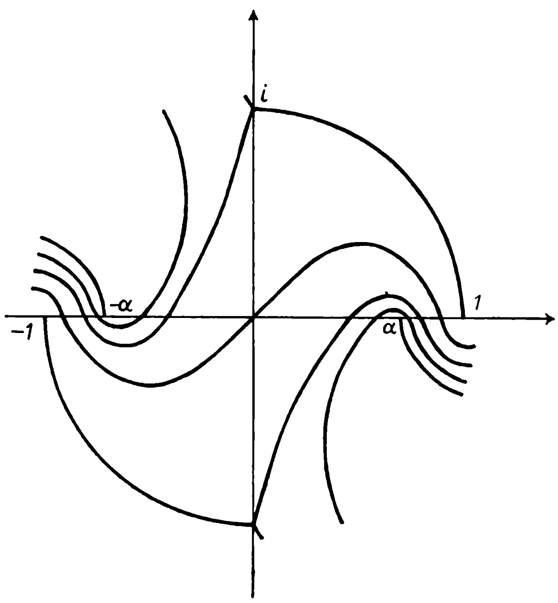

Fig. 1

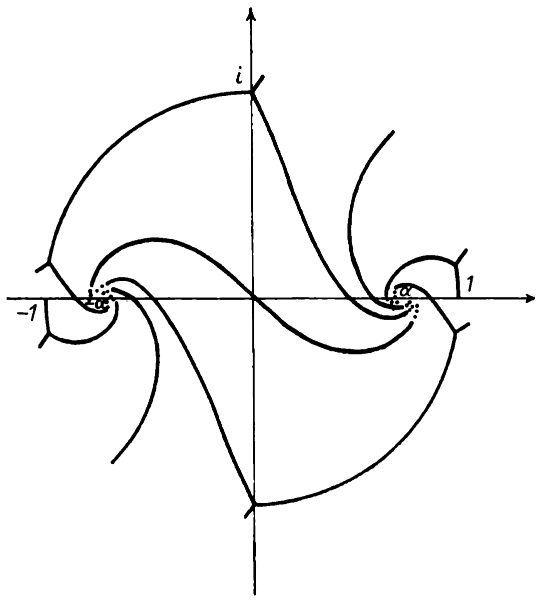

Fig. 2

The omitted continuum for the associated minimum problem is a portion of the orthogonal trajectory through the origin.

6. ANOTHER ROTATION PROBLEM

Another way of measuring the rotation at a point z is to regard $\operatorname{Im}\log g'(z)$. In the class F_3, we can choose the branch of the logarithm for which $\operatorname{Im}\log g'(1) = 0$, and the problem is meaningful. Suppose f is an extremal function for the problem

$$\max_{g\in F_3} \operatorname{Im}\log g'(z),$$

and set $a = f(z)$. Comparison of $f(z)$ and $V_\varrho^{(3)}(f(z))$ leads to the relation

$$\operatorname{Re}\left[a\varrho^2\left(i\,\frac{1-3\bar{a}^2+\bar{a}^2w_0^2+\bar{a}^4w_0^2}{(1-\bar{a}^2w_0^2)^2(1-w_0^2)} - i\,\frac{3a^2w_0^2-w_0^2-a^2-a^4}{(a^2-w_0^2)^2(1-w_0^2)}\right)+O(\varrho^3)\right] \leqslant 0.$$

Hence Γ_f satisfies the differential condition $w'(t)^2 s(w(t)) > 0$, where

$$s(w) = i\,\frac{a_0+a_1w^2+\bar{a}_1w^4+\bar{a}_0w^6}{(1-w^2)(1-\bar{a}^2w^2)^2(a^2-w^2)^2},$$

$$a_0 = a^2(1+2a^2-3|a|^4),$$

$$a_1 = 1+4|a|^4+|a|^8-a^2(5+|a|^4).$$

This differential condition is more complicated than the one in Section 5. To allow comparison, we have computed the direction field, again for $a^2 = 1/2$ (see Fig. 2).

If $a = i/\sqrt{3}$, then $a_0 = 0$, and the origin is a zero of order 2 of the quadratic differential. For all other values of a, the origin is a regular point. A computation shows that for all values of a, trajectories near $w = a$ and $w = -a$ are asymptotically logarithmic spirals ([4], p. 32, Theorem 3.4).

Remark. We have attempted to find the extrema for the curvature of the image of a circle $|z| = r$ by functions in F_2. But the resulting differential condition is not noticeably simpler than the differential condition for the whole class F_0 (the main difficulty is that too many parameters remain in the differential equation).

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Reçu par la Rédaction le 21. 3. 1972;
en version modifiée le 8. 6. 1972