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ON THE FORM OF PRINCIPAL TORUS-BUNDLES OVER TORUSES

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1. In [3] Palais and Stewart have shown that a smooth manifold M is a nilmanifold of class $\leqslant 2$ if and only if M is a total space of a principal T^n-bundle over torus T^k for some natural n and k. Their proof was geometrical in character and, apparently to avoid technical difficulties, they restricted one essential step in the argument to dimension $n = 1$ only.

In this paper we shall give another and complete proof of a somewhat more general result: *a CW-complex M is homeomorphic to a nilmanifold of class $\leqslant 2$ if and only if M is a total space of a principal T^n-bundle over torus T^k for some n and k.* Our methods, which are those of algebraic topology, allow to obtain also some results on the fundamental group of M.

The argument runs roughly as follows. Given a principal T^n-bundle $\xi = (M, p, T^k)$, we define, using only the characteristic class of ξ, a Lie algebra L_ξ and a Lie group G_ξ for which L_ξ is the Lie algebra. Next we distinguish in G_ξ a certain discrete subgroup D and define a manifold $X = G_\xi/D$, thus having $\pi_1(X) = D$. In a natural way X becomes the total space of a certain bundle $\eta = (X, q, T^k)$. Main result of this paper states that ξ and η are isomorphic to each other (thus yielding to M the structure of a nilmanifold). In particular, $\pi_1(M) = \pi_1(X) = D$ which gives a method for calculating a fundamental group of a total space of a principal T^n-bundle over T^k. As it turns out, this group is always a nilpotent, torsion-free group of class $\leqslant 2$.

Notions and notation used in this paper come from [1], [2], [5] and [6].

2. First step consists in the following. Given a principal T^n-bundle $\xi = (M, p, T^k)$, we define a real $(k+n)$-dimensional Lie algebra L_ξ (in fact, it is even a covariant functor).

The universal coefficient theorem and additivity of tensor product imply the following isomorphisms:

$$(1) \qquad H^2(T^k, Z^n) = H^2(T^k, Z)\otimes Z^n = \underbrace{H^2(T^k, Z)\oplus \ldots \oplus H^2(T^k, Z)}_{n \text{ times}}.$$

Thus the characteristic class $c(\xi)$ of the bundle ξ can be written as the sequence $c^1(\xi), c^2(\xi), \ldots, c^n(\xi)$ of elements of the group $H^2(T^k, Z)$.

To choose a basis in $H^2(T^k, Z)$ recall [5] that the cohomology algebra $H^*(T^k, Z)$ of the torus T^k is the exterior algebra generated by the group $H^1(T^k, Z)$. Therefore, if $\mathscr{H}_1, \mathscr{H}_2, \ldots, \mathscr{H}_k$ is a basis of the group $H^1(T^k, Z)$, then the set $\{\mathscr{H}_i \wedge \mathscr{H}_j\}_{i<j}$ will be a basis of $H^2(T^k, Z)$ and

$$c^p(\xi) \doteq \sum_{i<j} c^p_{ij} \mathscr{H}_i \wedge \mathscr{H}_j,$$

where $c^p_{ij} \in Z$ for $p = 1, 2, \ldots, n$.

To define the Lie algebra L_ξ take the following collection of numbers b^p_{rs}, where $1 \leqslant p, r, s \leqslant k+n$ (they will play a role of structural constants of L_ξ):

$$(2) \qquad b^p_{rs} = \begin{cases} 0 & \text{if either } r = s \text{ or } 1 \leqslant p \leqslant k \text{ or } r > k \text{ or } s > k, \\ c^{p-k}_{rs} & \text{if } p > k \text{ and } 1 \leqslant r < s \leqslant k, \\ -c^{p-k}_{sr} & \text{if } p > k \text{ and } 1 \leqslant s < r \leqslant k. \end{cases}$$

Since $b^p_{is} b^s_{jm} = 0$ for all s, i, j, m, p, this collection satisfies the condition

$$(3) \qquad \sum_{s=1}^{k+n} (b^p_{rs} b^s_{qu} + b^p_{qs} b^s_{ur} + b^p_{us} b^s_{rq}) = 0.$$

Take a basis $e_1, e_2, \ldots, e_{k+n}$ in a $(k+n)$-dimensional real linear space V and for $x, y \in V$ define

$$[x, y] = \sum_{i,j,p=1}^{k+n} x_i y_j b^p_{ij} e_p,$$

where $x_1, x_2, \ldots, x_{k+n}$ and $y_1, y_2, \ldots, y_{k+n}$ are coordinates of x and y, respectively. The rule $[\ ,\ ]\colon V \times V \to V$ is obviously bilinear, skew-symmetric by (2), and satisfies the Jacobi condition by (3) (cf. [4], p. 385). Hence the space V with the commutator $[\ ,\ ]$ is a Lie algebra.

It is not difficult to see that starting with another basis in $H^1(T^k, Z)$ we get an isomorphic Lie algebra. Hence this algebra is defined (up to isomorphism) uniquely, we denote it by L_ξ.

LEMMA 1. *L_ξ is a nilpotent Lie algebra of class $\leqslant 2$.*

Proof. Let

$$x^i = \sum_{j=1}^{k+n} x^i_j e_j \quad \text{for } i = 1, 2, 3$$

be any elements of V. Then

$$[x^1, [x^2, x^3]] = \Big[x^1, \sum_{j,l,p=1}^{k+n} x_j^2 x_l^3 b_{jl}^p e_p\Big] = \sum_{j,l,p=1}^{k+n} x_j^2 x_l^3 b_{jl}^p [x^1, e_p]$$

$$= \sum_{j,l,p=1}^{k+n} x_j^2 x_l^3 b_{jl}^p \sum_{m,q=1}^{k+n} x_m^1 b_{mp}^q e_q = \sum_{q=1}^{k+n} \sum_{j,l,m=1}^{k+n} x_j^1 x_l^2 x_m^3 \Big(\sum_{p=1}^{k+n} b_{lm}^p b_{jp}^q\Big) e_q = 0,$$

because $b_{lm}^p b_{jp}^q = 0$ for $p = 1, 2, \ldots, k+n$.

3. Now we proceed from the Lie algebra L_ξ to the Lie group G_ξ, for which L_ξ is the Lie algebra. To do it, define in V a new operation $\times$ by the formula

$$x \times y = x+y+\tfrac{1}{2}[x, y] \quad \text{for } x, y \in V.$$

LEMMA 2. *The group G_ξ is a nilpotent, torsion-free Lie group of class $\leqslant 2$.*

In fact, in virtue of the Campbell-Hausdorff formula we infer that V, whith the operation $\times$, is a nilpotent group for which L_ξ is the Lie algebra. By Lemma 1, it is a nilpotent Lie group of class $\leqslant 2$. And if $x \in G_\xi$, then the mapping

$$R \to G_\xi\colon\ t \to tx$$

is a 1-parameter subgroup of G_ξ. Hence G_ξ is torsion-free.

4. To construct a manifold X, consider mappings

$$\gamma_i\colon R \to G_\xi\colon\ t \to te_i \quad \text{for } i = 1, 2, \ldots, k+n,$$

where R is the reals. They define 1-parameter groups in G_ξ and collection $\gamma_1, \gamma_2, \ldots, \gamma_{k+n}$ of these groups clearly generates G_ξ.

In G_ξ we now take the subgroup D generated by the elements $e_1, e_2, \ldots$ $\ldots, e_{k+n}$. Since the structural constants b_{rs}^p of L_ξ are integers, D is a discrete subgroup of G_ξ and so the coset space $G_\xi/D = X$ is a compact nilmanifold [2]. As is known [5], D is the fundamental group of X, $\pi_1(X) = D$.

LEMMA 3. *Let $\mathcal{N}$ be a 1-connected, nilpotent Lie group of class 2 and let Γ be its uniform subgroup. If $\mathcal{N}_1 \subset \mathcal{N}$ is a 1-connected closed central subgroup of $\mathcal{N}$ such that the commutator group $\mathcal{N}^2$ of $\mathcal{N}$ is contained in $\mathcal{N}_1$, then the nilmanifold $\mathcal{N}/\Gamma$ is a principal T^n-space and the coset space $(\mathcal{N}/\Gamma)/T^n$ is the torus T^k, where $n = \dim \mathcal{N}_1$ and $k = \dim \mathcal{N}/\mathcal{N}_1$.*

Proof. Since $\mathcal{N}_1$ is a central subgroup of $\mathcal{N}$ and $\mathcal{N}^2 \subset \mathcal{N}_1$, the group $G = \Gamma \cdot \mathcal{N}_1$ is a normal subgroup of $\mathcal{N}$ and $\mathcal{N}/G = T^k$, where $k = \dim \mathcal{N}/\mathcal{N}_1$ (see [2]). Again, by [2], Γ is a normal subgroup of G and $G/\Gamma = T^n$, where $n = \dim \mathcal{N}_1$.

Since the group G acts smoothly on the manifold $\mathcal{N}/\Gamma$ by the right translations with Γ as an isotropy subgroup, the quotient group $G/\Gamma = T^n$

acts smoothly and freely on the manifold $\mathcal{N}/\Gamma$. And since T^n is a compact Lie group, $\mathcal{N}/\Gamma$ is a principal T^n-space by the Gleason theorem.

In view of the communitativity of the diagram

$$\begin{array}{ccccc} G/\mathcal{N}_1 & \longrightarrow & \mathcal{N}/\mathcal{N}_1 & \longrightarrow & (\mathcal{N}/\mathcal{N}_1)/(G/\mathcal{N}_1) \\ \uparrow & & \uparrow & & \uparrow \\ G & \longrightarrow & \mathcal{N} & \longrightarrow & T^k \\ \downarrow & & \downarrow & & \downarrow \\ G/\Gamma & \longrightarrow & \mathcal{N}/\Gamma & \longrightarrow & (\mathcal{N}/\Gamma)/(G/\Gamma) \end{array} \tag{4}$$

we infer that the coset space $(\mathcal{N}/\Gamma)/(G/\Gamma)$ is equal to $\mathcal{N}/G = T^k$.

5. Let G' be a subgroup of G_ξ generated by the subgroups $\gamma_{k+1}, \gamma_{k+2}, \ldots, \gamma_{k+n}$. By virtue of Lemma 3, since G' is a center subgroup of G_ξ and $G_\xi^2 \subset G'$, X is a principal T^n-space. Thus we can take the unique principal T^n-bundle $\eta = (X, q, T^k)$, where $q\colon X \to T^k$ is a canonical mapping.

By (4) we get the commutative diagram

$$\begin{array}{ccc} G_\xi & \xrightarrow{\varphi} & R^k \\ {\scriptstyle\varrho}\downarrow & & \downarrow{\scriptstyle\nu} \\ X & \xrightarrow{q} & T^k \end{array} \tag{5}$$

We shall show that η is isomorphic to ξ, whence it will follow, in particular, that the total spaces, X of η and M of ξ, are homeomorphic.

THEOREM. *The principal T^n-bundles $\xi = (M, p, T^k)$ and $\eta = (X, q, T^k)$ are isomorphic.*

Proof. To construct an isomorphism of η and ξ it is sufficient to define a homeomorphism $h\colon T^k \to T^k$ such that the characteristic class $c(\xi)$ of the bundle ξ and the characteristic class $c(h^!(\eta))$ of the bundle $h^!(\eta)$ are identical ([1] and [5]).

Let us calculate the characteristic class of η.

Let I^r be the r-dimensional cube and

$$\varphi_{i_1,\ldots,i_r}\colon I^r \to R^k\colon (t_1, \ldots, t_r) \to t_1\varphi(e_{i_1}) + \ldots + t_r\varphi(e_{i_r})$$

be mappings defined for $1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant k$. Denote by $\sigma_{i_1,\ldots,i_r}$ the image of the interior of I^r by the composition $\nu\varphi_{i_1,\ldots,i_r}$ and put $\sigma_0 = \varrho(0)$. Consider the cell complex K, r-cells of which are $\sigma_{i_1,\ldots,i_r}$ with the characteristic mappings $\nu\varphi_{i_1,\ldots,i_r}$. Let K^1 be the 1-skeleton of K. Define a mapping

$$\chi\colon K^1 \to X\colon \nu\varphi_i(t) \to \varrho(te_i),$$

where $\nu\varphi_i(t) \in \sigma_i$. The mapping χ is continuous, because $\chi\nu\varphi_i(0) = \varrho(0) = \varrho(e_i) = \chi\nu\varphi_i(1)$, and it is a cross-section of the bundle $\eta|_{K^1}$ by (5). The mapping

$$\chi\nu\varphi_i\colon I \to X$$

is a loop at the point $\varrho(0)$ and its homotopy class $\{\chi\nu\varphi_i\}$ is $e_i \in D$. Since the bundle of coefficients $\eta(\pi_1(T^n))$ is a product-bundle (cf. [6]) and the inclusion map $i\colon T^n \to X$ induces a monomorphism of the fundamental groups, $i_{\#}\colon \pi_1(T^n) \to \pi_1(X)$, we may calculate the obstruction cocycle $c(\chi) \in Z^2(T^k, Z^n)$ to extending χ over K^2 as follows (here $\circ$ denotes the juxtaposition of loops):

$$\begin{aligned} i_{\#}\big(\langle c(\chi), \sigma_{ij}\rangle\big) &= \{\chi\nu\varphi_{ij} \,|\, \partial I^2\} \\ &= \{(\chi\nu\varphi_i)\circ(\chi\nu\varphi_j)\circ(\chi\nu\varphi_i)^{-1}\circ(\chi\nu\varphi_j)^{-1}\} \\ &= \{\chi\nu\varphi_i\} \times \{\chi\nu\varphi_j\} \times \{\chi\nu\varphi_i\}^{-1} \times \{\chi\nu\varphi_j\}^{-1} \\ &= (e_i \times e_j) \times (e_i^{-1} \times e_j^{-1}) \\ &= (e_i + e_j + \tfrac{1}{2}[e_i, e_j]) \times (-e_i - e_j + \tfrac{1}{2}[e_i, e_j]) \\ &= \sum_{p=k+1}^{k+n} b_{ij}^p e_p = \sum_{p=1}^{n} c_{ij}^p e_{k+p}. \end{aligned}$$

Let $\delta_1, \delta_2, \ldots, \delta_k$ be cocycles of $Z^1(K, Z)$ such that

$$\langle \delta_i, \sigma_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Take $e_{k+1}, \ldots, e_{k+n}$ as a basis of $\pi_1(T^n) \subset D$. Then

$$c(\chi) = \sum_{p=1}^{n} c^p(\chi) e_{k+p}, \qquad \text{where } c^p(\chi) = \sum_{i<j} c_{ij}^p \delta_i \wedge \delta_j.$$

Since the group $B^2(K, Z)$ of 2-coboundaries of K is trivial, we have $H^2(T^k, Z) = Z^2(K, Z)$ and $c(\eta) = c(\chi)$.

Since $\delta_1, \ldots, \delta_k$ is a basis of $H^1(T^k, Z)$, the mapping

$$A\colon H^1(T^k, Z) \to H^1(T^k, Z)\colon \delta_i \mapsto \mathscr{H}_i, \qquad i = 1, \ldots, k,$$

is an isomorphism. There exists a homeomorphism $h\colon T^k \to T^k$ such that A is the induced homomorphism h^* and $A = h^*$. Principal T^n-bundles η and $h^!(\eta)$ are isomorphic (cf. [1]). And since

$$\begin{aligned} c^p\big(h^!(\eta)\big) &= h^*\big(c^p(\eta)\big) = \Lambda^2 A\Big(\sum_{i<j} c_{ij}^p \delta_i \wedge \delta_j\Big) = \sum_{i<j} c_{ij}^p A(\delta_i) \wedge A(\delta_j) \\ &= \sum_{i<j} c_{ij}^p \mathscr{H}_i \wedge \mathscr{H}_j = c^p(\xi) \end{aligned}$$

and, by (1), $c(\xi) = c(h^!(\eta))$, the principal T^n-bundles ξ and η are isomorphic (cf. [1] and [5]).

COROLLARY 1. *The spaces M and X are homeomorphic.*

COROLLARY 2. *The fundamental group $\pi_1(M)$ of the space M is isomorphic to D.*

Indeed, the spaces M and X are homeomorphic by Corollary 1, and so $\pi_1(M) = \pi_1(X) = D$.

COROLLARY 3. *The fundamental group of a total space of a principal T^n-bundle over torus is a nilpotent, torsion-free group of class $\leqslant 2$. It is abelian if and only if the bundle is trivial.*

In fact, $\pi_1(M) = D$ by Corollary 2, and since D is a subgroup of G_ξ, we infer by Lemma 2 that D must be nilpotent, torsion-free, of class $\leqslant 2$. And the bundle ξ is trivial iff its characteristic class is zero iff the Lie algebra L_ξ is abelian iff the group G_ξ, and thus the group D, is abelian (one implication of the last equivalence is obvious, another follows by the definition of D (cf. [2], Lemma 5)).

By Lemma 3 and Corollary 1, we have

COROLLARY 4. *A compact CW-complex M is homeomorphic to a nil-manifold of class $\leqslant 2$ (i. e., to a coset space of a 1-connected, nilpotent Lie group of class $\leqslant 2$) if and only if M is a total space of a principal T^n-bundle over torus T^k for some n and k.*

6. Consider principal T^1-bundles over 2-torus T^2. The 2-cohomology group $H^2(T^2, Z)$ of T^2 is isomorphic to Z. If $\xi = (M, p, T^2)$ is the principal T^1-bundle, then its characteristic class (Chern class) will be $k\mathscr{H}_1 \wedge \mathscr{H}_2$. The Lie algebra L_ξ is given by $[e_1, e_3] = [e_2, e_3] = 0$, $[e_1, e_2] = -[e_2, e_1] = ke_3$.

If N is the group of nilpotent 3-matrices, i. e.,

$$N = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in R \right\},$$

then the mapping

$$\psi\colon G_\xi \to N\colon xe_1 + ye_2 + ze_3 \to \begin{bmatrix} 1 & x & \dfrac{z}{k} + \dfrac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

is an isomorphism. In fact, it is obviously homeomorphism and so it remains to show that it is also a homomorphism. We have

$$\psi\big((xe_1+ye_2+ze_3)\times(x'e_1+y'e_2+z'e_3)\big)$$

$$= \psi\Big((x+x')e_1+(y+y')e_2+\big(z+z'+\tfrac{1}{2}k(xy'-x'y)\big)e_3\Big)$$

$$= \begin{bmatrix} 1 & x+x' & \dfrac{z+z'}{k}+xy'+\dfrac{1}{2}(xy+x'y') \\ 0 & 1 & y+y' \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & x & \dfrac{z}{k}+\dfrac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x' & \dfrac{z'}{k}+\dfrac{1}{2}x'y' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \psi(xe_1+ye_2+ze_3)\psi(x'e_1+y'e_2+z'e_3).$$

It is easy to see that the image of the group D under this isomorphism is the group of the matrices

$$N_k = \left\{ \begin{bmatrix} 1 & a & \dfrac{c}{k} \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in Z \right\}.$$

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