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CERTAIN TYPES OF AFFINE MOTION IN A FINSLER MANIFOLD. II

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1. Introduction. Certain types of affine motion generated by contra, concurrent, special concircular, recurrent, concircular, and torse forming vector fields in a non-Riemannian manifold of recurrent curvature were discussed by Takano [19], [20]. Following Takano, Sinha [18], Misra [6]–[9], Meher [5], [8], [9], and Kumar [2]–[4] discussed affine motions generated by some of the above types of vector fields in Finsler manifolds of recurrent curvature and some other special Finsler manifolds. These authors obtained various results using mostly the same techniques as those adopted by Takano. However, the problem to find the necessary and sufficient conditions for above vector fields to generate an affine motion could not attract these authors including Takano, and this problem thus remained undiscussed. The present author considered this problem for first four types of vector fields [15]. The aim of the present paper is to discuss the problem for the remaining two types of vector fields in a general Finsler manifold.

2. Preliminaries. Let $F_n(F, g, G)$ be an n-dimensional Finsler manifold of class at least C^7 equipped with a metric function F(1) satisfying the required conditions [17], the corresponding symmetric metric tensor g and Berwald's connection G. The coefficients of Berwald's connection G, denoted by G^i_{jk}, satisfy

$$(2.1)\qquad \text{(a) } G^i_{jk} = G^i_{kj}, \quad \text{(b) } G^i_{jk}\dot{x}^k = G^i_j, \quad \text{(c) } G^i_j\dot{x}^j = 2G^i,$$
$$\text{(d) } \dot{\partial}_j G^i = G^i_j, \quad \text{(e) } \dot{\partial}_k G^i_j = G^i_{jk},$$

(1) Unless otherwise stated, all the geometric objects are supposed to be functions of the line elements $(x^i, \dot{x}^i)$. The indices $i, j, k, \ldots$ take positive integer values from 1 to n.

where $\dot{\partial}_k$ stands for partial differentiation with respect to $\dot{x}^k$. The partial derivatives $\dot{\partial}_h G^i_{jk}$ of the connection coefficients G^i_{jk} constitute a tensor, say G^i_{jkh}, symmetric in its lower indices, and satisfy

(2.2) $$G^i_{jkh}\,\dot{x}^h = 0.$$

The covariant derivative of a tensor T^i_j for the connection G is given by

(2.3) $$\mathscr{B}_k T^i_j = \partial_k T^i_j - (\dot{\partial}_r T^i_j)\, G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk},$$

where $\partial_k \equiv \partial/\partial x^k$. The commutation formulae for the differential operators $\dot{\partial}_k$ and $\mathscr{B}_k$ are given by

(2.4) $$(\dot{\partial}_j \mathscr{B}_k - \mathscr{B}_k \dot{\partial}_j)\, T^i_h = G^i_{jkr} T^r_h - G^r_{jkh} T^i_r,$$

(2.5) $$(\mathscr{B}_j \mathscr{B}_k - \mathscr{B}_k \mathscr{B}_j)\, T^i_h = H^i_{jkr} T^r_h - H^r_{jkh} T^i_r - H^r_{jk} (\dot{\partial}_r T^i_h),$$

where H^i_{jkh} constitute Berwald's curvature tensor. This tensor is skew-symmetric in first two lower indices and positively homogeneous of degree zero in $\dot{X}^i$'s. The tensor H^i_{jk} appearing in (2.5) is connected with the curvature tensor by

(2.6) $$\text{(a) } H^i_{jkh}\,\dot{x}^h = H^i_{jk}, \quad \text{(b) } \dot{\partial}_h H^i_{jk} = H^i_{jkh}.$$

The tensor H^i_{jk} is related with the deviation tensor H^i_j by

(2.7) $$\text{(a) } H^i_{jk}\,\dot{x}^k = H^i_j, \quad \text{(b) } \tfrac{1}{3}(\dot{\partial}_k H^i_j - \dot{\partial}_j H^i_k) = H^i_{jk}.$$

The associate vector y_i of $\dot{x}^i$ satisfies

(2.8) $$\text{(a) } y_i\,\dot{x}^i = F^2, \quad \text{(b) } \dot{\partial}_i F^2 = 2y_i,$$
$$\text{(c) } \dot{\partial}_j y_i = g_{ij}, \quad \text{(d) } y_i H^i_{jk} = 0$$

(see [13]), where g_{ij} are components of the metric tensor g.

Let us consider an infinitesimal transformation

(2.9) $$\bar{x}^i = x^i + \varepsilon v^i(x^j)$$

generated by a vector $v^i(x^j)$, ε being an infinitesimal constant. The Lie derivatives of an arbitrary tensor T^i_j and connection coefficients G^i_{jk} with respect to the infinitesimal transformation (2.9) are given by (see [17] and [22])

(2.10) $$£T^i_j = v^r \mathscr{B}_r T^i_j - T^r_j \mathscr{B}_r v^i + T^i_r \mathscr{B}_j v^r + (\dot{\partial}_r T^i_j)\, \mathscr{B}_s v^r \dot{x}^s,$$

(2.11) $$£G^i_{jk} = \mathscr{B}_j \mathscr{B}_k v^i + H^i_{mjk} v^m + G^i_{jkr} \mathscr{B}_s v^r \dot{x}^s.$$

The operator £ commutes with the differential operators $\mathscr{B}_k$ and $\dot{\partial}_k$ according to

(2.12) $$(£\mathscr{B}_k - \mathscr{B}_k £)\, T^i_j = T^r_j\, £G^i_{rk} - T^i_r\, £G^r_{jk} - (\dot{\partial}_r T^i_j)\, £G^r_k,$$

(2.13) $$(\pounds\mathring{\partial}_k - \mathring{\partial}_k \pounds)\Omega = 0,$$

where Ω is any geometric object.

An infinitesimal transformation, say (2.9), is an *affine motion* if and only if (see [17] and [22])

(2.14) $$\pounds G^i_{jk} = 0;$$

and the vector field v^i is said to be the *generator* of the affine motion. A vector field v^i is said to be *torse forming* or *concircular* according as it satisfies (see [10], [11], [16])

(2.15) $$\mathscr{B}_k v^i = \mu_k v^i + \varrho\delta^i_k$$

or

(2.16) $$\mathscr{B}_k v^i = \mu_k v^i + \varrho\delta^i_k, \qquad \mathscr{B}_j \mu_k = \mathscr{B}_k \mu_j,$$

where μ_k and ϱ are a non-zero covariant vector field and a scalar field, respectively. Izumi [1] proved that the vector field μ_k appearing in (2.16), characterizing a concircular vector field, is in fact $v_k \overset{\text{def}}{=} g_{jk} v^j$. It is easy to see that if we take v_k in place of μ_k, then the second equation of (2.16) is automatically satisfied. Thus a concircular vector field may be characterized by

(2.17) $$\mathscr{B}_k v^i = v_k v^i + \varrho\delta^i_k.$$

An affine motion is said to be *torse forming* or *concircular* according as it is generated by a torse forming vector field or a concircular vector field, respectively.

3. Torse forming affine motion. Let us consider an infinitesimal transformation generated by a torse forming vector v^i characterized by (2.15). Misra et al. [10] proved that, for a Finsler manifold of dimension greater than two (F_n, $n > 2$), ϱ is a function of x^i's only and the vector μ_k satisfies

(3.1) $$\mathring{\partial}_j \mu_k = \mathring{\partial}_k \mu_j.$$

Transvecting (3.1) by $\dot{x}^j$ and using Euler's theorem of homogeneous functions, we get

(3.2) $$\mu_k = \mathring{\partial}_k \mu,$$

where $\mu \overset{\text{def}}{=} \mu_k \dot{x}^k$. Differentiating (2.15) covariantly with respect to x^j we get

(3.3) $$\mathscr{B}_j \mathscr{B}_k v^i = (\mathscr{B}_j \mu_k + \mu_j \mu_k) v^i + \varrho\mu_k \delta^i_j + \varrho_j \delta^i_k,$$

where $\varrho_j \overset{\text{def}}{=} \mathscr{B}_j \varrho$. If the torse forming vector v^i generates an affine motion, we have (2.14). Expanding the left-hand side of (2.14) with the help of (2.11) and (3.3) we get

$$(3.4)\qquad (\mathscr{B}_j \mu_k + \mu_j \mu_k) v^i + \varrho\mu_k \delta^i_j + \varrho_j \delta^i_k + H^i_{mjk} v^m + G^i_{jkr} \mathscr{B}_s v^r \dot{x}^s = 0,$$

which, in view of (2.15) and (2.2), gives

$$(3.5)\qquad (\mathscr{B}_j \mu_k + \mu_j \mu_k) v^i + \varrho\mu_k \delta^i_j + \varrho_j \delta^i_k + H^i_{mjk} v^m + \mu G^i_{jkr} v^r = 0.$$

Transvecting (3.5) by $\dot{x}^k$ and using (2.2) and (2.6a), we have

$$(3.6)\qquad (\mathscr{B}_j \mu + \mu\mu_j) v^i + \varrho\mu\delta^i_j + \varrho_j \dot{x}^i + H^i_{mj} v^m = 0.$$

Transvecting (3.6) by y_i and using (2.8a) and (2.8d), we get

$$(\mathscr{B}_j \mu + \mu\mu_j) y_i v^i + \varrho\mu y_j + \varrho_j F^2 = 0,$$

which implies

$$(3.7)\qquad \mathscr{B}_j \mu + \mu\mu_j = -(\varrho\mu y_j + \varrho_j F^2)/\varphi,$$

where $\varphi \overset{\text{def}}{=} y_i v^i$. From (3.6) and (3.7) we get

$$(3.8)\qquad -(\varrho\mu y_j + \varrho_j F^2) v^i/\varphi + \varrho\mu\delta^i_j + \varrho_j \dot{x}^i + H^i_{mj} v^m = 0.$$

Transvecting (3.8) by $\dot{x}^j$ and using (2.7a) and (2.8a), we have

$$(3.9)\qquad H^i_m v^m = (F^2 v^i - \varphi\dot{x}^i)(\varrho\mu + \varrho_j \dot{x}^j)/\varphi.$$

Thus we see that conditions (3.7) and (3.9) necessarily hold if the torse forming vector v^i generates an affine motion. Conversely, suppose that a torse forming vector v^i, characterized by (2.15), satisfies (3.7) and (3.9). In view of (2.11) and (2.15), the Lie derivative of connection coefficients G^i_{jk} with respect to an infinitesimal transformation generated by the vector v^i is given by

$$(3.10)\qquad \pounds G^i_{jk} = (\mathscr{B}_j \mu_k + \mu_j \mu_k) v^i + \varrho\mu_k \delta^i_j + \varrho_j \delta^i_k + H^i_{mjk} v^m + \mu G^i_{jkr} v^r.$$

Transvecting (3.10) by $\dot{x}^j \dot{x}^k$ and using (2.1b), (2.1c), (2.2), (2.6a), and (2.7a), we get

$$(3.11)\qquad 2\pounds G^i = \dot{x}^j (\mathscr{B}_j \mu + \mu\mu_j) v^i + (\varrho\mu + \varrho_j \dot{x}^j) \dot{x}^i + H^i_m v^m,$$

which, in view of (3.7) and (3.9), gives $\pounds G^i = 0$. Differentiating $\pounds G^i = 0$ partially with respect to $\dot{x}^j$ and using the commutation formula exhibited by (2.13), we get $\pounds G^i_j = 0$. By further partial differentiation with respect to $\dot{x}^k$ and repeating the same process we get (2.14), and hence the transformation considered is an affine motion. This leads to

THEOREM 3.1. *Conditions* (3.7) *and* (3.9) *together are necessary and sufficient for a torse forming vector* v^i *characterized by* (2.15) *to generate an affine motion.*

Let an affine motion be generated by a torse forming vector v^i characterized by (2.15). Operating the equation (2.15) by £, using the commutation formula exhibited by (2.12), and then using (2.14), we get

$$\pounds\mu_k v^i + \pounds\varrho\delta^i_k = 0,$$

which, after transvection by $\dot{x}^k$, gives

$$\pounds\mu v^i + \pounds\varrho\dot{x}^i = 0. \tag{3.12}$$

In his paper [14] the present author proved that if a non-zero vector $v^i(x^j)$ satisfies the equation $av^i + b\dot{x}^i = 0$, then $a = b = 0$. In view of this lemma, (3.12) implies

$$\text{(a)}\ \pounds\mu = 0, \quad \text{(b)}\ \pounds\varrho = 0. \tag{3.13}$$

Differentiating (3.13a) partially with respect to $\dot{x}^k$ and using the commutation formula (2.13), we get

$$\pounds\mu_k = 0. \tag{3.13c}$$

Thus, we have

THEOREM 3.2. *If a torse forming vector v^i characterized by* (2.15) *generates an affine motion, then the Lie derivatives of the vector μ_k and the scalar ϱ vanish.*

It is well known that the curvature tensor H^i_{jkh} is Lie invariant under an affine motion, i.e.,

$$\pounds H^i_{jkh} = 0. \tag{3.14}$$

Transvecting (3.14) by $\dot{x}^k\dot{x}^h$ and using (2.6a) and (2.7a), we get

$$\pounds H^i_j = 0. \tag{3.15}$$

By Theorems 3.1 and 3.2 we have (3.9) and (3.13). Operating the equation (3.9) by £ and using (3.15), we get

$$\varphi(\pounds F^2 v^i - \pounds\varphi\dot{x}^i)(\varrho\mu + \varrho_j\dot{x}^j) + \varphi(F^2 v^i - \varphi\dot{x}^i)(\varrho\pounds\mu + \pounds\varrho_j\dot{x}^j) - \\ -(F^2 v^i - \varphi\dot{x}^i)(\varrho\mu + \varrho_j\dot{x}^j)\pounds\varphi = 0. \tag{3.16}$$

Differentiating (3.13b) covariantly with respect to x^k and using (2.12), we get $\pounds\varrho_j = \pounds\mathscr{B}_j\varrho = \mathscr{B}_j\pounds\varrho = 0$. Using (3.13c) and $\pounds\varrho_j = 0$ in (3.16), we obtain

$$(\varrho\mu + \varrho_j\dot{x}^j)(\varphi\pounds F^2 - F^2\pounds\varphi) = 0, \tag{3.17}$$

which implies at least one of the following equalities:

$$\text{(a)}\ \varrho\mu + \varrho_j\dot{x}^j = 0, \quad \text{(b)}\ \pounds F^2 = \frac{\pounds\varphi}{\varphi}F^2. \tag{3.18}$$

If (3.18a) holds, then the partial differentiation of (3.18a) with respect to $\dot{x}^k$ gives

$$\varrho\mu_k + \varrho_k = 0. \tag{3.19}$$

Differentiating (3.19) covariantly with respect to x^j and using (3.19), we have

$$-\varrho\mu_j\mu_k + \varrho\mathscr{B}_j\mu_k + \mathscr{B}_j\varrho_k = 0. \tag{3.20}$$

Taking the skew-symmetric part of (3.20), using the commutation formula exhibited by (2.5) and using the independence of ϱ from $\dot{x}^i$'s, we conclude that

$$\mathscr{B}_j \mu_k = \mathscr{B}_k \mu_j, \tag{3.21}$$

since $\varrho \neq 0$. Transvecting (3.7) by $\dot{x}^j$ and using (3.18a), we get

$$\dot{x}^j (\mathscr{B}_j \mu + \mu\mu_j) = 0. \tag{3.22}$$

Differentiating (3.22) partially with respect to $\dot{x}^k$ and using (3.21), we obtain

$$\mathscr{B}_k \mu + \mu\mu_k = 0. \tag{3.23}$$

From (3.7) and (3.23) we find

$$\varrho\mu y_j + \varrho_j F^2 = 0. \tag{3.24}$$

Transvecting (3.24) by v^j and using $v^j \varrho_j = v^j \mathscr{B}_j \varrho = £\varrho = 0$ (due to (3.13b)), we have either $\varrho = 0$ or $\mu = 0$, since $y_j v^j \neq 0$. Thus we get a contradiction. Hence (3.18a) does not hold. Therefore (3.18b) necessarily holds. Calculating the Lie derivative of φ with the help of (2.10) and substituting it in (3.18b), we get

$$£F^2 = \left(\varrho + \frac{\mu}{\varphi} v^2\right) F^2,$$

where $v^2 = v_i v^i$, which, in view of (2.10), (2.15) and (2.8b), gives

$$(\mu v^2 - \varrho\varphi) F^2 = 2\mu\varphi^2. \tag{3.25}$$

$\mu v^2 - \varrho\varphi$ cannot be zero, for $\mu v^2 - \varrho\varphi = 0$ in (3.25) implies $\mu = 0$ or $\varphi = 0$. Putting $\mu = 0$ in formula (3.2) we have $\mu_k = 0$, which contradicts the fact that μ_k is non-zero. Again, $\varphi = 0$ and $\mu v^2 - \varrho\varphi = 0$ give $\mu v^2 = 0$. Since $\mu \neq 0$, we have $v^2 = 0$. This means v^i is a zero vector, which is not the case. Dividing (3.25) by $\mu v^2 - \varrho\varphi$, we get

$$F^2 = \frac{2\mu\varphi^2}{\mu v^2 - \varrho\varphi}. \tag{3.26}$$

Thus we have

THEOREM 3.3. *If a torse forming vector field v^i characterized by* (2.15) *generates an affine motion in a Finsler manifold, then the square of the metric function of the manifold is given by* (3.26).

Now we shall point out some conditions which whenever satisfied by a torse forming vector, it cannot generate an affine motion. These conditions are given in the following

THEOREM 3.4. *A torse forming vector v^i characterized by* (2.15) *and* *satisfying one of the following conditions cannot generate an affine motion in a Finsler manifold:*

$$\text{(i) } H^i_{jkh} v^h = 0, \quad \text{(ii) } \mathscr{B}_j \mu_k + \mu_j \mu_k = 0.$$

Proof. It is established that for a vector $v^i(x^j)$ the conditions (i) and $H^i_{hjk} v^h = 0$ are equivalent [15]. Hence (i) implies $H^i_{mjk} v^m = 0$, which after transvection by $\dot{x}^j \dot{x}^k$ gives $H^i_m v^m = 0$. If the torse forming vector v^i generates an affine motion, we have (3.9), which, in view of $H^i_m v^m = 0$, gives at least one of the conditions: (3.18a) and

$$F^2 v^i - \varphi \dot{x}^i = 0. \tag{3.27}$$

We have already seen in the proof of Theorem 3.3 that (3.18a) does not hold. By Lemma 5.1 in [14], condition (3.27) implies $v^i = 0$. Thus we get a contradiction.

Transvecting the condition (ii) by $\dot{x}^k$ and using it in (3.7), we get (3.24), which after transvection by v^j implies either $\varrho = 0$ or $\mu = 0$. Thus we get a contradiction.

We have already seen that a recurrent Finsler manifold does not admit any torse forming vector field [16]. Therefore, the question for existence of an affine motion generated by a torse forming vector field in a recurrent Finsler manifold does not arise.

4. Concircular affine motion. Let us consider a concircular vector field characterized by (2.17). Adopting the procedure of Section 3, we may easily prove the following:

If a concircular vector field v^i characterized by (2.17) generates an affine motion, we have

$$\text{(a) } \pounds v_k = 0, \quad \text{(b) } \pounds F^2 = \frac{\pounds\varphi}{\varphi} F^2. \tag{4.1}$$

These equations are analogues of (3.13c) and (3.18b), respectively.

In this case $\varphi = y_i v^i = v_k \dot{x}^k = \mu$. Transvecting (4.1a) by $\dot{x}^k$, we get $\pounds\varphi = 0$. Consequently, (4.1b) reduces to $\pounds F^2 = 0$. Expanding $\pounds F^2 = 0$ with the help of (2.10) and using (2.8) and the fact that the metric function F is a covariant constant, we get

$$\mu^2 + \varrho F^2 = 0. \tag{4.2}$$

Differentiating (4.2) partially with respect to $\dot{x}^k$ and using the fact that ϱ is a function of x^i's only, we have

$$2\mu \dot{\partial}_k \mu + \varrho \dot{\partial}_k F^2 = 0. \tag{4.3}$$

In this case, equation (3.2) reduces to $\dot{\partial}_k \mu = v_k$. Using this and (2.8b), we get

$$\mu v_k + \varrho y_k = 0. \tag{4.4}$$

Transvecting (4.4) by g^{ik} (the associate of g_{ij}), we have

$$\mu v^i + \varrho \dot{x}^i = 0. \tag{4.5}$$

In view of Lemma 5.1 in [14], equation (4.5) gives $\varrho = 0$, a contradiction. Hence there exists no concircular vector field which generates an affine motion in a Finsler manifold. Thus we have

THEOREM 4.1. *A concircular vector field characterized by* (2.17) *cannot generate an affine motion in a Finsler manifold.*

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