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ON SOME PROBLEMS OF J. SŁOMIŃSKI CONCERNING EQUATIONS IN QUASI-ALGEBRAS

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In this paper we give a solution of problems 5 and 6 of Słomiński [5].

All needed definitions which are not given in this paper can be found in [2]-[5].

Let $\boldsymbol{P}^a(\boldsymbol{G})$ be the Peano-algebra of type G generated by the set $X = (x_\xi, \xi < a)$ considered as the set of distinct variables. The pairs $\langle p, q\rangle$, where p and q belong to $P^a(G)$, are called *a-ary equations of type G*, or, to be short, simply *equations*. If $\langle p, q\rangle$ is an a-ary equation of type G, then it will also be denoted by $\ulcorner p = q \urcorner$. Let $\ulcorner p = q \urcorner$ be an arbitrary a-ary equation of type G and let $\boldsymbol{A}$ be an arbitrary quasi-algebra of type G.

Słomiński [5] has given three concepts of validity of equations for quasi-algebras.

Definition 1. Equation $\ulcorner p = q \urcorner$ is said to be *weakly valid in the quasi-algebra* $\boldsymbol{A}$ if for all sequences $\boldsymbol{a} = (a_\xi, \xi < a)$ in A^a we have ${}_{\boldsymbol{A}}p(a_\xi, \xi < a) = {}_{\boldsymbol{A}}q(a_\xi, \xi < a)$ provided $\boldsymbol{a} \in D({}_{\boldsymbol{A}}p) \cap D({}_{\boldsymbol{A}}q)$.

Definition 2. Equation $\ulcorner p = q \urcorner$ is said to be *valid in quasi-algebra* $\boldsymbol{A}$ if $D({}_{\boldsymbol{A}}p) = D({}_{\boldsymbol{A}}q)$ and ${}_{\boldsymbol{A}}p(a_\xi, \xi < a) = {}_{\boldsymbol{A}}q(a_\xi, \xi < a)$ for each sequence $(a_\xi, \xi < a)$ in $D({}_{\boldsymbol{A}}p) = D({}_{\boldsymbol{A}}q)$.

Definition 3. Equation $\ulcorner p = q \urcorner$ is said to be *strongly valid in the quasi-algebra* $\boldsymbol{A}$ if we have $p \sim_{\boldsymbol{a}} q$ for each sequence $\boldsymbol{a}$ in A^a, where $\sim_{\boldsymbol{a}}$ denotes the least congruence relation $\sim$ of the Peano algebra $\boldsymbol{P}^a(\boldsymbol{G})$ such that for all p and q in $P^a(G)$ we have

(s) $\quad p \sim q$ if $\boldsymbol{a} \in D({}_{\boldsymbol{A}}p) \cap D({}_{\boldsymbol{A}}q)$ and ${}_{\boldsymbol{A}}p(a_\xi, \xi < a) = {}_{\boldsymbol{A}}q(a_\xi, \xi < a)$ (1).

All three concepts of validity of equations in quasi-algebras are equivalent in the case of algebras.

(1) In [5], p. 17, in condition (b_9) there is a misprint. Obviously (b_9) should have the form (s).

Let E_0 be any set of equations of type G. Denote by $\mathrm{Cn}(E_0)$ the set of all consequences of E_0. By $\mathrm{wG}(E_0)$, $\mathrm{wG}(\mathrm{Cn}(E_0))$, $\mathrm{G}(E_0)$, $\mathrm{G}(\mathrm{Cn}(E_0))$, $\mathrm{sG}(E_0)$, $\mathrm{sG}(\mathrm{Cn}(E_0))$ and $\mathrm{G}^*(E_0)$ we denote the classes of all weak E_0-quasi-algebras, weak $\mathrm{Cn}(E_0)$-quasi-algebras, E_0-quasi-algebras, $\mathrm{Cn}(E_0)$-quasi-algebras, strong E_0-quasi-algebras, strong $\mathrm{Cn}(E_0)$-quasi-algebras and E_0-algebras, respectively. A quasi-algebra $\boldsymbol{A}$ is said to be a (*weak*, *strong*) *E_0-quasi-algebra* if all the equations belonging to E_0 are (weakly strongly) valid in the quasi-algebra $\boldsymbol{A}$.

In what follows we give a characterization of those sets E_0 of equations for which we have

$$\mathrm{G}(\mathrm{Cn}(E_0)) = \mathrm{wG}(E_0) \quad \text{(Problem 5 in [5])}, \tag{1}$$

$$\mathrm{G}(E_0) = \mathrm{wG}(E_0) \quad \text{(Problem 6 in [5])}. \tag{2}$$

The following relations are obvious:

$$\mathrm{G}^*(E_0) \subset \mathrm{G}(\mathrm{Cn}(E_0)) \subset \mathrm{G}(E_0) \subset \mathrm{wG}(E_0).$$

Moreover, let us observe that there does not exist any set E_0, for which $\mathrm{G}^*(E_0) = \mathrm{wG}(E_0)$ (Słomiński [5]), and that for every set E_0 of equations we have $\mathrm{G}^*(\mathrm{Cn}(E_0)) = \mathrm{G}^*(E_0)$.

The last observation is trivial.

THEOREM 1. *Let E_0 be an arbitrary set of equations of type G. Then each weak E_0-quasi-algebra $\boldsymbol{A}$, where $\overline{\overline{A}} \geqslant 2$, is E_0-quasi-algebra (problem (2)) if and only if the set E_0 is empty, or E_0 contains only equations of the form* $\ulcorner p = p \urcorner$.

Proof. The sufficiency is trivial.

Necessity. Let E_0 be an arbitrary set of equations and let $\ulcorner p = q \urcorner \in E_0$, where $p \neq q$ and p, q are not the variables at the same time. We show that there is a weak E_0-quasi-algebra which is not E_0-quasi-algebra. Let P_0 denote the set of the terms which are in the equations belonging to E_0 (i.e. if $\ulcorner r = s \urcorner \in E_0$, then $r, s \in P_0$). The set P_0 is not empty, because $p, q \in P_0$. The set P_0 contains the minimal elements (with respect to the relation of partialy order "to be subterm" in the algebra $\boldsymbol{P}^{a}(G)$). Let p_0 be a minimal element belonging to P_0. If P_0 contains variables, then let p_0 denote a variable, else — an arbitrary minimal term. For to prove the existence of the quasi-algebra in which the equations with the set E_0 there are only weak valid, it suffices to show that there is a quasi-algebra $\boldsymbol{A}$ of type G in which $D({}_{A}p_0) \neq \emptyset$ and for each term $q \in P_0$, where $p_0 \neq q$ and q is not a variable (term q is not, obviously, subterm of the term p_0), we have $D({}_{A}q) = \emptyset$.

Such a quasi-algebra is, for instance, the following relative subquasi-algebra $\boldsymbol{P}(p_0)$ of the Peano-algebra $\boldsymbol{P}^{a}(G)$:

1° $X \subset P(p_0)$, where X denotes the set of generators of the Peano-algebra $\boldsymbol{P}^a(\boldsymbol{G})$.

2° Moreover, the set $P(p_0)$ contains only the term p_0 and each subterm of the term p_0.

Because each term $p \in P^a(G)$ may be reckoned as a value of the operation ${}_{\boldsymbol{P}^a(G)}p$ (induced by the term p in the Peano-algebra $\boldsymbol{P}^a(\boldsymbol{G})$) for the argument $\boldsymbol{x} = (x_\xi, \xi < a)$, for which each element in X occurs once and only once, so $\boldsymbol{x} \in D({}_{\boldsymbol{P}(p_0)}p_0)$ and $D({}_{\boldsymbol{P}(p_0)}q) = \emptyset$, where p_0 and q are the terms described above. Since each term $p \in P^a(G)$ may be, for some set E_0 of equations, the minimal term, we must prove that a quasi-algebra $\boldsymbol{P}(p)$ always exists.

Let P_1 denote the set of terms $p \in P^a(G)$ for which the corresponding quasi-algebra $\boldsymbol{P}(p)$ exists. We prove, by induction, that $P_1 = P^a(G)$, namely we show that P_1 contains X and $\boldsymbol{P}_1$ is a subalgebra of the Peano-algebra $\boldsymbol{P}^a(\boldsymbol{G})$.

1° For each $x \in X$, $x \in P_1$, because the corresponding quasi-algebra $\boldsymbol{P}(x)$ always exists. This is, for instance, the relative subquasi-algebra of the Peano-algebra $\boldsymbol{P}^a(\boldsymbol{G})$ such that $P(x) = X$.

2° Let $p_\sigma \in P_1$, $\sigma < n(g)$ $(g \in G)$. We denote by $\boldsymbol{P}(p_\sigma)$, $\sigma < n(g)$, the corresponding quasi-algebras and set $p = g_{\boldsymbol{P}^a(G)}(p_\sigma, \sigma < n(g))$.

We define $\boldsymbol{P}(p)$ as a relative subquasi-algebra of the Peano-algebra $\boldsymbol{P}^a(\boldsymbol{G})$, where

$$P(p) = \bigcup_{\sigma < n(g)} P(p_\sigma) \cup \{p\}.$$

Obviously, $\boldsymbol{x} \in D({}_{\boldsymbol{P}(p)}p)$ and $D({}_{\boldsymbol{P}(p)}q) = \emptyset$ for every term q which is not a subterm of the term p.

Thus $\boldsymbol{P}_1$ is the subalgebra of the Peano-algebra $\boldsymbol{P}^a(\boldsymbol{G})$ and $X \subset P_1$. Therefore $\boldsymbol{P}_1 = \boldsymbol{P}^a(G)$. This completes the proof of Theorem 1.

THEOREM 2. *Each weak E_0-quasi-algebra is a* $\mathrm{Cn}(E_0)$*-quasi-algebra (problem* (1)*) if and only if the set E_0 of equations is empty or contains only equations of the form* $\ulcorner p = p \urcorner$.

Proof. The sufficiency is obvious.

Necessity. If E_0 contains an equation $\ulcorner p = q \urcorner$, where $p \neq q$, then:

1° If E_0 does not contain the equation $\ulcorner x = y \urcorner$, where x and y are distinct variables, then it follows from Theorem 1 that there exist a weak E_0-quasi-algebra which is not an E_0-quasi-algebra. Thus it is not a $\mathrm{Cn}(E_0)$-quasi-algebra.

2° If $\ulcorner x = y \urcorner \in E_0$, where x and y are variables (obviously distinct), then the one-element discrete quasi-algebra $\{\boldsymbol{a}\}_d$ of type G determined by any set $\{a\}$ is a weak E_0-quasi-algebra, but it is not a $\mathrm{Cn}(E_0)$-quasi-algebra, because for any $g \in G$ the expression on the right-hand side of equation $\ulcorner x = g_{P\alpha(G)}(x_\sigma, \sigma < n(g))\urcorner$ is not defined for any sequence $(a_\sigma, \sigma < n(g))$, where $a_\sigma = a$, $\sigma < n(g)$.

Theorem 2 is thus proved.

REFERENCES

[1] G. Grätzer, *Universal algebra*, D. van Nostrand, 1968.

[2] J. Słomiński, *The theory of abstract algebras with infinitary operations*, Rozprawy Matematyczne 18 (1959).

[3] — *A theory of extensions of quasi-algebras to algebras*, Rozprawy Matematyczne 40 (1964).

[4] — *A theory of P-homomorphisms*, Colloquium Mathematicum 14 (1966), p. 135-162.

[5] — *Peano-algebras and quasi-algebras*, Dissertationes Mathematicae 57 (1968).

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