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THE STRUCTURE OF L-IDEALS OF MEASURE ALGEBRAS. II

BY

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1. Introduction. Let G be an L.C.A. group and let $\hat{G}$ be its dual group. Let $M(G)$ be the measure algebra on G and let $L^1(G)$ be the group algebra on G. By [12], there exist a compact topological semigroup S and an isomorphism θ of $M(G)$ into $M(S)$ such that:

(i) $\theta(M(G))$ is weak*-dense in $M(S)$;

(ii) $\hat{S}$, the set of all non-zero continuous semicharacters on S, separates the points of S;

(iii) $M(G) \ni \mu \to \int_S f d\theta\mu$ $(f \in \hat{S})$ is a non-zero complex homomorphism of $M(G)$;

(iv) for a non-zero complex homomorphism F of $M(G)$, there is an $f \in \hat{S}$ such that

$$F(\mu) = \int_S f d\theta\mu \quad \text{for every } \mu \in M(G).$$

We can consider that $\hat{S}$ is the maximal ideal space of $M(G)$ and

$$\hat{S} \supset \hat{G} = \{f \in \hat{S};\ |f| = 1 \text{ on } S\}.$$

The Gelfand transform of $\mu \in M(G)$ is given by

$$\hat{\mu}(f) = \int_S f d\theta\mu \quad (f \in \hat{S}).$$

For $\mu \in M(G)$, we put

$$L^1(\mu) = \{\lambda \in M(G);\ \lambda \text{ is absolutely continuous with respect to } \mu\ (\lambda \ll \mu)\}.$$

A closed subspace (ideal, subalgebra) I of $M(G)$ is said to be an *L-subspace* (*L-ideal*, *L-subalgebra*) if $L^1(\mu) \subset I$ for every $\mu \in I$. An L-ideal (L-subalgebra) N is called *prime* if

$$N^{\perp} = \{\mu \in M(G);\ \mu \text{ is singular with respect to } N\ (\mu \perp N)\}$$

is an L-subalgebra (L-ideal). We denote by $M_0(G)$ the set of all measures whose Fourier-Stieltjes transforms vanish at infinity of $\hat{G}$. Then $M_0(G)$ is an L-ideal, but $M_0(G)$ is not prime. For a subset N of $M(G)$, we put

$$Z(N) = \{f \in \hat{S};\ \hat{\mu}(f) = 0 \text{ for every } \mu \in N\}.$$

For a subset E of $\hat{S}$, we put

$$I(E) = \{\mu \in M(G);\ \hat{\mu} = 0 \text{ on } E\};$$

then $I(E)$ is a closed ideal of $M(G)$. A subset E of $\hat{S}$ is called an *ideal* of $\hat{S}$ if $fE \subset E$ for every $f \in S$. By [11], if N is an L-subspace, then $Z(N)$ is a closed ideal of $\hat{S}$, and if E is an ideal of $\hat{S}$, then $I(E)$ is an L-ideal. We note that if N is an L-ideal, then $N \subset I(Z(N))$ and $Z(N) = Z(I(Z(N)))$.

For a closed ideal N of $M(G)$, we put $\tilde{N} = \{\mu \in N;\ L^1(\mu) \subset N\}$. Then $\tilde{N}$ is the largest L-ideal of $M(G)$ which is contained in N. Since $L^1(G)$ is the smallest L-ideal of $M(G)$, we have

$$L^1(G) \subset \tilde{N} \subset N \quad \text{and} \quad Z(\tilde{N}) \supset Z(N) \quad \text{if } \tilde{N} \neq \{0\}.$$

There are many non-L, closed ideals N of $M(G)$ such that $Z(\tilde{N}) = Z(N)$ (see [7] and [8]).

In Section 2, we prove

THEOREM A. *There exists $\mu \in M(G)$ such that*

(a) *$Z(I(\mu))$ is a closed ideal of $\hat{S}$,*

(b) *$Z(\tilde{I}(\mu)) \supsetneqq Z(I(\mu))$,*

where $I(\mu)$ is the closed ideal of $M(G)$ generated by μ, and $\tilde{I}(\mu) = (I(\mu))^{\sim}$.

In the previous paper [8], the author showed that there exist L-ideals I_1 and I_2 of $M(G)$ which satisfy

(v) $I_1 \subsetneqq I_2$ and $Z(I_1) = Z(I_2)$,

(vi) there are no L-ideals I such that $I_1 \subsetneqq I \subsetneqq I_2$.

On the other hand, there is a closed ideal I of $M(G)$, $I \subset L^1(G)$, which satisfies

(c) for a closed ideal I_1, $Z(I_1) = Z(I)$ implies $I_1 = I$.

Let us consider an analogous condition:

(d) for an L-ideal I_1 of $M(G)$, $Z(I_1) = Z(I)$ implies $I_1 = I$.

It is natural to ask the following two questions:

Is there a proper closed ideal I of $M(G)$ such that $I \not\subset L^1(G)$ and I satisfies (c)? **(P 1184)**

Is there a non-zero proper L-ideal I of $M(G)$ which satisfies (d)? **(P 1185)**

We cannot answer these questions for L.C.A. groups, but in Section 3 we prove

THEOREM B. *Let G be a non-metrizable compact Abelian group. Then there exists a proper L-ideal I of $M(G)$ which satisfies* (d).

In Section 4 we study $M_0(G)$-type L-ideals. For $E \subset \hat{G}$, we put

$$M_E^0 = \{\mu \in M(G);\ L^1(\mu)^{\wedge}|_E \subset C_0(E)\},$$

where $C_0(E)$ is the set of all continuous functions on E which vanish at infinity of E. Then we have $M_0(G) \subset M_E^0 \subset M(G)$. If $E = \hat{G}$, then $M_E^0 = M_0(G)$. And if E is a compact subset of $\hat{G}$, then $M_E^0 = M(G)$. It is easy to show that M_E^0 is an L-ideal of $M(G)$. It is a question whether there exists an E satisfying

(e) $M_0(G) \subsetneqq M_E^0 \subsetneqq M(G)$.

THEOREM C. *If G is a compact metrizable Abelian group, then* (e) *holds if and only if E is an infinite subset of $\hat{G}$ and satisfies*

(f) $\hat{G} \neq \bigcup_{i=1}^{n} \{\gamma_i + (E \cup -E)\}$ *for each finite subset* $\{\gamma_1, \ldots, \gamma_n\}$ *of* $\hat{G}$.

In such a case, M_E^0 is not a prime L-ideal (cf. [5]).

2. Proof of Theorem A. First, we assume that G is an infinite compact metrizable Abelian group. We put $\hat{G} = \{\gamma_1, \gamma_2, \ldots\}$. For a subset E of $\hat{G}$ and a positive integer n, we put

$$E(0) = \{0\}$$

and

$$E(n) = \{\delta_1 x_1 + \ldots + \delta_n x_n;\ \delta_i = \pm 1, x_i \in E\ (i = 1, \ldots, n) \text{ and } x_i \neq x_j \text{ if } i \neq j\}.$$

A subset D of $\hat{G}$ is said to be ***dissociate*** if it does not contain 0 and if the equality

$$\varepsilon_1 x_1 + \ldots + \varepsilon_m x_m = 0, \quad \text{where } \varepsilon_i \in \{-2, -1, 0, 1, 2\},\ x_i \in D,$$

implies the equations

$$\varepsilon_1 x_1 = \varepsilon_2 x_2 = \ldots = \varepsilon_m x_m = 0.$$

Let D' be an infinite dissociate subset of $\hat{G}$ ([9], p. 21). Let $D \subset D'$ be such that D and $D' \setminus D$ are infinite sets. Then D is also a dissociate set. It is well known that there is a Riesz product $\lambda \in M(G)$ ($\lambda \geqslant 0$) such that

$$\hat{\lambda}(\gamma) = \begin{cases} \left(\dfrac{1}{3}\right)^n & \text{if } \gamma \in D(n), \\ 0 & \text{if } \gamma \in \hat{G} \setminus [D], \text{ where } [D] = \bigcup_{n=0}^{\infty} D(n). \end{cases}$$

By the deep work of Brown [2] (Theorem 1), such a measure λ is a tame measure, that is, for each $f \in \hat{S}$ there exist $\gamma \in \hat{G}$ and a complex number a such that $|a| \leqslant 1$, $f = a\gamma$ a.e. $d\theta\lambda$, and there exists an $f \in \hat{S}$ such that

$0 < |f| < 1$ a.e. $d\theta\lambda$. We note that

$$(1) \qquad \bigcup_{i=1}^{N} (\gamma_i + [D]) \subsetneqq \hat{G} \quad \text{for every positive integer } N,$$

since D' is an infinite dissociate set. Using the above-mentioned facts and notation, we construct a measure μ which satisfies (a) and (b).

Let $f \in \hat{S}$ be such that $0 < f = a < 1$ a.e. $d\theta\lambda$. Then $\theta\lambda$ is concentrated on $E = \{x \in S;\ f(x) = a\}$. We put $E(c_1, c_2) = \{x \in S;\ c_1 < f(x) \leqslant c_2\}$ for positive numbers c_1 and c_2 $(c_1 < c_2)$.

LEMMA 1. *If I is a closed ideal of $M(G)$ and $Z(I) \cap \hat{G} \neq \emptyset$, then $L^1(\mu) \not\subset I$ for every $\mu \in I$.*

The lemma follows easily from [11].

LEMMA 2. *If $\nu \in I(\lambda)$ and $0 < c_1 < c_2$, then $\nu' \in I(\lambda)$, where ν' is the part of ν such that $\theta(\nu')$ is concentrated on $E(c_1, c_2)$.*

Proof. Since $\nu \in I(\lambda)$, there exists a sequence $\{\nu_n\}_{n=1}^{\infty}$ of $M(G)$ such that $\lambda * \nu_n \to \nu$ $(n \to \infty)$. We put $\nu_n = \nu_n^0 + \nu_n^{00}$, where $\theta\nu_n^0$ is concentrated on $E(c_1 a^{-1}, c_2 a^{-1})$ and $\theta\nu_n^{00}$ is concentrated on $S \setminus E(c_1 a^{-1}, c_2 a^{-1})$. Since $\theta(\nu_n^{00} * \lambda)$ is concentrated on $S \setminus E(c_1, c_2)$, we have $\nu_n^{00} * \lambda \perp \nu'$. Then $\nu_n^0 * \lambda \to \nu'$, so that we have $\nu' \in I(\lambda)$.

LEMMA 3. *We put*

$$I_0 = I\Big(\bigcup_{n=1}^{\infty} I(\gamma_n \lambda^n)\Big),$$

where $I(M)$, $M \subset M(G)$, is the closed ideal of $M(G)$ generated by M. Then

(2) $\quad Z(I_0)$ *is a closed ideal of* $\hat{S}$,

(3) $\quad Z(\tilde{I}_0) \supsetneqq Z(I_0)$.

Proof. Let $g \in Z(I_0)$. Suppose that $g \neq 0$ a.e. $d\theta\lambda$. Since λ is a tame measure, there are $\gamma \in \hat{G}$ and $c \neq 0$ such that $g = c\gamma$ a.e. $d\theta\lambda$. Then

$$(\gamma_n \lambda^n)\hat{}\,(g) = c^n \big(\hat{\lambda}(\gamma - \gamma_n)\big)^n \neq 0$$

for some positive integer n. This contradicts $g \in Z(I_0)$. Hence $g \in Z(I_0)$ implies $g = 0$ a.e. $d\theta\lambda$. On the other hand, if $g \in \hat{S}$ with $g = 0$ a.e. $d\theta\lambda$, then $g \in Z(I_0)$. Thus (2) is proved.

Next we show (3). We put

$$M(f) = \{\mu \in M(G);\ \theta\mu \text{ is concentrated on } J(f)\},$$

where $J(f) = \{x \in S;\ f(x) = 0\}$. Then we have $Z\big(M(f)\big) \not\subset Z(I_0)$. If we show that $M(f) \supset \tilde{I}_0$, then $Z(\tilde{I}_0) \supsetneqq Z(I_0)$. Suppose that $M(f) \not\supset \tilde{I}_0$. There exists a $\nu \in I_0$ $(\nu \neq 0)$ such that $\theta\nu$ is concentrated on $\{x \in S;\ 0 < f(x) < 1\}$ and $L^1(\nu) \subset I_0$. We may assume that $\theta\nu$ is concentrated on $E(c_1, c_2)$ for some c_1, c_2, $0 < c_1 < c_2 \leqslant a$. Let $\sigma \in L^1(\nu)$. Since I_0 coincides with the linear span of

$$\Big\{\sum_{k=1}^{n} I(\gamma_k \lambda^k);\ n = 1, 2, \ldots\Big\},$$

there is a sequence $\{\sigma_n\}_{n=1}^{\infty}$ of $M(G)$ such that

$$\sigma_n \in \sum_{k=1}^{n} I(\gamma_k \lambda^k) \quad \text{and} \quad \sigma_n \to \sigma \quad (n \to \infty).$$

We put

$$\sigma_n = \sum_{k=1}^{n} \sigma_{n,k}, \quad \text{where } \sigma_{n,k} \in I(\gamma_k \lambda^k),$$

and

$$\sigma_{n,k} = \sigma'_{n,k} + \sigma''_{n,k},$$

where $\theta\sigma'_{n,k}$ is concentrated on $E(c_1, c_2)$ and $\theta\sigma''_{n,k}$ is concentrated on $S \setminus E(c_1, c_2)$ for $1 \leqslant k \leqslant n$ $(n = 1, 2, \ldots)$. Then, by Lemma 2, we have $\sigma'_{n,k} \in I(\gamma_k \lambda^k)$. Since $\theta\lambda^k$ is concentrated on $\{x \in S; f(x) = a^k\}$, we conclude that if $a^k \leqslant c_1$, then $\sigma'_{n,k} = 0$. We choose a natural integer $s > 0$ such that $a^s \leqslant c_1$ and $a^{s-1} > c_1$. Then

$$\sigma'_n = \sum_{k=1}^{n} \sigma'_{n,k} \in \sum_{k=1}^{s} I(\gamma_k \lambda^k) \quad \text{and} \quad \sigma'_n \to \sigma.$$

Hence

$$\sigma \in I\Big(\sum_{k=1}^{s} I(\gamma_k \lambda^k)\Big) \quad \text{and} \quad L^1(\nu) \subset I\Big(\sum_{k=1}^{s} I(\gamma_k \lambda^k)\Big).$$

Since, on account of (1),

$$\begin{aligned} \hat{G} \cap Z\Big(I\Big(\sum_{k=1}^{s} I(\gamma_k \lambda^k)\Big)\Big) &= \hat{G} \cap \bigcap_{k=1}^{s} Z(\gamma_k \lambda^k) = \bigcap_{k=1}^{s} \big(\hat{G} \setminus (\gamma_k + [D])\big) \\ &= \hat{G} \setminus \bigcup_{k=1}^{s} (\gamma_k + [D]) \neq \emptyset, \end{aligned}$$

we have

$$L^1(\nu) \not\subset I\Big(\sum_{k=1}^{s} I(\gamma_k \lambda^k)\Big)$$

by Lemma 1. This is a contradiction, so that $\tilde{I}_0 \subset M(f)$, and the proof is completed.

PROPOSITION 1. *If G is an infinite compact metrizable Abelian group, then there exists a $\mu \in M(G)$ which satisfies* (a) *and* (b).

Proof. We put

$$\mu = \sum_{n=1}^{\infty} a_n \gamma_n \lambda^n, \quad \text{where } a_n > 0 \ (n = 1, 2, \ldots) \text{ and } \sum_{n=1}^{\infty} a_n < \infty.$$

Then $I(\mu) \subset I_0$ and $Z(I_0) = Z(I(\mu))$, since $\hat{\mu}(\gamma) \neq 0$ for every $\gamma \in \hat{G}$. This shows (a). Since

$$Z(\tilde{I}(\mu)) \supset Z(\tilde{I}_0) \supsetneqq Z(I_0),$$

we have $Z(\tilde{I}(\mu)) \supsetneqq Z(I(\mu))$.

Let λ be a tame measure on the real line R which is constructed by Brown [1], p. 503. Then we have $\hat{\lambda}(\gamma) > 0$ for $\gamma \in \hat{R} = R$ and there exists an $f \in \hat{S}$ such that $f = a$ a.e. $d\theta\lambda$, where a is a positive number, $0 < a < 1$. Let $\bar{R}^B$ be the Bohr compactification of $\hat{R}$; then there is a continuous isomorphism β of $\hat{R}$ into $\bar{R}^B$ ([10], p. 30). Fix $x_0 \in \bar{R}^B \setminus \beta(\hat{R})$. Then there are neighborhoods $\{U_n\}_{n=1}^{\infty}$ of x_0 in $\bar{R}^B$ such that

$$\Big(\bigcap_{n=1}^{\infty} U_n\Big) \cap \beta(\hat{R}) = \emptyset \quad \text{and} \quad U_n \supset U_{n+1} \quad (n = 1, 2, \ldots).$$

In fact, let E_n $(n = 1, 2, \ldots)$ be compact subsets of $\hat{R}$ such that

$$\bigcup_{n=1}^{\infty} E_n = \hat{R} \quad \text{and} \quad E_n \subset E_{n+1} \quad (n = 1, 2, \ldots).$$

Let U_n be a neighborhood of x_0 in $\bar{R}^B$ such that $\beta^{-1}(U_n) \cap E_n = \emptyset$. We can take $\{U_n\}_{n=1}^{\infty}$ such that $U_n \supset U_{n+1}$. Then

$$\Big(\bigcap_{n=1}^{\infty} U_n\Big) \cap \beta(\hat{R}) = \emptyset.$$

We can choose a sequence $\{\mu_n\}_{n=1}^{\infty}$ of $M_d(R)$ such that $\|\mu_n\| \leqslant 10$, $\hat{\mu}_n(\gamma) \geqslant 0$ for every $\gamma \in \bar{R}^B$, $\hat{\mu}_n = 0$ on a neighborhood of $x_0 \in \bar{R}^B \setminus \beta(\hat{R})$ and $\hat{\mu}_n(\gamma) > 0$ for every $\gamma \in \bar{R}^B \setminus U_{n+1}$ (see [10], p. 49), where $M_d(R)$ is the set of all discrete measures on R. We put

$$I_0 = I\Big(\bigcup_{n=1}^{\infty} I(\mu_n * \lambda^n)\Big).$$

We note that

$$\hat{R} \cap Z\Big(\bigcup_{n=1}^{N} I(\mu_n * \lambda^n)\Big) \neq \emptyset$$

for every positive integer N and

$$\hat{R} \cap Z\Big(\bigcup_{n=1}^{\infty} I(\mu_n * \lambda^n)\Big) = \emptyset.$$

LEMMA 4. *I_0 satisfies* (2) *and* (3).

The proof proceeds like that of Lemma 3.

PROPOSITION 2. *There exists a $\mu \in M(R)$ which satisfies* (a) *and* (b).

This follows from Lemma 4 by an argument analogous to that used in the proof of Proposition 1.

It remains to consider the general case.

LEMMA 5. *Let $H \subset G$, where H is an open subgroup. If there is a $\mu_0 \in M(H)$ which satisfies* (a) *and* (b), *then there exists a $\mu \in M(G)$ which satisfies* (a) *and* (b).

Proof. We can consider that $M(H) \subset M(G)$ and $\mu_0 \in M(G)$. Let $f \in \hat{S}$, the maximal ideal space of $M(G)$, be such that $f \neq 0$ a.e. $d\theta\lambda$ for some $\lambda \in I(\mu_0)$. Then $f \neq 0$ a.e. $d\theta\mu_0$. Since $\tilde{f} = f|_{M(H)}$ is a non-zero complex homomorphism of $M(H)$, we have $\hat{\mu}_0(\tilde{f}) \neq 0$. This implies that $Z(I(\mu_0))$ is a closed ideal of $\hat{S}$. Let $\hat{S}_1$ be the maximal ideal space of $M(H)$ and let $I_1(\mu_0)$ be the closed ideal of $M(H)$ generated by μ_0. Since $Z(\tilde{I}_1(\mu_0)) \supsetneqq Z(I_1(\mu_0))$ in $\hat{S}_1$, there is a $g \in \hat{S}_1$ such that $g \in Z(\tilde{I}_1(\mu_0)) \setminus Z(I_1(\mu_0))$. Since H is an open subgroup, there is a $\tilde{g} \in \hat{S}$ such that $\hat{\lambda}(g) = \hat{\lambda}(\tilde{g})$ for every $\lambda \in M(H)$ ([3], Lemma 2) and $\hat{\sigma}(\tilde{g}) = 0$ for every $\sigma \in \tilde{I}(\mu_0)$. Then we have $\tilde{g} \in Z(\tilde{I}(\mu_0))$. Since $g \notin Z(I_1(\mu_0))$, we obtain $\tilde{g} \notin Z(I(\mu_0))$. Hence $Z(\tilde{I}(\mu_0)) \supsetneqq Z(I(\mu_0))$.

LEMMA 6. *Let H_1 and H_2 be non-discrete L.C.A. groups. If there is a $\mu_0 \in M(H_1)$ which satisfies* (a) *and* (b), *then there exists a $\mu \in M(G)$ which satisfies* (a) *and* (b), *where $G = H_1 \times H_2$.*

Proof. We put $\mu = \mu_0 \times \delta_0$, where δ_0 is a unit point mass at $0 \in H_2$. We change the topology of G so that $H_1 \cong H_1 \times \{0\}$ is an open subgroup. We denote its L.C.A. group by G_{H_1} (see [10], p. 64). By Lemma 5, for $I_2(\mu)$ being the closed ideal in $M(G_{H_1})$ generated by μ, $Z(I_2(\mu))$ is a closed ideal of $\hat{S}_2$ and $Z(\tilde{I}_2(\mu)) \supsetneqq Z(I_2(\mu))$, where $\hat{S}_2$ is the maximal ideal space of $M(G_{H_1})$. For $g \in \hat{S}$ such that $g \neq 0$ a.e. $d\theta\mu$, there exists a $g_1 \in \hat{S}_2$ such that $g_1 \neq 0$ a.e. $d\theta\mu$ and $\hat{\lambda}(g) = \hat{\lambda}(g_1)$ for every $\lambda \in M(G_{H_1})$. Since $\hat{\mu}(g) = \hat{\mu}(g_1) \neq 0$, we have $g \notin Z(I(\mu))$. Consequently, $Z(I(\mu))$ is a closed ideal of $\hat{S}$. Let $h \in Z(\tilde{I}_2(\mu)) \setminus Z(I_2(\mu))$. We put $\tilde{h}(\sigma) = h(\sigma')$ for every $\sigma \in M(G)$, where σ' is the part of σ which is contained in $M(G_{H_1})$. Then we have $\tilde{h} \in Z(\tilde{I}(\mu)) \setminus Z(I(\mu))$, which completes the proof.

LEMMA 7. *Let G be an infinite compact Abelian group. Then there exists a $\mu \in M(G)$ which satisfies* (a) *and* (b).

Proof. By [10], p. 45, there is an infinite compact metrizable subgroup G_0 of G. By Proposition 1, we can prove Lemma 7 in the same way as Lemma 6.

Proof of Theorem A. Since there exists an open subgroup $R^n \times K$ of G, where K is a compact group, the proof is completed by Propositions 1, 2 and Lemmas 5-7.

3. Proof of Theorem B. In this section we assume that G is a non-metrizable compact Abelian group, that is, $\hat{G}$ is an uncountable discrete group. We put

$$I = \{\mu \in M(G);\ \hat{\mu}(\gamma) = 0 \text{ for every } \gamma \in \hat{G} \text{ except a countable subset}\}.$$

Since $\gamma\mu \in I$ for every $\gamma \in \hat{G}$ and $\mu \in I$, we infer that I is an L-ideal of $M(G)$. We show that I has property (d). Let $\Gamma = \{\hat{G}_\alpha\}_{\alpha \in \Lambda}$ be the family of all countable subgroups of $\hat{G}$. We put $H_\alpha = \hat{G}_\alpha^\perp$, the annihilator of $\hat{G}_\alpha$ in G; then H_α is a compact subgroup of G.

LEMMA 8. *I coincides with the L-ideal I_0 of $M(G)$ generated by $\{m_{H_\alpha};\ \alpha \in \Lambda\}$, where m_{H_α} is the normalized Haar measure on H_α.*

Proof. For $\alpha \in \Lambda$, we have $m_{H_\alpha} \in I$, so that I contains I_0. Let $\mu \in I$. Since $\{\gamma \in \hat{G};\ \hat{\mu}(\gamma) \neq 0\}$ is a countable set, there exists a $\beta \in \Lambda$ such that

$$\hat{G}_\beta \supset \{\gamma \in \hat{G};\ \hat{\mu}(\gamma) \neq 0\}.$$

Consequently, $m_{H_\beta} * \mu = \mu$ and $\mu \in I_0$. Thus we have $I = I_0$.

Proof of Theorem B. We show that I satisfies (d). Let I_1 be an L-ideal of $M(G)$ such that $Z(I_1) = Z(I)$. For $\alpha \in \Lambda$, $M(G_{H_\alpha})$ is a prime L-subalgebra. For $\mu \in M(G)$, we denote by μ_α the part of μ which is contained in $M(G_{H_\alpha})$. We put $I_\alpha = \{\mu_\alpha;\ \mu \in I\}$. Then I_α is a non-zero L-ideal of $M(G_{H_\alpha})$ and $Z(I_\alpha) = Z(I_{1\alpha})$ holds in the maximal ideal space of $M(G_{H_\alpha})$. Since $m_{H_\alpha} \in I_\alpha$, we have $Z(I_\alpha) \cap \hat{G}_{H_\alpha} = \emptyset$. This shows that $L^1(G_{H_\alpha}) \subset I_{1\alpha}$ so that $m_{H_\alpha} \in I_{1\alpha}$. Thus we have $I \subset I_1$ by Lemma 8. Next suppose that $I \subsetneqq I_1$. Then there exists a $\mu \in I_1$ such that $\mu \notin I$. Since $\{\gamma \in \hat{G};\ \hat{\mu}(\gamma) \neq 0\}$ is uncountable, there exists an $\varepsilon > 0$ such that $E = \{\gamma \in \hat{G};\ |\hat{\mu}(\gamma)| \geqslant \varepsilon\}$ is uncountable. Let $\mathrm{cl}(E)$ be the closure in $\hat{S}$; then $\mathrm{cl}(E)$ is a compact subset of $\mathrm{cl}(\hat{G})$. We note that $\mathrm{cl}(\hat{G}_\alpha)$ is an open compact subset of $\mathrm{cl}(\hat{G})$. If

$$\mathrm{cl}(E) \subset \bigcup_{\alpha \in \Lambda} \mathrm{cl}(\hat{G}_\alpha),$$

then

$$\mathrm{cl}(E) \subset \bigcup_{i=1}^{n} \mathrm{cl}(\hat{G}_{\alpha_i}) \quad \text{for some } \alpha_i \in \Lambda \ (i = 1, \ldots, n)$$

and there is a $\hat{G}_0 \in \Gamma$ such that

$$E \subset \bigcup_{i=1}^{n} \hat{G}_{\alpha_i} \subset \hat{G}_0.$$

This contradicts that E is uncountable.

Thus there exists an f_0 such that

$$f_0 \in \mathrm{cl}(E)\setminus \bigcup_{\alpha\in\Lambda} \mathrm{cl}(\hat{G}_\alpha) \quad \text{and} \quad |\hat{\mu}(f_0)| \geqslant \varepsilon.$$

Since $\hat{\lambda}(f_0) = 0$ for every $\lambda \in I$, we have $Z(I_1) \subsetneqq Z(I)$. This is a contradiction which completes the proof.

It is not known whether Theorem B is true without the condition of compact metrizability. In this connection, we show

PROPOSITION 3. *Let H be an open subgroup of G. Suppose that there is an L-ideal I_0 of $M(H)$ such that*

(4) *for an L-ideal I_1 of $M(H)$, if $Z(I_0) = Z(I_1)$ holds in the maximal ideal space of $M(H)$, then $I_1 = I_0$.*

Then there is an L-ideal I which satisfies (4) (*replace H and I_0 by G and I, respectively*).

Proof. Let I be an L-ideal of $M(G)$ generated by I_0. Let I_1 be an L-ideal of $M(G)$ such that $Z(I_1) = Z(I)$. We note that I is the closed linear span of $\{\delta_x * I_0;\ x \in G\}$. We put $I_1' = I_1 \cap M(H)$; then I_1' is an L-ideal of $M(H)$. Since $Z(I_1') = Z(I_0)$, we have $I_1' = I_0$. Since I_1 is the closed linear span of $\{\delta_x * I_1';\ x \in G\}$, we have $I_1 = I$.

4. Proof of Theorem C. In this section we assume that G is a compact metrizable Abelian group, that is, $\hat{G}$ is a countable discrete group. For $E \subset \hat{G}$, we put

$$M_E^0 = \{\mu \in M(G);\ L^1(\mu)^{\wedge}|_E \subset C_0(E)\}.$$

Since $\gamma\mu \in M_E^0$ for every $\gamma \in \hat{G}$ and $\mu \in M_E^0$, M_E^0 is an L-ideal of $M(G)$ and $M_0(G) \subset M_E^0 \subset M(G)$. A subset E of $\hat{G}$ is called *symmetric* if $E = -E$. To prove Theorem C, we show some lemmas.

LEMMA 9. *Let E be a symmetric subset of $\hat{G}$ which satisfies* (f), *and let F be a finite subset of $\hat{G}$. Then there exists an $x \in \hat{G}\setminus E$ such that*

$$\{x\} \cup \{x \pm F\} \subset \hat{G}\setminus E.$$

Proof. Suppose that $\{x \pm F\} \cap E \neq \emptyset$ for every $x \in \hat{G}\setminus E$. Then we have

$$\hat{G} = E \cup \bigcup_{\gamma\in F} \{(E+\gamma) \cup (E-\gamma)\}.$$

Since F is a finite set, E does not have property (f). This is a contradiction.

LEMMA 10. *If E is a symmetric subset of $\hat{G}$ and E satisfies* (f), *then there is an infinite dissociate subset D of $\hat{G}$ such that*

(5) $(\gamma+E)\cap D(n)$ *is a finite set for every $\gamma \in \hat{G}$ and every non-negative integer n.*

Proof. Since $\hat{G}$ is a countable set, we put $\hat{G} = \{\gamma_1, \gamma_2, \ldots\}$. We set

(6) $$E_0 = E \quad \text{and} \quad E_N = E\cup\{E\pm\gamma_1\}\cup \ldots \cup\{E\pm\gamma_N\}.$$

Then E_N is a symmetric subset and has property (f) for $N = 0, 1, \ldots$ We will construct $x_n \in \hat{G}\setminus E_n$ and $A_n \subset \hat{G}\setminus E_n$ $(n = 0, 1, \ldots)$ by induction. We take $x_0 \in \hat{G}\setminus E_0$ and put $A_0 = \{\pm x_0\}$. For a positive integer k, by Lemma 9 there exists an $x_k \in \hat{G}\setminus E_k$ such that

(7) $$A_k = \{\pm x_k\}\cup\{\pm x_k + \bigcup_{i=0}^{k-1} A_i\} \subset \hat{G}\setminus E_k.$$

We may assume $x_0, x_1, \ldots$ are all distinct. We put $X = \{x_0, x_1, \ldots\}$. Then there exist an infinite dissociate subset D of $\hat{G}$ and $\gamma_0 \in \hat{G}$ such that $\gamma_0 + D \subset X$ (see [9], p. 21). We put $D = \{y_0, y_1, \ldots\}$ and $\gamma_0+y_k = z_k$ $(k = 0, 1, \ldots)$. We show that D satisfies (5). Let $\gamma_p \in \hat{G}$ and let n be a non-negative integer. Suppose that $(\gamma_p+E)\cap D(n)$ is an infinite set. Then $E_p\cap D(n)$ is an infinite set. If $y \in E_p\cap D(n)$ and $y = \delta_1 y_{j_1} + \ldots + \delta_n y_{j_n}$, then

$$y = \delta_1 z_{j_1} + \ldots + \delta_n z_{j_n} - (\delta_1 + \ldots + \delta_n)\gamma_0 \qquad (\delta_i = 1 \text{ or } -1 \ (i = 1, \ldots, n)).$$

Since $\{(\delta_1 + \ldots + \delta_n)\gamma_0\}$ is a finite set, there are $\eta_1, \ldots, \eta_n$ $(\eta_i = 1$ or $-1)$ such that $\{E_p - (\eta_1 + \ldots + \eta_n)\gamma_0\}\cap X(n)$ is an infinite set. By the definition of E_N, there exists a positive integer p_0 such that

$$\{E_p - (\eta_1 + \ldots + \eta_n)\gamma_0\} \subset E_{p_0}.$$

Hence $E_{p_0}\cap X(n)$ is an infinite set. If

$$E_{p_0}\cap X(n) \ni x = \varepsilon_1 x_{i_1} + \ldots + \varepsilon_n x_{i_n} \qquad (\varepsilon_i = \pm 1, \ i_1 < i_2 < \ldots < i_n),$$

then $x \notin E_{i_n}$ by (7). Thus we have $i_n < p_0$ by (6). Consequently, $E_{p_0}\cap X(n)$ is a finite set. This is a contradiction. Thus D has the desired property.

LEMMA 11. *Let E be a symmetric subset of $\hat{G}$. Then E satisfies* (f) *if and only if there exists a non-zero measure $\mu \in M(G)$ such that $\mu \notin M_0(G)$ and $L^1(\mu)^\wedge|_E \subset C_0(E)$.*

Proof. Suppose that E has property (f). By Lemma 10, there exists an infinite dissociate subset D of $\hat{G}$ which satisfies (5). Let μ be a positive measure on G such that

$$\hat{\mu}(\gamma) = \begin{cases} (\frac{1}{2})^n & \text{if } \gamma \in D(n), \\ 0 & \text{if } \gamma \in \hat{G} \setminus [D]. \end{cases}$$

To prove that $L^1(\mu)^\wedge|_E \subset C_0(E)$, it is sufficient to show that $(\gamma\mu)^\wedge|_E \in C_0(E)$ for every $\gamma \in \hat{G}$. Since

$$\{x \in E;\ |(\gamma\mu)^\wedge(x)| \geqslant (\tfrac{1}{2})^s\}$$
$$= \{x \in E;\ |\hat{\mu}(x-\gamma)| \geqslant (\tfrac{1}{2})^s\} = (E-\gamma) \cap \Big(\bigcup_{k=0}^{s} D(k)\Big) + \gamma$$

is a finite set by (5), we have $(\gamma\mu)^\wedge|_E \in C_0(E)$.

On the other hand, suppose that there is a non-zero measure $\mu \in M(G)$ such that $\mu \notin M_0(G)$ and $L^1(\mu)^\wedge|_E \subset C_0(E)$, and assume that E does not satisfy (f). Then there exists a set $\{\gamma_1, \ldots, \gamma_n\} \subset \hat{G}$ such that

$$\hat{G} = \bigcup_{k=1}^{n} (E+\gamma_k).$$

Since $L^1(\mu)^\wedge|_E \subset C_0(E)$, we have $\hat{\mu}|_{E+\gamma_k} \in C_0(E+\gamma_k)$ $(k = 1, \ldots, n)$ and $\hat{\mu} \in C_0(G)$. This contradicts $\mu \notin M_0(G)$. Thus the proof is completed.

Proof of Theorem C. We put $F = E \cup -E$. By Lemma 11, it is sufficient to show that if there is a $\mu \notin M_0(G)$ such that $L^1(\mu)^\wedge|_E \subset C_0(E)$, then $L^1(\mu)^\wedge|_F \subset C_0(F)$. Let $\mu \notin M_0(G)$ be such that $L^1(\mu)^\wedge|_E \subset C_0(E)$. For $\lambda \in L^1(\mu)$, we put $\lambda = \lambda_1 + i\lambda_2$, where λ_1 and λ_2 are real measures. Since $\bar{\hat{\lambda}}_j(\gamma) = \hat{\lambda}_j(-\gamma)$ and $\lambda_j \in L^1(\mu)$, we have $\hat{\lambda}_j|_{-E} \in C_0(-E)$ $(j = 1, 2)$. Thus we have $\hat{\lambda}_j|_F \in C_0(F)$ and $\hat{\lambda}|_F \in C_0(F)$ for every $\lambda \in L^1(\mu)$. Following Graham [5], p. 558-559, we can show that there are $\mu, \nu \perp M_E^0$ such that $\mu * \nu \in M_0(G)$. This proves that M_E^0 is not prime.

For $f \in \hat{S}$, we put

$$J(f) = \{x \in S;\ f(x) = 0\}$$

and

$$I(f) = \{\mu \in M(G);\ \theta\mu \text{ is concentrated on } J(f)\}.$$

We note that $M_0(G) = \bigcap\{I(f);\ f \in \mathrm{cl}(\hat{G}) \setminus \hat{G}\}$. It is easy to show that

$$M_E^0 = \bigcap\{I(f);\ f \in \mathrm{cl}(E) \setminus \hat{G}\}.$$

COROLLARY 1. *Let G be an infinite compact metrizable Abelian group. If E is an infinite symmetric subset of $\hat{G}$ and E satisfies* (f), *then*

$$M_0(G) \subsetneqq \bigcap\{I(f);\ f \in \mathrm{cl}(E) \setminus \hat{G}\}.$$

A subset E of $\hat{G}$ is called *Sidon* if for every bounded function f on E there is a measure $\mu \in M(G)$ such that $\hat{\mu}|_E = f$. If E is a Sidon subset of $\hat{G}$, then $E \cup -E$ is a symmetric Sidon set [4] and satisfies (f). Then we have

COROLLARY 2. *If E is an infinite Sidon subset of $\hat{G}$, then*

$$M_0(G) \subsetneqq M_E^0 \subsetneqq M(G) \quad \text{and} \quad M_0(G) \subsetneqq \bigcap\{I(f); f \in \mathrm{cl}(E) \setminus \hat{G}\}.$$

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