COLLOQUIUM MATHEMATICUM

XLII DÉDIÉ À LA MÉMOIRE D'EDWARD MARCZEWSKI 1979

INTERPOLATION IN CONGRUENCE PERMUTABLE ALGEBRAS

BY

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1. Introduction. The classical problem of interpolation can be stated for universal algebras as follows: Let $\mathfrak{A} = \langle A, \Omega \rangle$ be a universal algebra and $f\colon A^k \to A$ a given k-ary function. Is there, for any finite subset $N \subset A^k$, a k-ary algebraic function g on A such that $f|N = g|N$?

Algebras for which the problem of interpolation is solvable for any function $f\colon A^k \to A$ ($k \in N$, arbitrary) are often called *locally functionally complete* (see, e.g., Pixley [13]).

Recently Gumm has proved a theorem having the following consequence, noted by Pixley ([6], Theorem 6.1):

Let $\mathfrak{A} = \langle A, \Omega \rangle$ be a simple algebra in a congruence permutable variety; then either $\mathfrak{A}$ is locally functionally complete or $\mathfrak{A}$ is affine with respect to an abelian group $\langle A, + \rangle$.

This generalizes a result of McKenzie [11] to the infinite case.

A next logical step in the development of this theory is to study interpolation properties of single algebras and algebras in congruence permutable varieties (or, more generally, in congruence permutable local varieties) with special attention to the ranks of the functions and the size of the sets on which they may be interpolated. The present paper* initiates this study.

Affine algebras and their structure have been studied in some detail by McKenzie ([11] and [12]) and Gumm [6]. Questions of interpolation have already been considered by a number of authors (see, for example, Baker and Pixley [1], Hule and Nöbauer [8] or Kaiser [10]).

For concepts used in this paper without definition see Grätzer [5].

2. Definitions and results. Let $\mathfrak{A} = \langle A, \Omega \rangle$ be a universal algebra and k a positive integer. We say that a k-ary function $f\colon A^k \to A$ has the *n-interpolation property* if, for every subset $N \subset A^k$ of given finite cardinality n, there is an algebraic function $g\colon A^k \to A$ such that $g|N = f|N$. Since every function has the 1-interpolation property, we will assume

* This work was performed when the second-named author was a visiting-professor in Kassel. He is indebted to Prof. B. Bosbach for the invitation.

$n \geq 2$ throughout this paper. An algebra $\mathfrak{A}$ has the *n-interpolation property for k-ary functions* if every k-ary function $f: A^k \to A$ has the n-interpolation property. If this holds for functions of arbitrary arity $k \in N$, we say that $\mathfrak{A}$ has the *n-interpolation property*. If $\mathfrak{A}$ has the n-interpolation property for all $n \in N$, then we say: $\mathfrak{A}$ has the *interpolation property*. Such algebras are often called *locally functionally complete*.

Let $\mathfrak{A}$ be a universal algebra. A function $m: A^3 \to A$ is called a *Mal'cev function* if the following equations hold for all $x, y \in A$:

$$m(x, x, y) = y, \quad m(x, y, y) = x.$$

A function $d: A^3 \to A$ is called a (*ternary*) *majority function* if the following equations hold for all $x, y \in A$:

$$d(x, x, y) = d(x, y, x) = d(y, x, x) = x.$$

Next, we give the definition of a local variety (see Hu [7]):

Let K be a class of similar algebras and let D, H, S, P_f denote the operators of forming, respectively, directed unions, homomorphic images, subalgebras and direct products of finite families. Then we call $DHSP_f(K)$ the *local variety $V_f(K)$ generated by K*. The class K is a *local variety* if $K = V_f(K)$.

Our first principal result, Theorem 3.2, is a characterization of universal algebras having the n-interpolation property and it can be viewed as a generalization of a theorem in [9]. As an interesting corollary we obtain a characterization of single algebras which have the 2-interpolation property (Corollary 3.4).

In Section 4 we investigate the n-interpolation property in local varieties. Our main result here is Theorem 4.1. As a corollary to this result we get a sharper form of the consequence of Gumm's theorem, stated in the introduction (Theorem 4.2). In addition we characterize all abelian groups which have the n-interpolation property for some integer $n \geq 2$ (Theorem 4.3).

Finally, for a local variety V_f, which is congruence permutable, we describe all algebras $\mathfrak{A} \in V_f$ which have the n-interpolation property for some integer $n \geq 2$, except for the case of algebras which have the 3-interpolation property but not the 4-interpolation property. This case remains as an open problem.

3. The n-interpolation property. Let $\mathfrak{A} = \langle A, \Omega \rangle$ be a universal algebra which has the n-interpolation property for all k-ary functions $f: A^k \to A$ and let $t \leq k$, $t \in N$. Since every t-ary function $g: A^t \to A$ can be considered as a k-ary function, the algebra $\mathfrak{A}$ has the n-interpolation property for all t-ary functions, $t \leq k$. By a theorem of Sierpiński [14] we know that every function $f: A^k \to A$ $(k \in N,\ k \geq 2)$ can be obtained by composition of binary functions $h: A^2 \to A$. Therefore we have

LEMMA 3.1. *Let n be a positive integer and let $\mathfrak{A}$ be a universal algebra. Then exactly one of the following three cases occurs:*

(a) *$\mathfrak{A}$ has the n-interpolation property for all functions $f: A^k \to A$ ($k \in N$, arbitrary).*

(b) *$\mathfrak{A}$ has the n-interpolation property for exactly the set of all unary functions $f: A \to A$.*

(c) *$\mathfrak{A}$ fails to have the n-interpolation property for k-ary functions for any $k \in N$.*

The alternatives (a) and (c) obviously do occur; when they occur is the subject of our subsequent discussion. Alternative (b) also occurs; for example, consider an algebra $\mathfrak{A}$ with a universe A of n elements, $n \geqslant 3$, and with fundamental operations just all unary functions $f: A \to A$. Then $\mathfrak{A}$ clearly has the n-interpolation property for all unary functions $g: A \to A$, but no binary function $h: A^2 \to A$, which depends on both arguments, can be represented on any n points of its domain by an algebraic function.

Next, we observe that an algebra which has the n-interpolation property for some $n \geqslant 2$ cannot have nontrivial congruence relations.

LEMMA 3.2. *Let $n \geqslant 2$ be an integer. If $\mathfrak{A} = \langle A, \Omega \rangle$ has the n-interpolation property for all unary functions $f: A \to A$, then $\mathfrak{A}$ is simple.*

Proof. Let $\Theta \neq 0$ be a congruence relation on $\mathfrak{A}$ and let $a, b \in A$ ($a \neq b$) such that $a\Theta b$. Let c be any element of A. Since $\mathfrak{A}$ has the n-interpolation property for all unary functions, there is an algebraic function $p: A \to A$ such that $p(a) = c, p(b) = b$, hence $c\Theta b$ for all $c \in A$. Thus we have $\Theta = I$.

Let $n \geqslant 2$ be an integer. To formulate Theorem 3.1, a subalgebra $\mathfrak{M} \subset \mathfrak{A}^n$ is called an *algebraic subdirect power* if the following properties hold:

1. $\mathfrak{M}$ contains the diagonal $\Delta \subset \mathfrak{A}^n$,

2. for every $1 \leqslant i < j \leqslant n$ there is an element $(a_1, \ldots, a_n) \in \mathfrak{M}$ such that $a_i \neq a_j$.

THEOREM 3.1. *Let $n \geqslant 2$ be an integer and let $\mathfrak{A} = \langle A, \Omega \rangle$ be a simple algebra such that:*

(a) *there is a Mal'cev-function $m: A^3 \to A$ which has the n-interpolation property;*

(b) *there is an element $a \in A$ and a nonconstant function $q: A^2 \to A$ which has the n-interpolation property and such that $q(a, x) = q(x, a) = a$ for all $x \in A$.*

If $\mathfrak{M} \subset \mathfrak{A}^n$ is an algebraic subdirect power, then $\mathfrak{M} = \mathfrak{A}^n$.

Proof. First we observe that $\mathfrak{M}$ is a subdirect power of $\mathfrak{A}$, since $\mathfrak{M}$ contains $\Delta \subset \mathfrak{A}^n$. Hence the kernels $\pi_1, \ldots, \pi_n$ of the projections of $\mathfrak{M}$ onto $\mathfrak{A}$ satisfy $\pi_1 \wedge \ldots \wedge \pi_n = 0$. In view of Birkhoff's [2] characterization of direct products it suffices to show that for all $2 \leqslant i \leqslant n$ the following properties hold:

(i) $\pi_1 \wedge \ldots \wedge \pi_{i-1}$ and π_i permute,

(ii) $(\pi_1 \wedge \ldots \wedge \pi_{i-1}) \vee \pi_i = I$.

First we show property (i). Let $\bar{x} = (x_1, \ldots, x_n)$, $\bar{y} = (y_1, \ldots, y_n)$, $\bar{z} = (z_1, \ldots, z_n)$ be elements of $\mathfrak{M}$ and let $\bar{x}(\pi_1 \wedge \ldots \wedge \pi_{i-1})\bar{y}\pi_i\bar{z}$. Hence $x_j = y_j$ for $j = 1, \ldots, i-1$ and $y_i = z_i$. Let $\hat{m}$ be an algebraic function which represents a Mal'cev function on the following n elements of A^3: $(x_1, y_1, z_1), \ldots, (x_n, y_n, z_n)$. Since $\Delta \subset \mathfrak{M}$, $\hat{m}$ can be extended to an algebraic function $\overline{m}: \mathfrak{M}^3 \to \mathfrak{M}$, defined by

$$\overline{m}(\bar{x}, \bar{y}, \bar{z}) = \big(\hat{m}(x_1, y_1, z_1), \ldots, \hat{m}(x_n, y_n, z_n)\big) \in \mathfrak{M},$$

we have

$$\bar{x}\pi_i \overline{m}(\bar{x}, \bar{y}, \bar{z})(\pi_1 \wedge \ldots \wedge \pi_{i-1})\bar{z}$$

which implies

$$(\pi_1 \wedge \ldots \wedge \pi_{i-1}) \circ \pi_i \leqslant \pi_i \circ (\pi_1 \wedge \ldots \wedge \pi_{i-1}).$$

By symmetry the reverse inclusion follows.

Now we show property (ii). Since the projections are maximal ($\mathfrak{A}$ is simple), we have

$$(\pi_1 \wedge \ldots \wedge \pi_{i-1}) \vee \pi_i = I \text{ or } \pi_i.$$

By induction on n we will prove that $\pi_1 \wedge \ldots \wedge \pi_{i-1} \nleqslant \pi_i$.

Let $n = 2$. Since $\mathfrak{M}$ is an algebraic subdirect power in $\mathfrak{A}^2$, there are $(a_1, a_1) \in \mathfrak{M}$ and $(a_1, a_2) \in \mathfrak{M}$, $a_1 \neq a_2$. Then $(a_1, a_2)\pi_1(a_1, a_1)$ holds, but not $(a_1, a_2)\pi_2(a_1, a_1)$, hence $\pi_1 \nleqslant \pi_2$.

Now let $n > 2$ and suppose our assertion is true for all $k < n$. By the induction hypothesis the projection of $\mathfrak{M}$ into any of its $n-1$ coordinates is $\mathfrak{A}^{n-1}$. Since $q: A^2 \to A$ is not constant, there are $b, c \in A$ such that $q(b, c) \neq a$. Therefore the following elements are in $\mathfrak{M}$ for some $x, y \in A$:

$$\bar{x}_1 = (x, a, a, \ldots, a, b), \quad \bar{x}_2 = (a, y, a, \ldots, a, c).$$

Since $\mathfrak{M}$ is an algebraic subdirect power in $\mathfrak{A}^n$, q extends to an algebraic function $\bar{q}: \mathfrak{M}^2 \to \mathfrak{M}$, defined by $\bar{q}(\bar{x}, \bar{y}) = \big(q(x_1, y_1), \ldots, q(x_n, y_n)\big)$. Then $\bar{q}(\bar{x}_1, \bar{x}_2) = \big(a, \ldots, a, q(b, c)\big)$ and $q(b, c) \neq a$. Hence we have $\pi_1 \wedge \ldots \wedge \pi_{n-1} \nleqslant \pi_n$, which completes our proof.

Now we can give a characterization of algebras which have the n-interpolation property.

THEOREM 3.2. *Let $n \geqslant 2$ be an integer. A universal algebra $\mathfrak{A} = \langle A, \Omega \rangle$ has the n-interpolation property if and only if $\mathfrak{A}$ has the following properties:*

(1) *A is simple;*

(2) *there is a function* $q\colon A^2 \to A$ *such that:*

(α) q *is not constant,*

(β) *there is an element* $a \in A$ *such that* $q(x, a) = q(a, x) = a$ *for all* $x \in A$,

(γ) q *has the n-interpolation property;*

(3) *there is a Mal'cev function* $m\colon A^3 \to A$ *which has the n-interpolation property.*

Proof. If $\mathfrak{A}$ has the n-interpolation property, conditions (1)-(3) obviously hold.

Now let $\mathfrak{A}$ have properties (1)-(3). According to Lemma 3.1 it suffices to show that every binary function $f\colon A^2 \to A$ has the n-interpolation property. Let S be an n-element subset of A^2. Since every constant function $\varkappa\colon A^2 \to A$ and the two projections $p_1(x, y) = x$ and $p_2(x, y) = y$ (for all $x, y \in A$) are algebraic functions, the restrictions of all algebraic functions $g\colon A^2 \to A$ to the n-element subset S can be considered as an algebraic subdirect power $\mathfrak{M}$ in $\mathfrak{A}^n$. By Theorem 3.1 we have $\mathfrak{M} = \mathfrak{A}^n$. Hence every $f\colon A^2 \to A$ has the n-interpolation property.

COROLLARY 3.1. *Let* $n \geqslant 2$ *be an integer. A universal algebra* $\mathfrak{A} = \langle A, \Omega\rangle$ *has the n-interpolation property if and only if the discriminator* $t\colon A^3 \to A$, *defined by*

$$t(x, y, z) = \begin{cases} x & \text{if } x \neq y, \\ z & \text{if } x = y, \end{cases}$$

has the n-interpolation property.

Proof. If $\mathfrak{A}$ has the n-interpolation property, then t has the n-interpolation property.

If t has the n-interpolation property, then $\mathfrak{A}$ is simple. t is a Mal'cev function and if we set $q(x, y) := t\big(x, t(x, y, a), a\big)$, $a \in A$ arbitrary, our assertion follows from Theorem 3.2.

Remark. Condition (2) of Theorem 3.2 can be modified to the existence of a ternary majority function which has the n-interpolation property. Indeed, if $t\colon A^3 \to A$ is the discriminator (which has the n-interpolation property because of Corollary 3.1), then $t\big(x, t(x, y, z), z\big)$ is a majority function having the n-interpolation property. In this form Theorem 3.2 is a sharpened version of an immediate corollary to Theorem 4.3 of Pixley [13].

COROLLARY 3.2. *Let* $n \geqslant 2$ *be an integer. A universal algebra* $\mathfrak{A}$ *has the n-interpolation property if and only if*

(1) A *is simple;*

(2) *there are binary functions* $p, t, q\colon A^2 \to A$ *such that:*

(α) p, q, t *have the n-interpolation property,*

(β) $p(x, x) = p(y, y)$ *for all* $x, y \in A$,

(γ) *$t(p(x, y), y) = x$ for all $x, y \in A$,*

(δ) *q is not constant,*

(ε) *there is an element $a \in A$ such that $q(x, a) = q(a, x) = a$ for all $x \in A$.*

Proof. By Theorem 3.2 we only have to show that there is a Mal'cev function $m: A^3 \to A$ which has the n-interpolation property. We set $m(x, y, z) =: t(p(x, y), z)$. Then $m(x, x, z) = t(p(x, x), z) = t(p(z, z), z) = z$ and $m(x, z, z) = t(p(x, z), z) = x$.

COROLLARY 3.3. *Let $n \geqslant 2$ be an integer. A finite universal algebra $\mathfrak{A}$ has the n-interpolation property if and only if*

(1) *$\mathfrak{A}$ is simple;*

(2) *there are binary functions p, $q: A^2 \to A$ such that:*

(α) *$p(x, x) = p(y, y)$ for all $x, y \in A$,*

(β) *$p(x, a)$ is bijective for every $a \in A$,*

(γ) *q is not constant,*

(δ) *there is an element $a \in A$ such that $q(x, a) = q(a, x) = a$ for all $x \in A$.*

Proof. In view of Corollary 3.2 we only have to show the existence of a function $t: A^2 \to A$ such that $t(p(x, y), y) = x$ for all $x, y \in A$ and which has the n-interpolation property. Let $|A| = s$. Since $p(x, a) =: p_a(x)$ is bijective, $p_a^{s!}(x)$ is the identity function on A for all $a \in A$. Now we set $t(x, y) := p_y^{s!-1}(x)$. Then

$$t(p(x, y), y) = p_y^{s!-1}(p_y(x)) = p_y^{s!}(x) = x.$$

COROLLARY 3.4. *A universal algebra $\mathfrak{A}$ has the 2-interpolation property if and only if*

(1) *$\mathfrak{A}$ is simple;*

(2) *there is a Mal'cev function $m: A^3 \to A$ having the 2-interpolation property.*

Proof. Observe that in the proof of Theorem 3.1 the function q is only used in the case of $n > 2$.

COROLLARY 3.5. *A universal algebra $\mathfrak{A}$ has the interpolation property (in other words: $\mathfrak{A}$ is locally functionally complete) if and only if*

(1) *$\mathfrak{A}$ is simple;*

(2) *there is a Mal'cev function $m: A^3 \to A$ which has the interpolation property;*

(3) *there is a nonconstant function $q: A^2 \to A$ which has the interpolation property and such that there is an element $a \in A$ with $q(x, a) = q(a, x) = a$ for all $x \in A$.*

The Remark following Corollary 3.1 applies here as well.

4. Interpolation in local varieties.

Definition. A local variety V_f is called *congruence permutable* if the congruences of each algebra $\mathfrak{A} \in V_f$ permute.

LEMMA 4.1. *Let $n \geqslant 2$ be an integer and let V_f be a local variety such that for every algebra $\mathfrak{A} \in V_f$ there is a Mal'cev function $m: A^3 \to A$ which has the n-interpolation property with respect to polynomials (i.e. for each n-element subset $N \subset A^3$ there is a polynomial $g: A^3 \to A$ such that $g|N = m|N$). Then V_f is congruence permutable.*

Proof. Let $\mathfrak{A} \in V_f$; Θ_1, Θ_2 congruence relations on $\mathfrak{A}$ and $x\,\Theta_1\,y\,\Theta_2\,z$. Let p be a polynomial which interpolates a Mal'cev function on $\{(x, x, z), (x, z, z)\}$. Hence $z = p(x, x, z)\,\Theta_1 p(x, y, z)\Theta_2 p(x, z, z) = x$.

Following Burris [3] and Werner [15] a congruence on a finite product of algebras will be called *skew* if it does not arise as a product of congruences on the factors.

LEMMA 4.2. *Let $\mathfrak{A}$ be an algebra which has the 4-interpolation property. Then there are no skew congruences on $\mathfrak{A}^2$.*

Proof. According to Fraser and Horn [4] it suffices to show that for any nontrivial congruence Θ on $\mathfrak{A}^2$ the following condition holds:

$(a, b)\,\Theta(c, d)$ implies $(a, x)\,\Theta(c, x)$ and $(y, b)\,\Theta(y, d)$ for all $x, y \in A$.

Suppose the four elements a, b, c, d are different. Then, since $\mathfrak{A}$ has the 4-interpolation property, there is an algebraic function $f: A \to A$ such that $f(a) = a$, $f(b) = x$, $f(c) = c$, $f(d) = x$ ($x \in A$, arbitrary). Then $f \times f$: $A^2 \to A^2$, defined by $f \times f(x, y) = \big(f(x), f(y)\big)$ for all $(x, y) \in A^2$, is a unary algebraic function on $\mathfrak{A}^2$, hence we have

$$(a, x) = f \times f(a, b)\Theta f \times f(c, d) = (c, x).$$

In a similar fashion one obtains:

$(a, b)\,\Theta(c, d)$ implies (for all $y \in A$) $(y, b)\,\Theta(y, d)$.

In case some (or all) of the four elements coincide analogous arguments and using the fact that Θ is a nontrivial congruence show that the condition of Fraser and Horn holds.

Let $\mathfrak{A} = \langle A, \Omega\rangle$ be a universal algebra. Let k be a positive integer and S a subset of A^k. A function $f: S \to A$ is called *conservative* if, for any $(a_1, \ldots, a_k) \in S$, $f(a_1, \ldots, a_k)$ is an element of the subalgebra of $\mathfrak{A}$ which is generated by $\{a_1, \ldots, a_k\}$. f is *isomorphism preserving* if, for every internal isomorphism φ of $\mathfrak{A}$ and element $(a_1, \ldots, a_k)$ in S, $\varphi\big(f(a_1, \ldots, a_k)\big) = f(\varphi a_1, \ldots, \varphi a_k)$ whenever both sides of the equation are defined. Here an internal isomorphism is understood to be any isomorphism between (not necessarily distinct) subalgebras of $\mathfrak{A}$. An algebra $\mathfrak{A}$ is called *locally quasiprimal* (see Pixley [13]) if $\mathfrak{A}$ is nontrivial and for each finite subset

$F \subset A^k$ and function $f\colon F \to A$, which is both conservative and isomorphism preserving, there is a polynomial p such that $p|F = f|F$.

LEMMA 4.3. *Let $n \geqslant 2$ be an integer and V_f a local variety such that for all algebras $\mathfrak{A} \in V_f$ there is a Mal'cev function $m\colon A^3 \to A$ which has the n-interpolation property with respect to polynomials. Suppose $\mathfrak{A} \in V_f$ is simple, has proper subalgebras of at most one element and $\mathfrak{A}^2$ has no skew congruences. Then $\mathfrak{A}$ is locally quasiprimal.*

Proof. By the implication (iii) $\Rightarrow$ (i) of Theorem 4.3 of Pixley [13] it suffices to show that the local variety $V_f(\mathfrak{A})$, generated by $\mathfrak{A}$, is arithmetical (in other words: is congruence permutable and congruence distributive). By Lemma 4.1, $V_f(\mathfrak{A})$ is congruence permutable. Next observe, since $\mathfrak{A}^2$ has no skew congruences and $V_f(\mathfrak{A})$ is congruence modular, it follows from [3] that no finite direct power of $\mathfrak{A}$ has skew congruences. Hence, since $\mathfrak{A}$ is simple, for each integer $m \geqslant 1$, the congruence lattice of $\mathfrak{A}^m$ is isomorphic to the distributive lattice $\mathbf{2}^m$. Since $\mathfrak{A}$ is simple and has only one-element proper subalgebras, $SP_f(\mathfrak{A}) \subset P_f(\mathfrak{A})$ so that $V_f(\mathfrak{A}) = DHP_f(\mathfrak{A}) = DP_f(\mathfrak{A})$, the latter equality following from the congruence distributivity for each algebra in $P_f(\mathfrak{A})$. Hence $V_f(\mathfrak{A}) = DP_f(\mathfrak{A})$, where $P_f(\mathfrak{A})$ is arithmetical. Finally, it is trivial that a direct limit of arithmetical algebras is arithmetical and hence we have the conclusion.

The following theorem generalizes a result of Werner [15]:

THEOREM 4.1. *Let $n \geqslant 2$ be an integer and V_f a local variety such that for every algebra $\mathfrak{A} \in V_f$ there is a Mal'cev function $m\colon A^3 \to A$ which has the n-interpolation property with respect to polynomials. Then a simple algebra $\mathfrak{A} \in V_f$ has the interpolation property if and only if $\mathfrak{A}^2$ has no skew congruences.*

Proof. If $\mathfrak{A}$ has the interpolation property, then $\mathfrak{A}^2$ has no skew congruences by Lemma 4.2. Now let $\mathfrak{A}^2$ have no skew congruences. Then by adding the elements of A as nullary fundamental operations, we obtain an algebra $\mathfrak{A}^+$ which has no proper subalgebras. By Lemma 4.3, $\mathfrak{A}^+$ is locally quasiprimal, and hence $\mathfrak{A}$ has the interpolation property.

COROLLARY 4.1. *Let V_f be a local variety as in Theorem 4.1. Then $\mathfrak{A}$ has the interpolation property if and only if it has the 4-interpolation property.*

Proof. If $\mathfrak{A}$ has the interpolation property, then, by definition, $\mathfrak{A}$ has the 4-interpolation property. If $\mathfrak{A}$ has the 4-interpolation property, then $\mathfrak{A}$ is simple and $\mathfrak{A}^2$ has no skew congruences (Lemma 4.3). Hence, by Theorem 4.1, it has the interpolation property.

Next we determine the structure of simple algebras in congruence permutable local varieties.

Definition. Let $\mathfrak{A}$ be a universal algebra. A function $f\colon A^k \to A$ is called *affine* (with respect to an abelian group $\mathfrak{G} = \langle A, +\rangle$) if for all elements $x_1, \ldots, x_k, y_1, \ldots, y_k \in A$ the following equation holds:

$$f(x_1, \ldots, x_k)+f(y_1, \ldots, y_k) = f(x_1+y_1, \ldots, x_k+y_k)+f(0, \ldots, 0)$$

(0 denotes the neutral element of $\mathfrak{G}$).

An algebra $\mathfrak{A} = \langle A, \Omega\rangle$ is called *affine* (with respect to an abelian group $\mathfrak{G} = \langle A, +\rangle$) if every $\omega \in \Omega$ is an affine function.

THEOREM 4.2. *Let $n \geqslant 2$ be an integer and V_f a local variety such that for every $\mathfrak{A} \in V_f$ there is a Mal'cev function $m\colon A^3 \to A$ which has the n-interpolation property with respect to polynomials. Then a simple algebra $\mathfrak{A} \in V_f$ either has the interpolation property or is affine with respect to an abelian group $\mathfrak{G} = \langle A, +\rangle$ which is an elementary p-group (p prime) or torsion free.*

Proof. By inspecting the proofs of the theorems leading to Theorem 6.1 of Gumm [6] one sees that compatibility of the Mal'cev function $m\colon A^3 \to A$, which has the n-interpolation property, is all that is required. Hence Gumm's proof applies equally in the present setting.

Remark. Note that if $\mathfrak{A}$ is a simple affine algebra in a local variety V_f as in Theorem 4.2, the group operation $+$ can only be chosen from the set of binary functions having the n-interpolation property.

THEOREM 4.3. *There is no abelian group which has the n-interpolation property for any $n \geqslant 4$. Only the cyclic group of order 2 has the 3-interpolation property. Only the groups of prime order have the 2-interpolation property.*

Proof. The first statement follows from the fact (Lemma 4.2) that if an algebra $\mathfrak{A}$ has the n-interpolation property for $n \geqslant 4$, then $\mathfrak{A}^2$ cannot have skew congruences. If $\mathfrak{A}$ is an abelian group, then $\mathfrak{A}^2$ always has a skew congruence (indeed, $\mathfrak{A}$ is abelian just in case the diagonal is a normal subgroup of $\mathfrak{A}^2$).

The cyclic group Z_2 has the 3-interpolation property, because the function $q\colon Z_2^2 \to Z_2$, defined by

$$q(x, y) = \begin{cases} a & \text{if } (x, y) = (a, a), \\ 0 & \text{if } (x, y) \neq (a, a) \end{cases}$$

(a denotes the nonzero element of Z_2), has the 3-interpolation property as one easily checks. Our assertion then follows from Theorem 3.2. In order to show that there are no other abelian groups having the 3-interpolation property, we consider cyclic groups Z_t, t prime and $t > 2$. Let x be the generator of Z_t; then $x+x \neq 0$. Suppose Z_t has the 3-interpolation property; then there is an algebraic function $p\colon Z_t \to Z_t$ such that $p(x) = 0, p(2x) \neq 0, p(0) = 0$. Then p would have to be affine, but $p(x)+p(x) \neq p(2x)+p(0)$. The last statement follows immediately from Corollary 3.3.

THEOREM 4.4. *Let V_f be a local variety as in Theorem 4.2. Then every simple algebra of V_f has the 2-interpolation property.*

Proof. The assertion follows readily from Theorem 4.2 together with the Remark following Theorem 4.2 and Corollary 3.3.

Because of Corollary 4.1, we know that in the case of a local variety V_f as in Theorem 4.2 the 4-interpolation property implies the interpolation property. In order to give a complete list of all algebras $\mathfrak{A} \in V_f$, which have the n-interpolation property for some integer $n \geqslant 2$, the following problem remains:

PROBLEM. Let $n \geqslant 2$ be an integer and V_f a local variety such that for every $\mathfrak{A} \in V_f$ there is a Mal'cev function $m: A^3 \to A$ which has the n-interpolation property with respect to polynomials. Find all simple algebras $\mathfrak{A} \in V_f$ which have the 3-interpolation property but not the 4-interpolation property. (**P 1137**)

Regarding this problem one immediately makes the following observations:

a. Every $\mathfrak{A} \in V_f$ (V_f as above) which has the 3-interpolation property but not the 4-interpolation property is necessarily affine with respect to an elementary abelian 2-group.

b. Every $\mathfrak{A} \in V_f$ (V_f as above) which is affine with respect to the group Z_2 has the 3-interpolation property but not the 4-interpolation property.

c. On $Z_2 \times Z_2$ one can construct a simple affine algebra, by adding one suitable endomorphism of $Z_2 \times Z_2$ as fundamental operation, which does not have the 3-interpolation property. But by adding all endomorphisms of $Z_2 \times Z_2$ as fundamental operations one obtains a simple affine algebra which has the 3-interpolation property (at least for unary functions) but not the 4-interpolation property.

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Reçu par la Rédaction le 4. 6. 1977