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ON LATTICES OF VARIETIES OF UNIVERSAL ALGEBRAS

BY

J. PŁONKA (WROCŁAW)

0. Let N denote the set of non-negative integers. We shall consider only algebras of a given type $\tau\colon T \to N$. We shall denote by $\{f_t\}_{t \in T}$ the set of fundamental polynomial symbols associated with τ (see [4]). The type τ we shall call *nullary* if $\tau(T) = \{0\}$ or $T = \emptyset$ and τ will be called *non-nullary* otherwise. Algebras of the nullary type will be called *nullary*. The type τ will be called *unary* (see [7]) if $1 \in \tau(T) \subseteq \{0, 1\}$. Let φ be a term associated with τ. We denote by $V(\varphi)$ the set of variables occurring in φ. A term φ will be called *essentially n-ary* if $|V(\varphi)| = n$. Let φ and ψ be terms associated with τ. An equality $\varphi = \psi$ is called *regular* (see [5]) if $V(\varphi) = V(\psi)$. An equality which is not regular will be called *non-regular*.

Let $\boldsymbol{K}$ be a variety of type τ. We denote by $E(\boldsymbol{K})$ the set of all equalities satisfied in any algebra from $\boldsymbol{K}$, by $R(\boldsymbol{K})$ the set of all regular equalities from $E(\boldsymbol{K})$ and we denote by $\boldsymbol{K}_R$ the variety of algebras of type τ defined by $R(\boldsymbol{K})$. A variety $\boldsymbol{K}$ will be called *regular* if $E(\boldsymbol{K}) = R(\boldsymbol{K})$ and will be called *non-regular* otherwise. If $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are two varieties of type τ then it is customary to denote by $\boldsymbol{K}_1 \vee \boldsymbol{K}_2$ the variety of type τ defined by $E(\boldsymbol{K}_1) \cap E(\boldsymbol{K}_2)$ and to denote by $\boldsymbol{K}_1 \wedge \boldsymbol{K}_2$ the variety of type τ defined by $E(\boldsymbol{K}_1) \cup E(\boldsymbol{K}_2)$. It is known that the set $L(\tau)$ of all varieties of type τ together with the operations $\vee$, $\wedge$ is a lattice. We denote by $N(\tau)$ the set of all non-regular varieties of type τ and by $R(\tau)$ the set of all regular varieties of type τ. It was shown in [6] (Corollary 2) that if $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are non-regular then $\boldsymbol{K}_1 \vee \boldsymbol{K}_2$ is non-regular. Thus from the definition of $\vee$, $\wedge$ it follows that $(N(\tau); \vee, \wedge)$ and $(R(\tau); \vee, \wedge)$ are both sublattices of $(L(\tau); \vee, \wedge)$. Denote by $\boldsymbol{D}(\tau)$ the variety of type τ defined by the equality $x_1 = x_2$. Observe that if τ is a nullary type then $\boldsymbol{D}(\tau)$ is the only non-regular variety of type τ. Obviously $\boldsymbol{D}(\tau)$ is always the smallest variety in $L(\tau)$ and $N(\tau)$.

In this paper we prove (Theorem 1): if τ is a non-nullary type of algebras then there are no maximal elements in the lattice $(N(\tau); \vee, \wedge)$. Further we show (Theorem 2) that: the mapping ϱ defined by the formula $\varrho(\boldsymbol{K}) = \boldsymbol{K}_R$ is an embedding of the join semilattice $(N(\tau); \vee)$ into the join semilattice $(R(\tau); \vee)$.

These two theorems have been proved for unary algebras in [7]. J. Dudek and E. Graczyńska in [2] considered the so-called *strongly non-regular varieties of type* τ, i.e. varieties satisfying the equality $\varphi(x_1, x_2) = x_1$ for some essentially binary term φ.

1. Preliminaries. It was proved in [6], Lemma 2 that:

(i) if K is a non-regular variety of type τ: $T \to N$ where $\tau(T)\setminus\{0, 1\} \neq \emptyset$ then there exists an essentially binary term $\varphi(x_1, x_2)$ associated with τ such that the equality $\varphi(x_1, x_2) = \varphi(x_1, x_1)$ belongs to $E(K)$.

Let τ: $T \to N$ be a non-nullary type of algebras. We denote $T^* = \{t\colon \tau(t) \neq 0\}$. We define a new type τ^* putting τ^*: $T^* \to N$ where $\tau^* = \tau | T^*$ and we accept $\{f_t\}_{t\in T^*}$ as the set of fundamental polynomial symbols associated with τ^*. If $\mathfrak{A} = (A; \{f_t\}_{t\in T})$ is an algebra of type τ then we denote by $\mathfrak{A}^*$ the non-nullary reduct of $\mathfrak{A}$, i.e. $\mathfrak{A}^* = (A; \{f_t\}_{t\in T^*})$. So $\mathfrak{A}^*$ is of the type τ^*. Finally if K is a variety of type τ then we denote by K^* the variety of type τ^* defined by all equalities from $E(K)$ in which nullary polynomial symbols do not occur. It was proved in [6] (Theorem 4) that:

(ii) If K is a non-regular variety of algebras and there exists an essentially binary term $x \cdot y$ such that the equality $x \cdot y = x \cdot x$ belongs to $E(K)$, then any algebra $\mathfrak{A} = (A; \{f_t\}_{t\in T}) \in K_R$ is of the form: there exists a semilattice $(I; \leqslant)$; if there are nullary fundamental polynomials in $\mathfrak{A}$ this semilattice has the smallest element ω; there exist a family of subalgebras $\{\mathfrak{A}_i\}$ of $\mathfrak{A}^*$, where $i \in I\setminus\{\omega\}$; if there are nullary polynomials in $\mathfrak{A}$ then there exists a subalgebra $\mathfrak{A}_\omega$ of $\mathfrak{A}$ to which the values of all fundamental nullary polynomials belong; the carriers A_i of all $\mathfrak{A}_i$ are mutually disjoint; $\mathfrak{A}_i \in K^*$ for $i \neq \omega$ and $\mathfrak{A}_\omega \in K$ if $a_j \in A_{i_j}$ for $j = 1, 2, \ldots, \tau(t)$ then $f_t(a_1, \ldots, a_{\tau(t)})$ belongs to the subalgebra indexed by $\sup\{i_1, \ldots, i_{\tau(t)}\}$.

From (ii) it follows that:

(iii) If K is a non-regular variety of type τ: $T \to N$ where $\tau(T)\setminus\{0, 1\} \neq \emptyset$, $\mathfrak{A} \in K_R$ and $|I| > 1$ then $\mathfrak{A}$ satisfies only regular equalities.

The proof of (iii) is identical with the last part of the proof of Theorem 1 from [6] so we omit it here.

2. Lattices of regular and non-regular varieties. Our aim now is to prove the following:

THEOREM 1. *If* τ: $T \to N$ *is a non-nullary type of algebras then there are no maximal elements in the lattice* $(N(\tau), \vee, \wedge)$. *Moreover for any variety* $K \in N(\tau)$ *there exist a variety* $K' \in N(\tau)$ *and a finite algebra* $\mathfrak{A}$ *such that* $K \subseteq K'$, $\mathfrak{A} \in K'$ *and* $\mathfrak{A} \notin K$.

To prove this theorem we need some notions.

For any term φ associated with τ and any variable x_k we define the deepness $d(x_k, \varphi)$ of x_k in φ as follows:

(1) if f_t is a nullary fundamental polynomial symbol then $d(x_k, f_t) = -\infty$;

(2) $d(x_k, x_i) = \begin{cases} 0 & \text{if } k = i, \\ -\infty & \text{otherwise;} \end{cases}$

(3) if $\varphi_1, \ldots, \varphi_{\tau(t)}$ are n-ary terms, f_t is a fundamental polynomial symbol with $\tau(t) \neq 0$ and $d(x_k, \varphi_i)$ is already defined for $i = 1, \ldots, \tau(t)$ then:

$$d(x_k, f_t(\varphi_1, \ldots, \varphi_{\tau(t)})) = \max\{d(x_k, \varphi_1), \ldots, d(x_k, \varphi_{\tau(t)})\} + 1.$$

We accept $-\infty + m = -\infty$ for a non-negative integer m.

For a term φ associated with τ we still define the number $d_0(\varphi)$ putting:

(4) if $\tau(t) = 0$ then $d_0(f_t) = 0$;

(5) $d_0(x_i) = -\infty$;

(6) if $\tau(t) \neq 0$; $\varphi_1, \varphi_2, \ldots, \varphi_{\tau(t)}$ are terms for which $d_0(\varphi_i)$ is already defined, $i = 1, \ldots, \tau(t)$ then

$$d_0(f_t(\varphi_1, \ldots, \varphi_{\tau(t)})) = \max\{d_0(\varphi_1), \ldots, d_0(\varphi_{\tau(t)})\} + 1.$$

For any type τ, where $\tau(T)\backslash\{0, 1\} \neq \emptyset$ and a natural number $n > 0$ we define an algebra $\mathfrak{A}_n$ as follows:

$\mathfrak{A}_n = (\{0, 1, \ldots, n\}, \{f_t\}_{t \in T})$ putting $f_t = 0$ if $\tau(t) = 0$ and $f_t(a_1, \ldots, a_{\tau(t)}) = \min\{n, \max\{a_1, \ldots, a_{\tau(t)}\} + 1\}$ otherwise.

Let $P^2(x_1, x_2)$ be the set of all terms associated with τ constructed by means of at most two variables x_1, x_2.

LEMMA 1. *If* $\tau(T)\backslash\{0, 1\} \neq \emptyset$, $\varphi \in P^2(x_1, x_2)$ *and* $\bar{\varphi}(x_1, x_2)$ *is the realization of* φ *in* $\mathfrak{A}_n$ *then we have in* $\mathfrak{A}_n$:

$$\bar{\varphi}(c_1, c_2) = \min\{n, \max\{d_0(\varphi), c_1 + d(x_1, \varphi), c_2 + d(x_2, \varphi)\}\}.$$

Proof. We use induction on the length $L(\varphi)$ where $L(\varphi)$ is the number of signs in φ including parantheses. Obviously the lemma holds for nullary fundamental polynomial symbols and variables x_1, x_2. So the lemma is true for all terms of the length not exceeding 3.

Assume the lemma holds for all terms from $P^2(x_1, x_2)$ of the lengths not exceeding $q \geqslant 3$. Let $\varphi \in P^2(x_1, x_2)$, $L(\varphi) = q + 1$. So it must be: $\varphi = f_t(\varphi_1, \ldots, \varphi_{\tau(t)})$ for some $t \in T$, $\tau(t) \neq 0$, and some $\varphi_k \in P^2(x_1, x_2)$, $k = 1, \ldots, \tau(t)$.

Obviously $L(\varphi_k) \leqslant q$, for $k = 1, \ldots, \tau(t)$.

We shall use the following notations:

$$\alpha \vee \beta \quad \text{for } \max\{\alpha, \beta\};$$

$$\alpha \wedge \beta \quad \text{for } \min\{\alpha, \beta\};$$

$$\bigvee_{k=1}^{s} \alpha_k \quad \text{for } \max\{\alpha_1, \ldots, \alpha_s\}.$$

So we have by (1)–(6):

$$\bar{\varphi}(c_1, c_2) = n \wedge \Big[\Big(\bigvee_{k=1}^{\tau(t)} \bar{\varphi}_k(c_1, c_2)\Big) + 1\Big]$$

$$= n \wedge \Big\{\bigvee_{k=1}^{\tau(t)} \big[n \wedge \big(d_0(\varphi_k) \vee [c_1 + d(x_1, \varphi_k)] \vee [c_2 + d(x_2, \varphi_k)]\big)\big] + 1\Big\}$$

$$= n \wedge \Big\{\Big[n \wedge \Big\{\bigvee_{k=1}^{\tau(t)} \big(d_0(\varphi_k) \vee [c_1 + d(x_1, \varphi_k)] \vee$$

$$\vee [c_2 + d(x_2, \varphi_k)]\big)\Big\}\Big] + 1\Big\}$$

$$= n \wedge \Big\{(n+1) \wedge \Big[\bigvee_{k=1}^{\tau(t)} \big(d_0(\varphi_k) \vee [c_1 + d(x_1, \varphi_k)] \vee$$

$$\vee [c_2 + d(x_2, \varphi_k)]\big) + 1\Big]\Big\}$$

$$= n \wedge \Big[\Big(\bigvee_{k=1}^{\tau(t)} \big(d_0(\varphi_k) \vee [c_1 + d(x_1, \varphi_k)] \vee [c_2 + d(x_2, \varphi_k)]\big)\Big) + 1\Big]$$

$$= n \wedge \Big\{\Big[\bigvee_{k=1}^{\tau(t)} d_0(\varphi_k) \vee \bigvee_{k=1}^{\tau(t)} [c_1 + d(x_1, \varphi_k)] \vee \bigvee_{k=1}^{\tau(t)} [c_2 + d(x_2, \varphi_k)]\Big] + 1\Big\}$$

$$= n \wedge \Big\{\Big[\bigvee_{k=1}^{\tau(t)} d_0(\varphi_k) \vee \Big(c_1 + \bigvee_{k=1}^{\tau(t)} d(x_1, \varphi_k)\Big) \vee \Big(c_2 + \bigvee_{k=1}^{\tau(t)} d(x_2, \varphi_k)\Big)\Big] + 1\Big\}$$

$$= n \wedge \Big\{\Big[\bigvee_{k=1}^{\tau(t)} (d_0(\varphi_k)) + 1\Big] \vee \Big[c_1 + \Big(\bigvee_{k=1}^{\tau(t)} d(x_1, \varphi_k) + 1\Big)\Big] \vee$$

$$\vee \Big[c_2 + \Big(\bigvee_{k=1}^{\tau(t)} d(x_2, \varphi_k) + 1\Big)\Big]\Big\}$$

$$= n \wedge \big[d_0(\varphi) \vee [c_1 + d(x_1, \varphi)] \vee [c_2 + d(x_2, \varphi)]\big],$$

so the lemma holds for $q+1$ and therefore for arbitrary q. Q.E.D.

Proof of Theorem 1. For a unary type τ the theorem was proved in [7]. So we can assume that $\tau(T) \setminus \{0, 1\} \neq \emptyset$.

By (i) there exists an essentially binary term $\varphi(x_1, x_2)$ associated with τ such that the equality

$$\varphi(x_1, x_2) = \varphi(x_1, x_1) \tag{7}$$

belongs to $E(\boldsymbol{K})$. Let $\boldsymbol{K}''$ denote the variety of type τ defined by (7) only. Obviously $\boldsymbol{K} \subseteq \boldsymbol{K}''$. Since $\tau(T) \setminus \{0, 1\} \neq \emptyset$, there exists a non-nullary fundamental polynomial symbol f_{t_0} associated with τ. Denote

$$f^1(x) = f_{t_0}(x, \ldots, x), \quad f^{s+1}(x) = f(f^s(x)).$$

Let $m = \max\{d_0(\varphi), d(x_1, \varphi), d(x_2, \varphi)\}$. By (3) it must be $m > 0$. Let $r = m$

$+1-d(x_2, \varphi)$ and let $\psi(x_1, x_2) = \varphi(x_1, f^r(x_2))$. Denote by K' the variety of type τ defined by the equality:

(8) $$\psi(x_1, x_2) = \psi(x_1, x_1) \quad \text{only}.$$

Obviously $K'' \subseteq K'$ since (8) is a consequence of (7). To complete the proof we shall show that the algebra $\mathfrak{A}_n$ defined above for $n = m+1$ belongs to K' but does not belong to K''. It follows easily from (1)–(6) that

$$d(x_2, \psi) \geqslant d(x_2, \varphi) + d(x_2, f^r(x_2)) = m+1.$$

So by Lemma 1 we have: $\bar{\psi}(c_1, c_2) = \bar{\psi}(c_1, c_1) = m+1$ for any $c_1, c_2 \in \{0, \ldots, m+1\}$. However $\bar{\varphi}(0, m+1) = m+1$ but

$$\bar{\varphi}(0, 0) = \min\{m+1, \max\{d_0(\varphi), d(x_1, \varphi), d(x_2, \varphi)\}\} = m.$$

So $\mathfrak{A}_{m+1} \in K'$ and $\mathfrak{A}_{m+1} \notin K''$.

Let us denote by $S = S(\tau)$ the variety of type τ: $T \to N$ defined by all possible regular equalities which can be written down by means of terms associated with τ. The structure of any algebra $\mathfrak{A} = (A, \{f_t\}_{t \in T}) \in S$ is simple, namely: If $T = \emptyset$ then $\mathfrak{A} = (A, \emptyset)$. If there are nullary fundamental polynomials in $\mathfrak{A}$ then there exists a fixed element $O \in A$ such that $f_t = O$ for any t such that $\tau(t) = 0$. If $\tau(t) = 1$ then $f_t(x) = x$ since the last equality is regular, so any unary fundamental operation is the identity; if $\tau(t) \notin \{0, 1\}$, then let us denote $x \oplus y = f_t(x, y, \ldots, y)$, thus $\oplus$ is a semilattice operation since the equalities $x \oplus x = x$, $x \oplus y = y \oplus x$, $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ written by means of $f_t(x, y, \ldots, y)$ are regular, we have also $x \oplus f_t = x$ for any t such that $\tau(t) = 0$ and $f_t(x_1, \ldots, x_{\tau(t)}) = x_1 \oplus \ldots \oplus x_{\tau(t)}$.

Anyway, S is consistent and it is easy to show that S is equationally complete.

Algebras from S in the case where $0 \notin \tau(T)$ and $\tau(T) \setminus \{1\} \neq \emptyset$ were considered by J. Dudek in [1] and were called there *τ-semilattices*. The unary case was considered in [7].

Let us denote by $U(\tau)$ the variety of type τ defined by the empty set of equalities. Then we have:

(iv) S is the least element and $U(\tau)$ is the greatest element of the lattice $(R(\tau), \vee, \wedge)$.

Let $(A; \vee, \wedge)$ be a lattice with the greatest element 1. An element $a \in A$ is called a *coatom* if $a \neq 1$ and for any $b \in A$ such that $a \leqslant b$ we have $b = a$ or $b = 1$. It is known (see [8]) that if $\tau(T) \setminus \{0\} \neq \emptyset$ then there are no coatoms in the lattice $(L(\tau), \vee, \wedge)$. So we have:

COROLLARY 1. *There are no coatoms in the lattice* $(R(\tau), \vee, \wedge)$.

In fact, let $K \in R(\tau)$ and $K \neq U(\tau)$. By the result of Tarski quoted above there exists $K' \in R(\tau)$ such that $K \subsetneq K'$, but K' must be regular since $E(K') \subseteq E(K) = R(K)$.

3. Connections between join semilattices of regular and non-regular varieties. We want now to consider properties of the mapping ϱ: $L(\tau) \to L(\tau)$ defined by the formula $\varrho(\boldsymbol{K}) = \boldsymbol{K}_{\boldsymbol{R}}$. Obviously we have:

(v) $\varrho(\varrho(\boldsymbol{K})) = \varrho(\boldsymbol{K})$;

(vi) if $\boldsymbol{K} \in R(\tau)$ then $\varrho(\boldsymbol{K}) = \boldsymbol{K}$;

(vii) $\boldsymbol{K} \subseteq \varrho(\boldsymbol{K})$;

(viii) if $\boldsymbol{K} \in N(\tau)$, then $\varrho(\boldsymbol{K}) = \boldsymbol{K} \vee \boldsymbol{S}$. In fact

$$E(\boldsymbol{K}_{\boldsymbol{R}}) = R(\boldsymbol{K}) = E(\boldsymbol{K}) \cap E(\boldsymbol{S}) = E(\boldsymbol{K} \vee \boldsymbol{S}).$$

By (viii) we have:

(ix) if $\boldsymbol{K}$, $\boldsymbol{K}' \in L(\tau)$ then

$$\varrho(\boldsymbol{K} \vee \boldsymbol{K}') = \varrho(\boldsymbol{K}) \vee \varrho(\boldsymbol{K}') = \varrho(\boldsymbol{K}) \vee \boldsymbol{K}'.$$

In fact

$$\begin{aligned}\varrho(\boldsymbol{K} \vee \boldsymbol{K}') &= (\boldsymbol{K} \vee \boldsymbol{K}') \vee \boldsymbol{S} = (\boldsymbol{K} \vee \boldsymbol{S}) \vee (\boldsymbol{K}' \vee \boldsymbol{S}) = \varrho(\boldsymbol{K}) \vee \varrho(\boldsymbol{K}') \\ &= (\boldsymbol{K} \vee \boldsymbol{S}) \vee \boldsymbol{K}' = \varrho(\boldsymbol{K}) \vee \boldsymbol{K}'.\end{aligned}$$

LEMMA 2. *If* $\boldsymbol{K}'$, $\boldsymbol{K}'' \in N(\tau)$ *and* $\boldsymbol{K}' \neq \boldsymbol{K}''$ *then* $\varrho(\boldsymbol{K}') \neq \varrho(\boldsymbol{K}'')$.

Proof. For unary algebras the lemma was proved in [7]. Let $\tau(T)\setminus\{0, 1\} \neq \emptyset$. Assume $\varrho(\boldsymbol{K}') = \varrho(\boldsymbol{K}'')$. If $\mathfrak{A} \in \boldsymbol{K}'$ then $\mathfrak{A} \in \boldsymbol{K}'_{\boldsymbol{R}}$ and consequently $\mathfrak{A} \in \boldsymbol{K}''_{\boldsymbol{R}}$ so by (ii) it is of the form described in (ii) for $\boldsymbol{K} = \boldsymbol{K}''$. It cannot be $|I| > 1$ since then by (iii) $\mathfrak{A}$ satisfies only regular equalities contrary to the assumption $\boldsymbol{K}' \in N(\tau)$. Thus $|I| = 1$ and by (ii) $\mathfrak{A} \in \boldsymbol{K}''$. By symmetry we get $\boldsymbol{K}'' \subseteq \boldsymbol{K}'$. Finally $\boldsymbol{K}' = \boldsymbol{K}''$.

An endomorphism h of a semilattice $(B, \cdot)$ is called a *retraction* if it is idempotent, i.e. $h(h(a)) = h(a)$ for any $a \in B$.

An endomorphism h of $(B, \cdot)$ is called *splitting* if it satisfies $h(ab) = h(a)\cdot b = a\cdot h(b)$ for any $a, b \in B$. Further h is called *extensive* if $a \leqslant h(a)$ for any $a \in B$, where $\leqslant$ is the semilattice order.

The results of this chapter we can now formulate as follows

THEOREM 2. *The mapping* ϱ: $L(\tau) \to L(\tau)$ *has the following properties*:

(a_1) ϱ *is an extensive splitting retraction of* $L(\tau)$;

(a_2) ϱ *is an embedding of* $(N(\tau), \vee)$ *into* $(R(\tau), \vee)$;

(a_3) ϱ *maps the least element of* $N(\tau)$ *onto the least element of* $R(\tau)$, *i.e.*, $\varrho(\boldsymbol{D}(\tau)) = \boldsymbol{S}(\tau)$.

(a_4) *the greatest element* $\boldsymbol{U}(\tau)$ *of* $R(\tau)$ *has no counterimage in* $N(\tau)$.

Proof. (a_1) follows by (v), (vii) and (ix). The condition (a_2) in the case where τ is nullary is obvious since we have only 1 non-regular variety namely $\boldsymbol{D}(\tau)$. If τ is unary then (a_2) follows from [7] (thereom 3). If $\tau(T)\setminus\{0, 1\} \neq \emptyset$ then (a_2) follows by lemma 2 and (a_1). (a_3) follows from the definition of $\boldsymbol{S}(\tau)$. The property (a_4) can be easily obtained from Theorem 1.

Remark. Any of the sets $N(\tau)$ and $R(\tau)$ is a sublattice of $L(\tau)$, so one could expect that ϱ is an embedding of the lattice $N(\tau)$ into the lattice $R(\tau)$. However, it is not the case in general since ϱ need not preserve $\wedge$, see [3].

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