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A COUNTABLE CONNECTED URYSOHN SPACE WITH A DISPERSION POINT THAT IS REGULAR ALMOST EVERYWHERE

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It is well known that a countable topological space that is regular or merely regular at every point of some open set cannot be connected [11]. However, examples have been given of countable connected Hausdorff spaces (see [1]-[4], [7]-[9], and [11]) and other examples (see [5], [6], [9], and [10]) have been given of countable connected spaces that satisfy the Urysohn separation axiom in the sense that for every pair of distinct points a and b there are open sets U and V for which $a \in U$, $b \in V$ and $\overline{U} \cap \overline{V} = \emptyset$. Some of these spaces are locally connected (see [6], [7], and [9]), others (see [4], [8]-[10]) have a dispersion point and the example due to Roy [10] is regular at the dispersion point but at no other point. He asks if a space exists with the properties of his space, i.e., countable, connected, Urysohn, a dispersion point, that is also regular at every point of some dense subset. It is the purpose of this note to construct such an example. First Roy's space is modified to obtain a topology τ very similar to his. Then a topology τ' coarser than τ, i.e., $\tau' \subset \tau$, is constructed so that the resulting space is the desired one.

Construction of the space Y, τ.

Let $C_0, C_1, C_{-1}, C_2, C_{-2}, \ldots$ be pairwise disjoint subsets of the rational numbers such that no one of them contains an integer and each is dense in the reals. Let Z be the set of integers and let $X = \{(x, y) \mid y \in Z, x \in C_y\}$. The set Y will be $X \cup Z \cup \{\omega\}$ where ω is an ideal point. For a geometrical interpretation of Y it will be helpful to picture the points of X as points in the plane, the points of Z along a horizontal line at the foot of the plane (coordinates $(i, -\infty)$ for each i in Z) and the ideal point ω at the top of the plane $(0, +\infty)$.

For $p = (s, t) \in X$ and $\varepsilon > 0$ let

$$N_\varepsilon(p) = \begin{cases} \{(x, y) | x \in C_t;\ y = t;\ s-\varepsilon < x < s+\varepsilon\} & \text{for } t \text{ even}, \\ \{(x, y) | x \in C_y;\ y = t, t-1, t+1;\ s-\varepsilon < x < s+\varepsilon\} & \text{for } t \text{ odd}. \end{cases}$$

For $\varepsilon > 0$ and $i \in Z$ let

$$N_\varepsilon(i) = \{i\} \cup \{(x, y) | x \in C_y;\ y \in Z;\ y \leqslant -2[1/\varepsilon];\ i-\varepsilon < x < i+\varepsilon\}.$$

Finally, for $\varepsilon > 0$, let

$$N_\varepsilon(\omega) = \{\omega\} \cup \{(x, y) | x \in C_y,\ y \in Z;\ y \geqslant 2[1/\varepsilon]\}.$$

The collection $\{N_\varepsilon(y)\}$ of all these neighborhoods of points of Y forms a basis for a topology τ of which the subspace $X \cup \{\omega\}$ is the space described by Roy [10]. In a manner analogous to the proof given there it can be shown that Y is a countable connected Urysohn space and that ω is a dispersion point. The point ω is a dispersion point means that given distinct points $a, b \in Y - \{\omega\}$, there exists a set U, open and closed in $Y - \{\omega\}$, such that $a \in U$, $b \notin U$. It should be noted that Y is regular at ω and at each point of Z but at no point of X. It is clear that the regular points do not form a dense subset.

We will now describe a coarser topology τ' such that with this topology Z is dense in Y, Y is regular at each point of Z and all the other properties are preserved.

Construction of the space Y, τ'.

Take f to be any one to one function from X onto Z. For $y \in Y$ and $\varepsilon > 0$ let

$$\begin{aligned} U_0^\varepsilon(y) &= N_\varepsilon(y), \\ U_1^\varepsilon(y) &= \bigcup \{N_\varepsilon(i) | i \in f(U_0^\varepsilon(y) - \{y\})\}, \\ U_n^\varepsilon(y) &= \bigcup \{N_\varepsilon(i) | i \in f(U_{n-1}^\varepsilon(y) - Z)\} \quad \text{for } n \geqslant 2. \end{aligned}$$

Now let

$$U_\varepsilon(y) = \bigcup_{i=0}^{\infty} U_i^\varepsilon(y).$$

It is clear from this construction that $f(U_\varepsilon(y) - [\{y\} \cup Z]) \subset U_\varepsilon(y)$. The collection $\{U_\varepsilon(y)\}$ of all neighborhoods of points of Y forms a basis for a topology τ'. For let $U_\varepsilon(x)$ and $U_{\varepsilon'}(y)$ be neighborhoods of points $x, y \in Y$, respectively, and let $z \in U_\varepsilon(x) \cap U_{\varepsilon'}(y)$. There exists $\varepsilon_0 > 0$ such that

$$U_0^{\varepsilon_0}(z) = N_{\varepsilon_0}(z) \subset U_\varepsilon(x) \cap U_{\varepsilon'}(y).$$

Furthermore, if $\varepsilon_0 < \min(\varepsilon, \varepsilon')$, then

$$U_n^{\varepsilon_0}(z) \subset U_\varepsilon(x) \cap U_{\varepsilon'}(y) \quad \text{for } n \geqslant 1.$$

Therefore

$$U_{\varepsilon_0}(z) = \bigcup_{i=0}^{\infty} U_i^{\varepsilon_0}(z) \subset U_\varepsilon(x) \cap U_{\varepsilon'}(y).$$

Y, τ' is countable and connected.

Each neighborhood $U_\varepsilon(y)$ in τ' is the union of open sets in τ, so $\tau' \subset \tau$. Since Y, τ is connected, surely Y, τ' is connected.

Y, τ' is an Urysohn space.

If $A \subset Y$, denote the closure of A in τ and τ' by $\mathrm{Cl}A$ and $\overline{A}$, respectively. Now, if $0 < \varepsilon < \frac{1}{3}$ and i, j are integers such that $i \neq j$, it is clear that $\mathrm{Cl}\, N_\varepsilon(i) \cap \mathrm{Cl}\, N_\varepsilon(j) = \emptyset$. From this it follows that if $y \in Y$ and $0 < \varepsilon < \frac{1}{3}$, then

$$\overline{U_\varepsilon(y)} = \mathrm{Cl}\, N_\varepsilon(y) \cup \bigcup \{\mathrm{Cl}\, N_\varepsilon(i) \mid i \in f(U_n^\varepsilon(y) - [\{y\} \cup Z]);\ n = 0, 1, 2, \ldots\}.$$

To show Y, τ' is an Urysohn space, suppose $x \in X$, $i \in Z$; other cases are treated similarly. Choose $\varepsilon > 0$ such that the following four conditions hold:

(1) $\mathrm{Cl}\, N_\varepsilon(x) \cap \mathrm{Cl}\, N_\varepsilon(i) = \emptyset$;

(2) $f^{-1}(i) \notin \mathrm{Cl}\, N_\varepsilon(x)$ unless $f^{-1}(i) = x$;

(3) the y-coordinate of $f^{-1}(i) > -2[1/\varepsilon]$;

(4) the y-coordinate of $x > -2[1/\varepsilon]+3$.

Condition (1) follows because Y, τ is an Urysohn space; (2) and (3) guarantee that $i \notin U_\varepsilon(x)$; (4) guarantees that $\mathrm{Cl}\, N_\varepsilon(x) \cap \overline{U_\varepsilon(i)} = \emptyset$. If, in addition, $\varepsilon < \frac{1}{3}$, then these conditions together with the structure of $\overline{U_\varepsilon(x)}$ and $\overline{U_\varepsilon(i)}$ imply that $\overline{U_\varepsilon(x)} \cap \overline{U_\varepsilon(i)} = \emptyset$.

Y, τ' is regular almost everywhere.

Clearly Z is a dense subset of Y. Assume $i \in Z$ and let $U_\varepsilon(i)$ be a basic neighborhood of i. Let $\varepsilon' = \min(\varepsilon/2, \frac{1}{3})$. It is clear that $\mathrm{Cl}\, N_{\varepsilon'}(i) \subset N_\varepsilon(i)$. For each n ($n = 0, 1, 2, \ldots$) we have $f(U_n^{\varepsilon'}(i) - Z) \subset f(U_n^\varepsilon(i) - Z)$ and, consequently,

$$\bigcup \{\mathrm{Cl}\, N_{\varepsilon'}(j) \mid j \in f(U_n^{\varepsilon'}(i) - Z)\} \subset \bigcup \{N_\varepsilon(j) \mid j \in f(U_n^\varepsilon(i) - Z)\}.$$

It follows that $\overline{U_{\varepsilon'}(i)} \subset U_\varepsilon(i)$.

ω is a dispersion point of Y, τ'.

Let $p, q \in Y - \{\omega\}$. Assume $p = (s, t) \in X$ and $q \in Z$; similar arguments apply to other cases. Choose $\varepsilon > 0$ such that the following conditions hold:

(1) the x-coordinate of $f^{-1}(q)$ does not lie in any of the intervals $(i-\varepsilon, i+\varepsilon)$, $i \in Z$; nor does it lie in the interval $(s-\varepsilon, s+\varepsilon)$ unless $p = f^{-1}(q)$;

(2) the interval $(s-2\varepsilon, s+2\varepsilon)$ contains no integer;

(3) $\{(x, y) | x \in C_y; y \in Z; x = s-\varepsilon \text{ or } s+\varepsilon\} = \emptyset$ and if $i \in Z$, then we have $\{(x, y) | x \in C_y; y \in Z; x = i-\varepsilon \text{ or } i+\varepsilon\} = \emptyset$.

For each $i \in Z$ let $M_\varepsilon(i) = \{i\} \cup \{(x, y) | x \in C_y; y \in Z; i-\varepsilon < x < i+\varepsilon\}$. Furthermore, let

$$V_0^\varepsilon(p) = \{(x, y) | x \in C_y; y \in Z; s-\varepsilon < x < s+\varepsilon\},$$
$$V_1^\varepsilon(p) = \bigcup\{M_\varepsilon(i) | i \in f(V_0^\varepsilon(p) - \{p\})\},$$
$$V_n^\varepsilon(p) = \bigcup\{M_\varepsilon(i) | i \in f(V_{n-1}^\varepsilon(p) - Z)\} \quad \text{for } n \geqslant 2.$$

Finally, let

$$V_\varepsilon(p) = \bigcup_{i=0}^{\infty} V_i^\varepsilon(p).$$

Clearly $p \in V_\varepsilon(p)$ and $q \notin V_\varepsilon(p)$. For each $z \in V_\varepsilon(p)$ there exists $\varepsilon' > 0$ such that $U_{\varepsilon'}(z) \subset V_\varepsilon(p)$ since $V_\varepsilon(p)$ has the property that

$$f(V_\varepsilon(p) - [\{p\} \cup Z]) \subset V_\varepsilon(p).$$

Therefore $V_\varepsilon(p)$ is open. As before,

$$\overline{V_\varepsilon(p)} = \mathrm{Cl}\, V_0^\varepsilon(p) \cup \bigcup \{\mathrm{Cl}\, M_\varepsilon(i) | i \in f(V_n^\varepsilon(p) - [\{p\} \cup Z]),\ n = 0, 1, 2, \ldots\}.$$

But, from (3),

$$\mathrm{Cl}\, V_0^\varepsilon(p) = V_0^\varepsilon(p) \cup \{\omega\} \quad \text{and} \quad \mathrm{Cl}\, M_\varepsilon(i) = M_\varepsilon(i) \cup \{\omega\}, \quad i \in Z;$$

therefore $\overline{V_\varepsilon(p)} = V_\varepsilon(p) \cup \{\omega\}$. Thus $V_\varepsilon(p)$ is closed in $Y - \{\omega\}$ and ω is a dispersion point.

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