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CONTINUA WHOSE CONNECTED SUBSETS ARE ARCWISE CONNECTED

BY

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1. A *continuum* is a non-degenerate, compact, connected, metric space. A continuum is said to be *hereditarily locally connected* if each of its subcontinua is locally connected. A continuum is said to be *regular* if it has a basis of open sets with finite boundaries. An *arc* is a homeomorph of the closed unit interval $[0, 1]$. A space X is said to be *arcwise connected* if for each $x, y \in X$ there is an arc in X which contains x and y.

Let A be a connected subset of a connected space X. A set C in X is said to *locally separate* A in X if there is an open set U in X such that $U \cap A$ is connected and $(U \cap A) - C$ meets at least two components of $U - C$. A point $x \in X$ is said to be a *local cutpoint of* X if $\{x\}$ locally separates X in X.

This paper* contains some partial solutions to the problem of characterizing the continua whose connected subsets are arcwise connected. It follows readily from a result in [4], p. 249, that these continua are hereditarily locally connected. In this paper, it is proved that if every connected subset of a continuum X is arcwise connected, then the set of local cutpoints of X is uncountable and locally separates every proper subcontinuum of X. These results are used to prove that certain known hereditarily locally connected continua contain connected subsets that are not arcwise connected.

Most of our notation is taken from Dugundji [1]. It is with pleasure that I acknowledge the help of Prof. L. Mohler who made many suggestions regarding this problem.

2. The following is the key theorem of this paper. The proof is modeled on one given by Kuratowski and Knaster in [5].

THEOREM 1. *Let X be a continuum such that $X = A_0 \cup A_1 \cup A_2 \cup \ldots$, where $A_0 \neq \emptyset$ and*

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(i) *A_0 contains at most a countable number of local cutpoints of X,*

(ii) *the sets A_i are pairwise disjoint,*

(iii) *A_i is closed for each $i = 1, 2, \ldots$*

Then X contains a connected subset that is not arcwise connected.

Proof. By an earlier comment, we may assume X is locally connected.

By a result in [2], p. 201, $X = E \cup F$, where neither E nor F contains a perfect subset.

Let B denote the set of points in A_0 which are local cutpoints of X and let $Y = E \cup B \cup A_1 \cup A_2 \cup \ldots$

We shall prove that Y is a connected subset of X that is not arcwise connected.

Suppose T is an arc in Y which meets both A_1 and A_2. Let $C = T - (B \cup A_1 \cup A_2 \cup \ldots)$. Then, C is a G_δ in the complete space T. By a theorem of Mazurkiewicz ([1], p. 308), C is topologically complete. By Sierpiński's Theorem ([4], p. 173), C is uncountable. Hence, $C \subset E$ contains a perfect set. This is a contradiction. Thus, Y is not arcwise connected.

Suppose that Y is not connected. Then, $Y = C \cup D$, where C is separated from D. Since X is completely normal, there exists a closed set G in X such that G separates C from D. Since F contains no perfect set and G is a compact subset of F, it follows that G is either countable or finite. Clearly, $X = \bar{E} \subset \bar{C} \cup \bar{D}$. Since G separates C from D, $\bar{C} \cap \bar{D}$ is a closed subset of G which separates X. Let x be an isolated point of $\bar{C} \cap \bar{D}$ and let U be a connected neighbourhood of x such that $\bar{C} \cap \bar{D} \cap U = \{x\}$. Since $x \notin C \cup D$, it follows that x separates U. Thus, x is a local cutpoint of X. Since $x \in A_0$, we have $x \in B$. This contradiction proves that Y is connected.

Theorem 1 may be restated as follows:

THEOREM 1'. *If X is a continuum such that every connected subset of X is arcwise connected and A_0 is a subset of X whose complement is the union of countably many pairwise disjoint compact sets, then A_0 contains uncountably many local cutpoints of X.*

THEOREM 2. *If X is a locally connected continuum which has only countably many local cutpoints, then X has a connected subset which contains no perfect set.*

Proof. Let $A_0 = X$ and imitate the proof of Theorem 1.

The triangular Sierpiński curve R is defined in [4], p. 276, and in [5]. It is easy to see that the local cutpoints of R are contained among the vertices of the countable number of triangles that are used to define R. From Theorem 2, we get the following theorem of Kuratowski and Knaster:

COROLLARY 3 (Kuratowski and Knaster [5]). *The triangular Sierpiński curve contains a connected subset which contains no perfect subset.*

3. In this section, we shall provide an application of Theorem 1. We shall need the following two preliminary results:

PROPOSITION 4. *Let X be a continuum. If there exists an $\varepsilon > 0$ and a sequence A_i of connected pairwise disjoint subsets of X such that for each $i = 1, 2, \ldots$ the diameter of A_i is greater than ε, then X contains a connected subset that is not arcwise connected.*

Proof. We may assume without loss of generality that each A_i is an arc and that X is hereditarily locally connected. Let $A = A_1 \cup A_2 \cup \ldots$ By a theorem in [4], p. 272, A has only a finite number of components. By Sierpiński's Theorem, each component of A which contains more than one of the arcs A_i is not arcwise connected.

Definition. Let ϱ be a metric for a space X. If $A \subset X$ and $r > 0$, we let

$$S(A, r) = \{x \in X \mid \varrho(x, y) < r \text{ for some } y \in A\}.$$

LEMMA 5. *Let X be a hereditarily locally connected continuum. If A is an arc in X and $\varepsilon > 0$ is given, then there exists $\delta > 0$ such that every arc in $S(A, \delta) - A$ has diameter less than ε.*

Proof. It is proved in [4], p. 269, that a hereditarily locally connected continuum contains no continuum of convergence.

THEOREM 6. *If a continuum X contains a subcontinuum A_0 such that no point of A_0 locally separates A_0 in X, then X contains a connected subset that is not arcwise connected.*

Proof. We may suppose that X is hereditarily locally connected and, by Proposition 4, that for every $\varepsilon > 0$, there exist at most finitely many pairwise disjoint connected subsets of X of diameter greater than ε.

We may also assume that A_0 is an arc. We identify A_0 with the closed unit interval $[0, 1]$ and we let A_0 have its usual linear order with initial point 0.

We construct inductively a continuum Y in X such that Y satisfies the hypotheses of Theorem 1.

Since the point $\frac{1}{2}$ does not locally separate $A_0 = [0, 1]$ in X, there exists an arc C in X such that $C \cap A_0 = \{a, b\}$, where $1/4 < a < 1/2 < b < 3/4$ and where a and b are the end points of C.

For $\varepsilon > 0$, let K_ε be the component of $\big(S(A_0, \varepsilon) \cup C\big) - A_0$ which meets C. Let a_ε (resp. b_ε) be the minimal (resp. maximal) point of $\overline{K}_\varepsilon \cap A_0$. We may assume that there is an arc C_ε in $K_\varepsilon \cup \{a_\varepsilon, b_\varepsilon\}$ with endpoints a_ε and b_ε. Otherwise, $K_\varepsilon \cup \{a_\varepsilon, b_\varepsilon\}$ would be a connected subset of X which is not arcwise connected. We can ensure, by Lemma 5, that if we take $\varepsilon > 0$ to be sufficiently small, then $1/4 < a_\varepsilon \leqslant a < 1/2 < b \leqslant b_\varepsilon < 3/4$.

Pick such an $\varepsilon > 0$ and denote C_ε by $C(1/2, 1)$, a_ε by $a\,(1/2, 1)$ and b_ε by $b\,(1/2, 1)$. Since $a\,(1/2, 1)$ does not locally separate A_0, there exist arcs of arbitrarily small diameter in $X-\{a(1/2,1)\}$ which meet both components of $A_0-\{a(1/2,1)\}$. From the way $C(1/2,1)$ was constructed it follows that $C(1/2,1)$ is disjoint from these small diameter arcs. A similar situation exists at $b(1/2,1)$. This argument proves that $C(1/2,1)$ does not locally separate A_0 in X.

Let $A_1 = C(1/2,1)$. Suppose that $A_1,\ldots,A_{n-1}$ have been constructed to be pairwise disjoint sets such that, for each $i=1,\ldots,n-1$,

(i) A_i is the union of a finite number of arcs each of which meets A_0 in precisely two points,

(ii) no point of $[1/2^i, 1-1/2^i]$ separates $A_0 \cup A_i$,

(iii) A_i does not locally separate A_0 in X,

(iv) $A_0 \cap A_i \subset\,]1/2^{i+1}, 1-1/2^{i+1}[$.

For each $x\in[1/2^n, 1-1/2^n]$, let C_x be an arc in the complement of $A_1\cup\ldots\cup A_{n-1}$ with end points a_x and b_x such that $1/2^{n+1} < a_x < x < b_x < 1-1/2^{n+1}$ and $C_x \cap A_0 = \{a_x, b_x\}$. This is possible since the set $A_1\cup\ldots\cup A_{n-1}\cup\{x\}$ does not locally separate A_0 in X. By the method used to construct $C(1/2,1)$, we construct an arc $C(x,n)$ in the complement of the compact set $A_1\cup\ldots\cup A_{n-1}$ with end points $a(x,n)$ and $b(x,n)$ such that

$$C(x,n)\cap A_0 = \{a(x,n), b(x,n),\}$$

$$\frac{1}{2^{n+1}} < a(x,n) \leqslant a_x < x < b_x \leqslant b(x,n) < 1-\frac{1}{2^{n+1}}$$

and $C(x,n)$ does not locally separate A_0 in X.

The set of open intervals

$$\{]a(x,n), b(x,n)[\ \mid x\in[1/2^n, 1-1/2^n]\},$$

is an open cover for the compact set $[1/2^n, 1-1/2^n]$. Hence, there exist $x_1,\ldots,x_m\in[1/2^n, 1-1/2^n]$ such that

$$[1/2^n, 1-1/2^n] \subset\,]a(x_1,n), b(x_1,n)[\ \cup\ldots\cup\ ]a(x_m,n), b(x_m,n)[.$$

Let $A_n = C(x_1,n)\cup\ldots\cup C(x_m,n)$. It is not difficult to check that conditions (i)-(iv) above hold for A_n.

Let $Y = A_0\cup A_1\cup A_2\cup\ldots$ By hypothesis, there exists for each $\varepsilon>0$ a natural number N_ε such that if $m>N_\varepsilon$, then every arc in A_m has length less than ε. Thus, if $m>N_\varepsilon$, then $A_m\subset S(A_0,\varepsilon)$. It follows that Y is a continuum. From the construction, it follows that the set $Y-(A_1\cup A_2\cup\ldots)$ contains no local cutpoints of Y. By Theorem 1, Y contains a connected subset that is not arcwise connected.

The following example is due to Knaster (see [4], p. 284):

Example. Let R be the plane continuum consisting of the segment $0\leqslant x\leqslant 1$, $y=0$, of the semi-circles

$$\left(x-\frac{2k-1}{2^n}\right)^2 + y^2 = \frac{1}{4^n}, \qquad y\geqslant 0,$$

where $n = 1, 2, \ldots$ and $k = 1, 2, \ldots, 2^{n-1}$, and of the semi-circles

$$\left(x - \frac{2k-1}{2\cdot 3^n}\right)^2 + y^2 = \frac{1}{4\cdot 9^n}, \quad y \leqslant 0,$$

where $n = 1, 2, \ldots$ and $k = 1, 2, \ldots, 3^n$.

A. Lelek has conjectured that the continuum R contains a connected subset that is not arcwise connected. The following theorem asserts that Lelek's conjecture is correct:

COROLLARY 7. *The continuum R in the above example contains a connected subset that is not arcwise connected.*

Proof. The segment $0 \leqslant x \leqslant 1$, $y = 0$, contains no points that locally separate it in R. Thus, Theorem 6 applies to R.

Remark. It is not also difficult to construct a regular continuum Y in the continuum R of the above example such that Y contains a connected subset that is not arcwise connected.

4. We now state a few general propositions and some open questions concerning continua whose connected subsets are arcwise connected.

Definition: A function $f: X \to Y$ is said to be *monotonic* if for each $y \in Y$, $f^{-1}(y)$ is connected.

PROPOSITION 8. *Let f be a continuous, monotonic function of a continuum X onto a Hausdorff space Y. If every connected subset of X is arcwise connected, then every connected subset of Y is arcwise connected.*

Proof. Let A be a connected subset of Y and let x and y be two points in A. Then, $f^{-1}(A)$ is a connected subset of X. Let B be an arc in $f^{-1}(A)$ which meets both $f^{-1}(x)$ and $f^{-1}(y)$. Then, $f(B)$ is a locally connected continuum in A which contains x and y. Hence, there exists an arc in A which contains both x and y.

Question 1. Let X be a continuum such that every connected subset of X is arcwise connected. Is X regular? (**P 766**)

Question 2. Let X be a continuum which cannot be embedded in any continuum Y such that Y is the union of a countable family of arcs. Does X contain a connected subset that is not arcwise connected? (**P 767**)

Question 2 was motivated by the following result:

PROPOSITION 9. *Let X be a continuum such that, for each $\varepsilon > 0$, there exist at most a finite number of pairwise disjoint, connected subsets of X of diameter greater than ε. Suppose that if Y is any continuous, monotonic, Hausdorff image of X, then Y can be embedded in a continuum Z such that Z is the union of a countable family of arcs. Then, for every subset C of X such that the arc components of C are compact, the arc-components of C coincide with the components of C.* (An arc-component of C is a maximal arcwise connected subset of C.)

Proof. Let C be a subset of X such that the path components of C are compact. Let $\sim$ be the equivalence relation defined on X by letting $x \sim y$ if and only if $x = y$ or x and y lie in the same arc-component of C. Let π be the natural projection of X onto the quotient space $X/\sim$. Then π is continuous and monotonic, and $X/\sim$ is Hausdorff. Let Z be a continuum which contains $X/\sim$ such that Z is the union of a countable family of arcs. It is easy to see that $\pi(C)$ has trivial arc-components. By the Sum Theorem, for dimension 0, it follows hat $\pi(C)$ is 0-dimensional.

Question 3. Suppose X is a continuum which satisfies the hypotheses of Proposition 9. Is every connected subset of X arcwise connected? (**P 768**)

Question 4. Let X be a continuum such that every subcontinuum C of X contains a point that locally separates C in X. Is X regular? (**P 769**)

By [4], p. 269, X is hereditarily locally connected because X, clearly, does not contain any non-trivial continua of convergence.

An affirmative solution to Question 4 would yield an affirmative solution to Question 1.

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