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ON DECOMPOSITION OF PSEUDOMEASURES ON SOME SUBSETS OF LCA GROUPS

BY

P. GŁOWACKI (WROCŁAW)

Let G be a locally compact abelian group, and Γ its dual. For a closed $F \subset G$, $A(F)$ denotes, as usually, the Fourier algebra on F ($A(F)$ is a Banach algebra). $A'(F)$ denotes the *dual Banach space* to $A(F)$, that is the space of those pseudomeasures on G which annihilate all functions from $A(G)$ vanishing on F. For such a pseudomeasure $S \in A'(F)$ its value on a function $f \in A(F)$ is equal to the value of S on any extension of f to a member of $A(G)$.

Suppose F_1 and F_2 to be closed disjoint subsets of G. If one of them is compact, it is easy to see that every pseudomeasure $S \in A'(F_1 \cup F_2)$ may be represented as follows:

(A) $$S = S_1 + S_2, \quad \text{where } S_1 \in A'(F_1),\ S_2 \in A'(F_2).$$

This decomposition is unique and topologically continuous, so we have

$$A'(F_1 \cup F_2) = A'(F_1) \oplus A'(F_2),$$

where $\oplus$ denotes the norm topology direct sum. This formula is not true if neither F_1 nor F_2 is compact.

If F_1 and F_2 are not disjoint, it makes no sense to require the unique decomposition of pseudomeasures, as it does not hold even for measures. However, passing to quotients, we may define another form of "unique decomposition"

(S) $$A'(F_1 \cup F_2)/A'(F_1 \cap F_2) = A'(F_1)/A'(F_1 \cap F_2) \oplus A'(F_2)/A'(F_1 \cap F_2),$$

where $\oplus$ denotes the quotient norm topology direct sum. Evidently, (S) implies (A). As it will be shown later, even (A) is in general "rather difficult" to be satisfied.

For discrete measures it seems reasonable to consider another kind of decomposition, namely

(M) $$M(F_1 \cup F_2) = M(F_1) \oplus M(F_2 \setminus F_1),$$

where $\oplus$ denotes the $A'(F_1 \cup F_2)$ topology direct sum.

In this paper we shall consider F_2 compact and such that $F_1 \cap F_2$ is just the set of all cluster points of $F_2 \setminus F_1$. In this case it turns out that there are some close relations concerning the existence of decompositions (A), (S) and (M). Our main result is Theorem 2, where we assume additionally the spectral synthesis of either F_1 or $F_1 \cap F_2$. This is the subject of this paper, more precisely — of Section 2. Section 1 is devoted to the proof of Theorem 1, which allows us to formulate (M) in terms of the algebra $A(G)$. We continue investigations initiated in [3]. Basic notions and notation are taken from [2] and [3]. Definitions of the decompositions mentioned above will be given once more, sometimes in somewhat different but, obviously, equivalent way.

We wish to thank Professor S. Hartman for his advice and encouragement.

1. Let K and E be disjoint subsets of G. We fix them throughout this section. Let us write condition (M) in the form used in [3]:

(M) There exists a constant $\varkappa$ such that for any finitely supported measure $\mu \in M(K \cup E)$ we have

$$\|\mu | E\|_{PM(G)} \leqslant \varkappa \|\mu\|_{PM(G)}.$$

(M) may also be formulated as follows:

(M′) There exists a measure $\nu \in M(\tilde{\Gamma})$ ($\tilde{\Gamma}$ denotes the Bohr compactification of Γ) such that $\hat{\nu}|K \equiv 0$ and $\hat{\nu}|E \equiv 1$, where $\hat{\nu}$ denotes the Fourier transform of the measure ν.

Here is another condition introduced in [3]:

(D) There exists a net $f_a \in A(G)$ such that $f_a | K \equiv 0$,

$$\sup_a \|f_a\|_{A(G)} < \infty \quad \text{and} \quad f_a(t) = 1 \text{ for any } t \in E,$$

whenever a is sufficiently large.

To prove our first theorem we need two lemmas.

LEMMA 1. *Let F be a finite subset of G. Then there is a measure μ_F with finite support contained in* GpF *(the group generated by F) and such that $\mu_F(\{x\}) = 1$ for $x \in F$ and $\mathfrak{M}(|\hat{\mu}_F|) < 2$, where $\mathfrak{M}$ denotes the mean value on almost periodic functions on Γ.*

Proof. For $G = \boldsymbol{R}$ this is proved in [1]. For the sake of completeness we reproduce here the proof *mutatis mutandis*.

Let $\{\lambda_j\}_{j=1}^n$ be a basis for GpF. Each $t \in$ GpF may uniquely be represented in the form

$$t = \sum_{j=1}^{n} t_j \lambda_j.$$

Put

$$b = \max_{\substack{t \in F \\ j=1,\ldots,n}} (|t_j|)$$

and take an integer $N \geqslant b$. As $\lambda_j \in G$ are characters of Γ, we may regard them as functions on Γ. It is trivial that for $s \in \Gamma$, $s \neq 0$,

$$D_j^N(s) = \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N}\right) \lambda_j^{-k}(s) = \frac{1}{N+1} \frac{2 - \lambda_j^{N+1}(s) - \lambda_j^{-(N+1)}(s)}{2 - \lambda_j(s) - \lambda_j^{-1}(s)} \geqslant 0$$

and

$$D_j^N(0) = \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N}\right) \geqslant 0,$$

so the functions D_j^N are non-negative. Thus

$$\mathfrak{M}\left(\left|D_j^N + \sum_{|k| \leqslant b} \frac{|k|}{N} \lambda_j^{-k}\right|\right) \leqslant \mathfrak{M}(D_j^N) + \sum_{|k| \leqslant b} \frac{|k|}{N} = 1 + \frac{b(b+1)}{N} < 2^{1/n}$$

for N sufficiently large. Let $\mu_j^N \in M(G)$ be such that

$$(\mu_j^N)^\wedge = D_j^N + \sum_{|k| \leqslant b} \frac{|k|}{N} \lambda_j^{-k}.$$

Then $\mu_F = \mu_1^N * \mu_2^N * \ldots * \mu_n^N$ has finite support contained in $\mathrm{Gp}\,F$ and $\mathfrak{M}(|\hat{\mu}_F|) < 2$. Moreover, $\mu_F(\{x\}) = 1$ for $x \in F$.

Now, let V be a compact neighbourhood of 0 in G. Put

$$\Delta_V = \frac{1}{|V|} \varphi_V * \varphi_V,$$

where $|\cdot|$ denotes Haar measure and φ_V is the characteristic function of V. Obviously, Δ_V has the following properties:

(i) $\Delta_V \in A(G)$, $\|\Delta_V\|_{A(G)} = \Delta_V(0) = 1$;

(ii) Δ_V vanishes outside $V + V$;

(iii) $\hat{\Delta}_V \geqslant 0$.

Denote by $\mathscr{V}$ the family of all compact neighbourhoods of 0 in G. The family $\mathscr{V}$ is directed by inclusion.

LEMMA 2 (cf. [1], lemme 2). *Let* $\mu \in M_d(G)$. *Then*

$$\lim_{V \in \mathscr{V}} \|\Delta_V * \mu\|_{A(G)} = \mathfrak{M}(|\hat{\mu}|).$$

Proof. Let $\varepsilon > 0$. There exists a trigonometric polynomial $w = \sum C_\gamma \gamma$ such that $\||\hat{\mu}| - w\|_\infty \leqslant \varepsilon$. Therefore, by (i),

$$\left| \int_G \hat{\Delta}_V |\hat{\mu}| - \int_G \hat{\Delta}_V w \right| \leqslant \varepsilon \quad \text{for any } V \in \mathscr{V}.$$

But, by (i) and (ii),

$$\int_G \hat{\Delta}_V w = \sum C_\gamma \int_G \hat{\Delta}_V \gamma = \sum C_\gamma \Delta_V(\gamma) = C_0$$

for $V \in \mathscr{V}$ sufficiently small. As $|C_0 - \mathfrak{M}(|\hat{\mu}|)| \leqslant \varepsilon$, we have

$$\Big| \int_G \hat{\Delta}_V |\hat{\mu}| - \mathfrak{M}(|\hat{\mu}|) \Big| \leqslant 2\varepsilon$$

for $V \in \mathscr{V}$ sufficiently small. But

$$\|\Delta_V * \mu\|_{A(G)} = \int_G \hat{\Delta}_V |\hat{\mu}|$$

by (iii), and so the proof is complete.

THEOREM 1. *If K is closed, then* (M) *is equivalent to* (D).

Proof. Let (f_a) satisfy (D),

$$\sup_a \|f_a\|_{A(G)} = \varkappa,$$

$\mu \in M(K \cup E)$ with finite support and $f \in A(G)$. Take a such that $f_a(t) = 1$ for $t \in \operatorname{supp} \mu$. Then

$$\langle \mu | E, f \rangle = \langle \mu | E, f_a f \rangle = \langle \mu, f_a f \rangle.$$

Thus

$$|\langle \mu | E, f \rangle| \leqslant \varkappa \|\mu\|_{PM(G)} \|f\|_{A(G)},$$

and so

$$\|\mu | E\|_{PM(G)} \leqslant \varkappa \|\mu\|_{PM(G)},$$

which completes the first part of the proof.

We now assume (M′). Let $\nu \in M(\tilde{\Gamma})$ be such that $\check{\nu} | K \equiv 0$, $\check{\nu} | E \equiv 1$, where $\check{\nu}(t) = \hat{\nu}(-t)$. Let F be a finite subset of E. Then $\nu_F = \check{\nu}\mu_F$ is a measure on G with finite support disjoint from K. We have

$$\mathfrak{M}(|\hat{\nu}_F|) = \mathfrak{M}(|\nu * \hat{\mu}_F|) < 2 \|\nu\|_{M(\tilde{\Gamma})}.$$

Hence there is a compact neighbourhood $V_0 \in \mathscr{V}$ such that

$$\|\Delta_V * \nu_F\|_{A(G)} \leqslant 2 \|\nu\|_{M(\tilde{\Gamma})}$$

for $V \in \mathscr{V}$ contained in V_0 (Lemma 2). Take $V_F \in \mathscr{V}$, $V_F \subset V_0$, such that

$$(t + V + V) \cap K = \emptyset \quad \text{and} \quad (t' + V + V) \cap (t + V + V) = \emptyset$$

for any $t, t' \in F$. We see that $f_F = \Delta_{V_F} * \nu_F$ vanishes in some neighbourhood of K and satisfies $\|f_F\|_{A(G)} \leqslant 2 \|\nu\|_{M(\tilde{\Gamma})}$ as well as $f_F(t) = 1$ for $t \in F$. The net $(f_F)_{F \in \mathscr{F}}$ ($\mathscr{F}$ is the family of all finite subsets of E directed by inclusion) satisfies (D). Our proof is thus complete.

PROPOSITION 1. (D) *is equivalent to somewhat "stronger" condition: functions f satisfy* (D) *and vanish not only on K but also on some neighbourhood of K.*

This is obvious from the proof of Theorem 1.

We end this section with a consequence of the Cohen theorem:

PROPOSITION 2. *If $K\cap \mathrm{Gp}E$ is finite, then* (M) *is satisfied.*

Proof. In fact, $\mathrm{Gp}E$ is an open subgroup of G_d (G with discrete topology), so $\mathrm{Gp}E\setminus K$ is a support of the Fourier transform of some idempotent measure on $\tilde{\Gamma}$ ([4], Chapter VI.8.2, Proposition 5). As $E\subset \mathrm{Gp}E\setminus K$, our assertion is proved.

2. From now we assume K to be closed, $\bar{E}$ to be compact, $E'\subset K$, and, as before, $E\cap K=\emptyset$ (by E' we denote the set of all cluster points of E). Hence for any neighbourhood U of E' there is only a finite number of elements of E outside U.

We introduce the Banach spaces (see [2])

$$A_0=\{f\in A(\bar{E}):f|E'\equiv 0\}\quad\text{and}\quad A_{00}=\{f\in A(K\cup E):f|K\equiv 0\}$$

which are closed subspaces of the Banach spaces $A(\bar{E})$ and $A(K\cup E)$, respectively, with usual norms. The map

$$A_{00}\ni f\mapsto f|\bar{E}\in A_0$$

is injective, so we shall regard A_{00} as a subspace of A_0 in the algebraic sense.

Let $S\in A'(\bar{E})$. Then $\langle S,f\rangle=0$ for all $f\in A_{00}$ if and only if $S\in A'(K)$. Hence we get the following

PROPOSITION 3. *A_{00} is dense in A_0 exactly if*

$$A'(K)\cap A'(\bar{E})=A'(E').$$

We have the equality $A_0=A_{00}$ if and only if

$$\text{(S)}\qquad A'(K\cup E)/A'(E')=A'(K)/A(E')\oplus A'(\bar{E})/A'(E'),$$

where $\oplus$ denotes the topological direct sum [3]. Evidently, (S) implies

$$\text{(A)}\qquad A'(K\cup E)=A'(K)+A'(\bar{E}).$$

PROPOSITION 4. *$A_0=A_{00}$ if and only if A_{00} is dense in A_0 and* (A) *holds.*

Proof. Of course, $A_0=A_{00}$ implies both density of A_{00} in A_0 and (A). Suppose then A_{00} to be dense in A_0 and (A) to be satisfied. Due to (A) we have

$$A'(E\cup K)/A'(E')=A'(K)/A'(E')+A'(\bar{E})/A'(E')$$

and, by Proposition 3, this is a direct sum. By (S) the proof is complete.

The chief purpose of this section is to examine relations between (M) and (A) and between (M) and the equality $A_0 = A_{00}$. At first we give an example of K and E satisfying $\mathrm{Gp}K \cap \mathrm{Gp}E = \{0\}$ while $A_0 \neq A_{00}$. This gives a negative answer to P 1054 and, combined with Theorem 1, also a negative answer to P 1053 in [3].

Definition ([4], p. 257, Definition 4). Let $(t_k)_{k=1}^{\infty}$ be a sequence of real numbers such that

$$\sum_{k=1}^{\infty} |t_k| < \infty.$$

We say that $(t_k)_{k=1}^{\infty}$ is *fully independent* if for any bounded sequence of rational integers $(p_k)_{k=1}^{\infty}$ the following implication holds:

$$\sum_{k=1}^{\infty} p_k t_k = 0 \Rightarrow p_k = 0 \quad \text{for every } k \geqslant 1.$$

Example 1. (Here we put $G = \boldsymbol{R}$.) Let $(t_k)_{k=1}^{\infty}$ be fully independent and such that

$$t_k \in [0, 1] \quad \text{and} \quad t_k > \sum_{j=k+1}^{\infty} t_j \quad \text{for every } k \geqslant 1$$

(see [4], p. 257, Proposition 5). Let

$$K_0 = \Big\{\sum_{k=2}^{\infty} \varepsilon_k t_k : \varepsilon_k = 0, 1\Big\}.$$

K_0 is a perfect symmetric set, so it contains a compact subset F for which spectral synthesis fails ([4], p. 255, Theorem VIII). By a known theorem of Herz, there exists a compact K of synthesis such that $F \subset K \subset \mathrm{lin}_{\boldsymbol{Q}} F$. Write

$$E_{(n)} = \left\{\frac{p}{n} t_1 : 0 < \mathrm{dist}\left(\frac{p}{n} t_1, F\right) < \frac{t_1}{n},\ 0 \leqslant p \leqslant n\right\}$$

and put

$$E = \bigcup_{n=1}^{\infty} E_{(n)}.$$

Evidently, $E' = F$. We have

$$\mathrm{lin}_{\boldsymbol{Q}} E \cap \mathrm{lin}_{\boldsymbol{Q}} K = \{0\}$$

because of the choice of $(t_k)_{k=1}^{\infty}$. At the same time, K and $\bar{E}$ are of synthesis (by construction) and for $E' = F$ spectral synthesis fails. Hence, by Proposition 3 (see also [3], Proposition 1), A_{00} is not dense in A_0.

However, it turns out that, in our example, (A) holds. That is an immediate consequence of Theorem 2.

THEOREM 2. (i) *If K is of synthesis, then* (M) *implies* (A).

(ii) *If E' is of synthesis, then* (M) *implies $A_0 = A_{00}$.*

Proof. Let (f_a) fulfill the "stronger" form of (D) (see Proposition 1). If $S \in A'(K\cup E)$, then $f_a S \in M_d(E)$ and

$$\|f_a S\|_{PM(G)} \leqslant \varkappa \|S\|_{PM(G)}, \qquad \text{where } \varkappa = \sup_a \|f_a\|_{A(G)}.$$

Assume that $(f_\beta S)$ is a subnet of $(f_a S)$ converging in the topology $\sigma(PM(G), A(G))$ to some $S_1 \in A'(\bar{E})$. Let

$$S_2 = S - S_1 \in A'(K\cup E).$$

If $\varphi \in A(G)$ vanishes on some neighbourhood of K, then $f_\beta\varphi | K\cup E = \varphi | K\cup E$ for β sufficiently large, and thus

$$\langle S_2, \varphi\rangle = \langle S, \varphi\rangle - \lim \langle S, f_\beta\varphi\rangle = 0,$$

which implies that $S_2 \in PM(K)$ ($PM(K) = A'(K)$ by synthesis of K). Since $S = S_1 + S_2$, the proof of (i) is complete.

Now we assume E' to be of synthesis. First we show that every $f \in A_0$ belongs to the closure in A_0 norm (i.e. $A(\bar{E})$ norm) of

$$B = \{\varphi \in A_{00}\colon \|\varphi\|_{A_{00}} \leqslant (\varkappa+1)\|f\|_{A_0}\}.$$

In fact, for arbitrary $0 < \varepsilon < \|f\|_{A_0}/2\varkappa$ take $\tilde{f} \in A(G)$ such that $\tilde{f} | \bar{E} = f$ and $\|\tilde{f}\|_{A(G)} \leqslant \|f\|_{A_0} + \varepsilon$. By synthesis of E' there exists $g \in A(G)$ which vanishes in some neighbourhood of E' and fulfills $\|g - \tilde{f}\|_{A(G)} < \varepsilon$. Since $g = f_a g$ on $\bar{E}$ for a sufficiently large, for some a we have

$$(*) \qquad \|f_a g | \bar{E} - f\|_{A_0} \leqslant \|g - \tilde{f}\|_{A(G)} \leqslant \varepsilon.$$

But $f_a g | (K\cup E) \in A_{00}$ and, moreover,

$$\|f_a g | K\cup E\|_{A_{00}} \leqslant \|f_a g\|_{A(G)} \leqslant \varkappa \|g\|_{A(G)} \leqslant \varkappa(\|\tilde{f}\|_{A(G)} + \varepsilon)$$
$$\leqslant \varkappa(\|f\|_{A_0} + 2\varepsilon) \leqslant (\varkappa+1)\|f\|_{A_0}.$$

Thus $f_a g | K\cup E$ belongs to B. Since ε is arbitrary, $f \in \bar{B}$ by $(*)$.

To complete the proof it is sufficient to show that the norms in A_0' and A_{00}' are equivalent. Evidently, we have $\|\cdot\|_{A_{00}'} \leqslant \|\cdot\|_{A_0'}$ Take now $S \in A_0'$ and $f \in A_0$. By the first part of the proof, $f = \lim \varphi_n$ in A_0 norm, where $\varphi_n \in A_{00}$ and $\|\varphi_n\|_{A_{00}} \leqslant (\varkappa+1)\|f\|_{A_0}$. Therefore

$$|\langle S, f\rangle| = \lim_n |\langle S, \varphi_n\rangle| \leqslant \|S\|_{A_{00}'}(\varkappa+1)\|f\|_{A_0}.$$

This shows that $\|S\|_{A_0'} \leqslant (\varkappa+1)\|S\|_{A_{00}'}$ and completes the proof.

COROLLARY. *Let E' be finite. Then* (M) *holds if and only if $A_0 = A_{00}$.*

Proof. (M) implies $A_0 = A_{00}$ by Theorem 2 (ii). Conversely, suppose that $A_0 = A_{00}$. Then by (S)

$$A'(K\cup E)/A'(E') = A'(K)/A'(E') \oplus A'(\bar{E})/A'(E').$$

But E' is finite and we have

$$A'(K\cup E) = A'(K)\oplus A'(\bar{E})/A'(E').$$

So

$$M(K\cup E) = M(K) \underset{PM(G)}{\oplus} M(E),$$

which is equivalent to (M).

An easy example (with $G = \boldsymbol{R}$) shows that (M) may fail although E' is countable and $A_0 = A_{00}$. In fact, take $v_n \in \boldsymbol{R}$, $0 \neq v_n \nearrow 0$ and $t_n \in \boldsymbol{R}$, $0 \neq t_n \searrow 0$ such that (M) fails for $K = \{v_n\}_{n=1}^{\infty} \cup \{0\}$ and $E = \{t_n\}_{n=1}^{\infty}$ ([3], le corollaire to théorème 5). Let us join a countable set F to E in such a way that $E\cup F$ be bounded, disjoint from K and $(E\cup F)' = K$. Then $A_0 = A_{00}$ for K and $E\cup F$ but (M) still fails.

In [3] it is shown that (D) implies

$$\text{(n)} \qquad \|\mu\|_{PM(G)} \leqslant \varkappa \|\mu\|_{A_{00}'}$$

for $\mu \in M_d(E)$ and some $\varkappa > 0$, where

$$\|\mu\|_{A_{00}'} = \inf_{S\in A'(K)} \|\mu + S\|_{PM(G)}.$$

Now we are going to answer the question P 1052 of [3].

PROPOSITION 5. (n) *is equivalent to* (D).

Proof. It is sufficient to prove that (n) implies (M). If $\mu \in M_d(K\cup E)$, then

$$\|\mu|E\|_{PM(G)} \leqslant \varkappa \|\mu|E\|_{A_{00}'}$$

by (n). At the same time, for an f which vanishes on K and belongs to $A(G)$ we have

$$|\langle \mu|E, f\rangle| = |\langle \mu, f\rangle| \leqslant \|\mu\|_{PM(G)}\|f\|_{A(G)}.$$

Thus $\|\mu\,|E\|_{A_{00}'} \leqslant \|\mu\|_{PM(G)}$ and, finally,

$$\|\mu|E\|_{PM(G)} \leqslant \varkappa \|\mu\|_{PM(G)}.$$

Our last proposition is a positive answer to P 1055 of [3].

PROPOSITION 6. *For any sequence of real numbers $t_n \neq 0$, $t_n \to 0$, there exists a compact $K \subset [-1, 0]$ such that the left-side metric density of K at* 0 *equals* 1, *K is disjoint from $E = \{t_n\}_{n=1}^{\infty}$ and* (M) ($A_0 = A_{00}$) *holds for K and E.*

Proof. In view of Proposition 2, for (M) to be satisfied it is sufficient to construct a compact $K \subset [-1, 0]$ with required density at 0 and disjoint from $\mathrm{Gp}E\setminus\{0\}$. But this is very easy, since E is countable, and thus $\mathrm{Gp}E$ is of Lebesgue measure zero. Since E' is a one-point set, we have also $A_0 = A_{00}$ by Theorem 2.

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