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# THE FIXED POINT PROPERTY FOR SET-VALUED MAPPINGS OF SOME ACYCLIC CURVES

BY

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All spaces considered in the paper are assumed to be metric, provided the opposite is not said. A curve means a 1-dimensional continuum. A hereditarily decomposable continuum such that every two its points can be joined by exactly one irreducible subcontinuum is said to be a $\lambda$-*dendroid* (see [4], p. 15 and 16).

**1. Acyclic curves.** The homology theory we use in this section is based on the nerves of coverings, first given by P. S. Aleksandrov and extended by E. Čech (see, e.g., [13], p. 135). A space $X$ is said to be *acyclic* if all its homology groups are trivial (see, e.g., [2], p. 35). Since every $\lambda$-dendroid is a curve ([4], (1.6), p. 16), the $n$-th homology group of a $\lambda$-dendroid is trivial for all $n \neq 1$. And it is known that every mapping of a $\lambda$-dendroid $X$ onto a circle is inessential ([3], Theorem XI, p. 217) which is equivalent (see, e.g., [13], p. 150) to the triviality of the first homology group of $X$. Therefore

(1.1) *Every $\lambda$-dendroid is an acyclic curve.*

This is an answer to the question asked by prof. K. Borsuk in a conversation.

Conversely ([1], p. 17),

(1.2) *Every acyclic curve is hereditarily unicoherent.*

Since every hereditarily decomposable continuum is 1-dimensional (see [21], Theorem II, p. 328), it follows from (1.2) that

(1.3) *Every hereditarily decomposable acyclic continuum is a $\lambda$-dendroid.*

**2. Set-valued mappings.** A set-valued mapping $F: X \to Y$ from a space $X$ to a space $Y$ is a point-to-set correspondence such that, for each $x \in X$, $F(x)$ is a closed subset of $Y$. A set-valued mapping $F: X \to Y$ is said to be *continuous* if the following conditions are satisfied:

(i) for each $x \in X$ and for each open set $V$ of $Y$ such that $F(x) \subset V$ there exists a neighbourhood $U$ of $x$ such that $F(x') \subset V$ for each $x' \in U$ (the upper semi-continuity of $F$);

(ii) for each $x \in X$ and for each open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$ there exists a neighbourhood $U$ of $x$ such that $F(x') \cap V \neq \emptyset$ for each $x' \in U$ (the lower semi-continuity of $F$).

For basic properties of upper and lower semi-continuous set-valued mappings see, e.g., [15], p. 173-182, and [16], p. 57-75. Recall that (see, e.g., [22], 9.2, p. 179) a set-valued mapping $F: X \to Y$ is *upper* (*lower*) *semi-continuous* if and only if the set $\{x: F(x) \cap A \neq \emptyset\}$ is closed (open) whenever $A$ is closed (open) in $Y$. In other words, $F$ is continuous if and only if the inverse images of closed sets are closed and of open sets are open.

Let $X$ be a topological space and $\mathscr{C}$ a class of set-valued mappings of $X$ into itself. We say that $X$ has the *fixed point property* for $\mathscr{C}$ (F.p.p. for $\mathscr{C}$) if, for each $F \in \mathscr{C}$, there exists a point $x \in X$ such that $x \in F(x)$. Two classes of set-valued mappings were investigated by a number of authors: the class $\mathscr{C}_1$ of all upper semi-continuous continuum-valued mappings, and the class $\mathscr{C}_2$ of all continuous closed set-valued mappings.

Consider three properties of continua $X$:

(I) $X$ has the F.p.p. for $\mathscr{C}_1$,

(II) $X$ is hereditarily unicoherent,

(III) $X$ has the F.p.p. for $\mathscr{C}_2$.

The following are two main problems:

PROBLEM 1. Characterize all continua $X$ which have the F.p.p. for $\mathscr{C}_1$. (**P 811**)

PROBLEM 2. Characterize all continua $X$ which have the F.p.p. for $\mathscr{C}_2$. (**P 812**)

Although natural to state, both problems are still open. There are known only some partial solutions, which are indicative of relations between continua having the F.p.p. for $\mathscr{C}_1$ or $\mathscr{C}_2$ and acyclic continua.

It seems natural to consider dendrites, dendroids and $\lambda$-dendroids as increasing classes of acyclic curves. One can define curves which belong to these classes as hereditarily unicoherent continua which are

(A) locally connected,

(B) arcwise connected,

(C) hereditarily decomposable,

respectively. These three classes of continua will be considered here in connection with properties (I), (II) and (III).

(A) Wallace has proved ([27], Theorem A, p. 757) that every tree (i.e., a locally connected acyclic compact Hausdorff continuum) has the

F.p.p. for $\mathscr{C}_1$. It is well known that, for locally connected continua $X$, the hereditary unicoherence is equivalent to the acyclicity of $X$; thus it follows from the Wallace result that, for locally connected continua, (II) implies (I). The inverse implication (also for locally connected continua) follows from a result due to Ward, Jr. (see [30], Lemma 4, p. 162):

(2.1) *If $Y$ is a compactum and $S$ is a simple closed curve contained in $Y$, then there exists a continuous continuum-valued mapping $R$ from $Y$ to $S$ such that $R(t) = \{t\}$ for each $t \in S$.*

Finally, Plunkett has proved the equivalence between (II) and (III) for locally connected continua (see [24], Theorems 1 and 2, p. 161 and 162). Therefore

(2.2) *If a continuum $X$ is locally connected, then* (I) $\Leftrightarrow$ (II) $\Leftrightarrow$ (III).

Since a locally connected continuum $X$ is a dendrite if and only if (II), (2.2) can be restated as follows (see [30], Theorem 3, p. 164):

(2.3) *For the class of locally connected continua, the property of being a dendrite is equivalent to having the F.p.p. for $\mathscr{C}_1$ as well as to having the F.p.p. for $\mathscr{C}_2$.*

(B) For arcwise connected continua, Ward, Jr., has proved that (I) is equivalent to (II) (see [30], Theorems 1 and 2, p. 162 and 163) and that (II) implies (III) (see [28], Theorem 2, p. 926). Therefore

(2.4) *If a continuum $X$ is arcwise connected, then* (I) $\Leftrightarrow$ (II) $\Rightarrow$ (III).

Since an arcwise connected and hereditarily unicoherent continuum is a dendroid, Ward's results (2.4) can be restated as follows (see [31], Theorems 6 and 7, p. 92):

(2.5) *For the class of arcwise connected continua, the property of being a dendroid is equivalent to having the F.p.p. for $\mathscr{C}_1$.*

(2.6) *Each dendroid has the F.p.p. for $\mathscr{C}_2$.*

The problem if, in the class of arcwise connected continua, dendroids can be characterized as continua which have the F.p.p. for $\mathscr{C}_2$, is still open. It was first asked in 1961 by Ward, Jr. ([30], p. 160; see also [31], p. 92). Since for arcwise connected continua (II) implies (III), the problem remains for the inverse implication only. We recall it here as

PROBLEM 3. Is it true that every arcwise connected continuum which has the F.p.p. for $\mathscr{C}_2$ is hereditarily unicoherent?

It is a conjecture that the answer is affirmative ([31], p. 92).

(C) The aim of this paper is to investigate relations between properties (I), (II) and (III) for hereditarily decomposable continua. It will be proved that, in this class of continua, (I) implies (II) and (II) implies (III). Thus

(2.7) *If a continuum $X$ is hereditarily decomposable, then* (I) $\Rightarrow$ (II) $\Rightarrow$ (III).

Since a hereditarily decomposable and hereditarily unicoherent continuum is a $\lambda$-dendroid, (2.7) can be restated as follows:

(2.8) *If a hereditarily decomposable continuum has the F.p.p. for $\mathscr{C}_1$, then it is a $\lambda$-dendroid.*

(2.9) *Every $\lambda$-dendroid has the F.p.p. for $\mathscr{C}_2$.*

To prove (2.8) it suffices to prove Proposition (2.8′) which is equivalent to (2.8) by a simple transposition.

(2.8′) *If a hereditarily decomposable continuum $X$ is not hereditarily unicoherent, then there exists an upper semi-continuous continuum-valued mapping $F: X \to X$ which is fixed point free.*

Proof of (2.8′). We adopt Ward's proof of Theorem 2 in [30], p. 163. If a continuum $X$ is hereditarily decomposable and not hereditarily unicoherent, then it contains (see [23], Theorem 2.6, p. 187) a subcontinuum $N$ such that $N$ is a simple closed curve with respect to the elements of some upper semi-continuous collection of mutually exclusive continua $N_s$ filling up $N$. Thus $N = \bigcup\{N_s: s \in S\}$, where $S$ is a simple closed curve. For each $x \in X$ we define $[x] = \{x\}$ if $x \in X \setminus N$ and $[x] = N_s$ if $x \in N_s \subset N$. Let $[X]$ be the space of all $[x]$, endowed with the quotient topology, that is, if $\sigma: X \to [X]$ is the natural mapping $\sigma(x) = [x]$, then $V$ is an open subset of $[X]$ if and only if $\sigma^{-1}(V)$ is open in $X$. We note that $\sigma$ is continuous and monotone and that $\sigma(N) = S$ is a simple closed curve in $[X]$. Thus the space $[X]$ satisfies the hypothesis of (2.1) and hence there exists a continuous continuum-valued mapping $R: [X] \to S$ such that $R([x]) = [x]$ for each $[x] \in S$. Let $h: S \to S$ be a fixed point free homeomorphism. Define $F(x) = \sigma^{-1}hR\sigma(x)$. By Lemma 3 in [30], p. 161, each $R\sigma(x)$ is a continuum and since $h$ is a homeomorphism, so is $hR\sigma(x)$. Therefore, $\sigma$ being monotone, each $F(x)$ is a continuum by the definition. Hence, the mapping $F$ is continuum-valued. To see that $F$ is upper semi-continuous we shall verify that the set $\{x: F(x) \cap A \neq \emptyset\}$ is closed for any closed set $A$ in $X$. Since $X$ is compact and $\sigma$ and $h$ are continuous, it follows that $h^{-1}\sigma(A)$ is closed. Since $R$ is continuous,

$$R^{-1}h^{-1}\sigma(A) = \{[x]: R([x]) \cap h^{-1}\sigma(A) \neq \emptyset\}$$

is closed, and, therefore, the set

$$\sigma^{-1}R^{-1}h^{-1}\sigma(A) = \{x: F(x) \cap A \neq \emptyset\}$$

is also closed. — Finally, suppose there exists $x \in F(x)$; then $\sigma(x) \in hR\sigma(x)$ and, since $\sigma(x) \in S$, we have $R\sigma(x) = \sigma(x)$, whence $\sigma(x) \in h\sigma(x)$, whereas $h$ was assumed to be fixed point free. Thus $F$ is without fixed points, and so the proof of (2.8′), and thereby of (2.8), is complete.

The proof of (2.9) will be presented in Section 3.

As a consequence of (2.9) we have

(2.10) *Every $\lambda$-dendroid has the fixed point property for continuous (single-valued) mappings.*

This is an answer to the question raised by prof. B. Knaster (The New Scottish Book, Problem 526). Result (2.10) partially generalizes some earlier results due to Gray [9] and [10], Hamilton [11], Holsztyński [12], Sieklucki [25], Ward, Jr., [29], the author [5]-[7] and others.

The question whether the inverse implications to those in (2.7) are true is open. So we have

PROBLEM 4. Is it true that every hereditarily decomposable continuum which has the F.p.p. for $\mathscr{C}_2$ is hereditarily unicoherent? (**P 813**)

PROBLEM 5. Is it true that every $\lambda$-dendroid has the F.p.p. for $\mathscr{C}_1$? (**P 814**)

It is our conjecture that the answer to both Problems 4 and 5 is positive. In that case we would get a characterization of $\lambda$-dendroids as hereditarily decomposable continua which have the F.p.p. for $\mathscr{C}_1$ as well as for $\mathscr{C}_2$.

**3. Inverse limits.** Let $\Pi = \{P_a\}, a \in A$, be a class of polyhedra (i.e., triangulable compacta). A compactum $X$ is said to be *$\Pi$-like* provided that for each $\varepsilon > 0$ there is a polyhedron $P_a \in \Pi$ and an $\varepsilon$-mapping of $X$ onto $P_a$ (see [20], p. 146). Mardešić and Segal have proved (see [20], Theorem 1*, p. 148) the following theorem:

(3.1) *Let $\Pi$ be a class of connected polyhedra. Then the class of $\Pi$-like compacta coincides with the class of limits of inverse systems $\{P_i, \pi_{ij}\}$ with mappings $\pi_{ij}$ onto and with $P_i \in \Pi$.*

Cook has proved ([8], p. 20) that

(3.2) *Every $\lambda$-dendroid is tree-like.*

Taking for $\Pi$ the class of finite metric trees (i.e., the class of finite dendrites), we get from (3.1) and (3.2)

(3.3) *Every $\lambda$-dendroid is the limit of some inverse sequence of finite dendrites with bonding mappings onto.*

This is an answer to the question asked by R. Duda (The New Scottish Book, Problem 828).

The following theorem is announced in [32]:

(3.4) *Let a compact space $X$ be the limit of an inverse system $\{X_i, \pi_{ij}, D\}$ of compact spaces $X_i$ with bonding mappings $\pi_{ij}$ onto. If each space $X_j$, $i \in D$, has the F.p.p. for $\mathscr{C}_2$, then $X$ has the F.p.p. for $\mathscr{C}_2$.*

The corresponding theorem for single-valued continuous mappings is not true (see [18], p. 252), but the known counter-examples are 2- or 3-dimensional (see [14] and [26]). Thus one can ask the following

PROBLEM 6. Does there exist a curve without the fixed point property for single-valued mappings which is the limit of an inverse system of curves which have the fixed point property for single-valued mappings? **(P 815)**

A theorem similar to (3.4) is proved in [17], but under some additional assumptions on $F$ (see [19]).

Now, assuming (3.4) to be true, we are able to prove (2.9).

Proof of (2.9). By (2.3) each dendrite has the F.p.p. for $\mathscr{C}_2$, and so we receive (2.9) from (3.3) and (3.4).

Let $X$ be the closed interval $[-1, 1]$ of reals, and let $F\colon X \to X$ be defined as follows:

$$F(x) = \begin{cases} \{x+1\} & \text{if } -1 \leqslant x < 0, \\ \{-1, 1\} & \text{if } x = 0, \\ \{x-1\} & \text{if } 0 < x \leqslant 1. \end{cases}$$

Thus $F(x)$ is closed for each $x$, by the definition, and it is easy to verify that $F$ is upper semi-continuous. But $F$ is not lower semi-continuous, and there is no point $x$ in $X$ with $x \in F(x)$. Observe that $F(0)$ is not connected. — This example shows that the hypothesis of the lower semi-continuity of $F$ is essential in (2.9) and that the hypothesis of the connectedness of $F(x)$ for each $x \in X$ is essential in the implication (II) $\Rightarrow$ (I) in Theorems (2.2) and (2.4).

One could answer Problem 5 in the affirmative (i.e., could prove that $\lambda$-dendroids have the F.p.p. for $\mathscr{C}_1$) exactly in the same way as (2.9) is proved, provided that the result similar to (3.4) is true for $\mathscr{C}_1$ in place of $\mathscr{C}_2$. Thus the following question seems to be interesting:

PROBLEM 7. Let a compact space $X$ be the limit of an inverse system $\{X_i, \pi_{ij}, D\}$ of compact spaces $X_i$ with bonding mappings $\pi_{ij}$ onto. Does it follow that if each $X_i$, $i \in D$, has the F.p.p. for $\mathscr{C}_1$, then $X$ has the F.p.p. for $\mathscr{C}_1$? **(P 816)**

**4. Remarks.** Let $X$ be a continuum. We shall show that property (I) can be formulated also in two other slightly different ways:

(I′) If $F_1, F_2\colon X \to X$ are two upper semi-continuous continuum-valued mappings of $X$ into itself, then there exist two points $x_1$ and $x_2$ in $X$ such that $x_2 \in F_1(x_1)$ and $x_1 \in F_2(x_2)$.

(I″) If $F\colon X \to X$ is an upper semi-continuous continuum-valued mapping of $X$ into itself, and if $g\colon X \to X$ is a monotone continuous (single-valued) mapping of $X$ onto itself, then there exists a point $x \in X$ such that $g(x) \in F(x)$.

(4.1) *For an arbitrary continuum $X$ we have* (I) $\Leftrightarrow$ (I′) $\Leftrightarrow$ (I″).

Proof of (4.1). (I) $\Rightarrow$ (I′). Let $F_1$ and $F_2$ be mappings as in (I′). The mapping $F_2F_1: X \to X$ defined by $F_2F_1(x) = \bigcup\{F_2(y): y \in F_1(x)\}$ is obviously upper semi-continuous and, by Lemma 3 in [30], p. 161, continuum-valued, thus by (I) there is a point $x_1 \in F_2F_1(x_1)$, and so, according to the definition of the mapping $F_2F_1$, there exists a point $x_2 \in F_1(x_1)$ such that $x_1 \in F_2(x_2)$.

(I′) $\Rightarrow$ (I″). Put $F_2(x) = g^{-1}(x)$ for $x \in X$. Since $g$ is continuous, $F_2$ is upper semi-continuous (see [15], Theorem 5, p. 177). Further, $g$ is monotone, whence $F_2$ is continuum-valued. Applying (I′) with $F_1 = F$, we obtain two points $x_1$ and $x_2$ such that $x_2 \in F(x_1)$ and $x_1 \in F_2(x_2) = g^{-1}(x_2)$, i.e., $x_2 = g(x_1)$, and, therefore, $g(x_1) \in F(x_1)$.

(I″) $\Rightarrow$ (I). Put $g(x) = x$ for $x \in X$.

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