

*ORBIT SPACES OF THE HYPERSPACE OF A GRAPH
WHICH ARE HILBERT CUBES*

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0. Introduction. Let $\alpha: G \times X \rightarrow X$ be a transformation group of a compact group G on a compact metric space X . Let 2^X denote the hyperspace of nonvoid, closed subsets of X , topologized by the Hausdorff metric, and let $\beta: G \times 2^X \rightarrow 2^X$ be the induced action, i.e.,

$$\beta(g, A) = \bigcup_{a \in A} \alpha(g, a).$$

Such actions are studied in [8] and [9]. In particular, in [8] Toruńczyk and the second-named author examined the translative action of S^1 on its own hyperspace, finding that the orbit space contains a lattice of naturally defined Hilbert cube manifold Eilenberg–MacLane spaces of the type $K(Z_{(\mathfrak{P})}, 2)$ corresponding to the lattice of finite subgroups of S^1 . (Here $Z_{(\mathfrak{P})}$ denotes the localization of the integers at a set \mathfrak{P} of primes.) In order to extend this result to other compact Lie groups we have had to analyze hyperspace actions of finite groups induced from actions on Peano continua, and, in particular, from actions on finite graphs.

In this paper*, we show (Corollary 2) that if X is a nondegenerate Peano continuum P and if β is as above, then $2^P/\beta$ is a Hilbert cube if it is an AR (absolute retract for metric spaces). This is an equivariant version of the Curtis–Schori Theorem [1]. [In general, it is unknown whether orbit spaces of infinite-dimensional ANR's by finite group actions are ANR's. For example, it is still unknown whether a semi-free action of Z_2 , on a Hilbert cube with a unique fixed point, has the orbit space of the homotopy type of a CW-complex (cf. [0], [9], [10]).] We do not here attempt any further analysis in the general case. However, we do show (Theorem 2) that, for a finite, connected, nondegenerate graph Γ , $2^\Gamma/\beta$ is a Hilbert cube. Our proof of Theorem 1 relies heavily on Toruńczyk's topological characterization of the

* This research was partially supported by N.S.F. grants.

Hilbert cube [7]. Theorem 2 should be thought of as an analog of [6], in which Schori and the second-named author established that 2^F is a Hilbert cube.

1. Preliminaries. Given a group action $\alpha: G \times X \rightarrow X$, X a metric space, we say that the metric d on X is G -equivariant (with respect to α) if, for every $g \in G$,

$$d(gx, gy) = d(x, y).$$

Here, as is customary, gx denotes $\alpha(g, x)$. If G is finite and if ϱ is any metric on X , then

$$d(x, y) = \max \{ \varrho(gx, gy) \mid g \in G \}$$

defines a G -equivariant metric d on X . Thus, in the remainder of this paper we may, and do, work only on G -equivariant metrics. For such a metric d on compact X , d^* denotes the induced Hausdorff metric on 2^X . The group action $\alpha: G \times X \rightarrow X$ induces a group action $\beta: G \times 2^X \rightarrow 2^X$ defined by

$$\beta(g, A) = \{ga \mid a \in A\}.$$

It is immediate that if d is a G -equivariant metric with respect to α , then d^* is a G -equivariant metric with respect to β . In this case, d^* induces a metric d' , on the orbit space $2^X/\beta$, defined by

$$d'([A], [B]) = \min \{ d^*(C, D) \mid \pi(C) = A, \pi(D) = B \},$$

where $\pi: 2^X \rightarrow 2^X/\beta$ is the projection.

Given a group action $\alpha: G \times X \rightarrow X$, a map $f: X \rightarrow X$ is G -invariant with respect to α if $f(gx) = gf(x)$ for each $x \in X$ and $g \in G$. Finally, for $A \in 2^X$,

$$2_A^X = \{ B \in 2^X \mid B \supset A \},$$

and, if also $g \in G$,

$$gA = \bigcup_{a \in A} \alpha(g, a) (= \beta A).$$

2. Moving 2^X G -equivariantly off sets with nonempty interior. In this section, let

$$G = \{g_0 = \text{id}, g_1, \dots, g_k\}$$

denote a finite group and let P denote a nondegenerate Peano continuum. Let $\alpha: G \times P \rightarrow P$ be a group action, $\beta: G \times 2^P \rightarrow 2^P$ the induced action. Let d be a metric on P which is G -equivariant with respect to α . Define $\alpha_i: P \rightarrow P$ by $\alpha_i(x) = \alpha(g_i, x)$. The main result of this section is that the identity on 2^P can be approximated by G -equivariant maps whose images miss sets with nonempty interior. This, in turn, allows us to approximate the identity on $2^P/\beta$ by maps missing sets with nonempty interior.

LEMMA 1. *Given $A \subset P$ such that $\text{Int } A \neq \emptyset$, there is a nonempty open set U contained in $\text{Int } A$ such that for $1 \leq i, j \leq k$ either*

$$\alpha_i|U = \alpha_j|U$$

or

$$\alpha_i(U) \cap \alpha_j(U) = \emptyset.$$

Note. The proof of Lemma 1 uses only that P is Hausdorff.

Proof. If $k = 0$, there is nothing to prove. If $k \geq 1$, let $U_0 = \text{Int } A$. If $\alpha_0|U = \alpha_1|U$, let $U_1 = U_0$. Otherwise, choose an open set U_1 such that

$$\emptyset \neq U_1 \subset U_0 \quad \text{and} \quad \alpha_0(U_1) \cap \alpha_1(U_1) = \emptyset.$$

If $k = 1$, then $U = U_1$ satisfies the conclusion of the lemma. If $k \geq 2$, consider $\alpha_2|U_1$. If $\alpha_2|U_1 = \alpha_i|U_1$, $i = 0$ or $i = 1$, then let $U_2 = U_1$. If $\alpha_2|U_1 \neq \alpha_i|U_1$, $i = 0, 1$, then choose an open set $U_{2,0}$ such that

$$\emptyset \neq U_{2,0} \subset U_1 \quad \text{and} \quad \alpha_2(U_{2,0}) \cap \alpha_0(U_{2,0}) = \emptyset.$$

If $\alpha_2|U_{2,0} = \alpha_1|U_{2,0}$, let $U_2 = U_{2,0}$. Otherwise, choose an open set $U_{2,1}$ such that

$$\emptyset \neq U_{2,1} \subset U_{2,0} \quad \text{and} \quad \alpha_2(U_{2,1}) \cap \alpha_1(U_{2,1}) = \emptyset.$$

Let $U_2 = U_{2,1}$. If $k = 2$, U_2 satisfies the requirements of the lemma. If $k \geq 3$, a continuation of the above arguments leads, eventually, to a suitable $U = U_k$.

LEMMA 2. *Let $A \in 2^P$, P a nondegenerate Peano continuum, be such that $\text{Int } A \neq \emptyset$. Let $\varepsilon > 0$ be given. Then there is a continuous G -equivariant map*

$$f: 2^P \rightarrow 2^P \setminus \bigcup \{2_{gA}^P \mid g \in G\}$$

such that $d^*(f, \text{id}) < \varepsilon$ and such that

$$f(B) = [B \setminus \bigcup \{gA \mid g \in G\}] \cup [\bigcup \{f(B \cap gA) \mid g \in G\}]$$

for each $B \in 2^P$ and $f(B \cap gA) \not\subseteq gA$ for each $B \in 2^P$ and $g \in G$. (We agree that $f(\emptyset) = \emptyset$.)

Proof. By Lemma 1 there is an open set U such that

$$\emptyset \neq U \subset \text{Int } A$$

and such that for $g_i, g_j \in G$ either $\alpha_i|U = \alpha_j|U$ or $\alpha_i(U) \cap \alpha_j(U) = \emptyset$. Choose a closed set $D \subset U$ with nonempty interior. Choose

$$\{h_j \mid j = 0, \dots, r\} \subset G$$

such that $\{h_j D \equiv D_j\} = \{gD \mid g \in G\}$ and such that $\{D_j\}$ is a collection of pairwise disjoint sets. We may, and do, assume that $h_0 = 1$, so that $D_0 = D$,

and, also, that $\varepsilon < \min \{d(D_i, D_j) \mid D_i \neq D_j\}$. By Lemma 5.4 of [1] and its proof, there is a continuous

$$f_0: 2^P \rightarrow 2^P \setminus 2_D^P$$

such that

$$d^*(f_0, \text{id}) < \varepsilon \quad \text{and} \quad f_0(B) = (B \setminus D) \cup f_0(B \cap D)$$

with $f_0(B \cap D) \not\subseteq D$. Let

$$f_j = h_j f_0 h_j^{-1}: 2^P \rightarrow 2^P \setminus 2_{D_j}^P, \quad j = 1, \dots, r.$$

Let

$$f = f_r \circ \dots \circ f_1 \circ f_0: 2^P \rightarrow 2^P \setminus \bigcup \{2_{D_j}^P \mid j = 0, \dots, r\} \subset 2^P \setminus \bigcup \{2_{gA}^P \mid g \in G\}.$$

Then f satisfies the conclusion of Lemma 2.

LEMMA 3. *Let $A_1, \dots, A_n \in 2^P$ have nonempty interiors. Then there exist closed sets D_1, \dots, D_s ($s \leq n$) such that each A_i contains gD_j for some $(g, j) \in G \times \{1, \dots, s\}$ and such that*

$$\left[\bigcup \{gD_i \mid g \in G\} \right] \cap \left[\bigcup \{gD_j \mid g \in G\} \right] = \emptyset \quad \text{if } i \neq j.$$

Note. The proof of Lemma 3 uses only that P is regular.

Proof. Let $U_i = \text{Int } A_i$, $i = 1, \dots, n$. For $x \in P$ let

$$I(x) = \{i \mid x \in \bigcup_{g \in G} g(U_i)\}.$$

Let $\text{Card } I(x)$ be the number of elements in $I(x)$. Choose $a_1 \in U_1$ such that

$$\text{Card } I(a_1) \geq \text{Card } I(x) \quad \text{for each } x \in U_1.$$

For $i \in I(a_1)$ choose $g_{1_i} \in G$ such that $a_1 \in g_{1_i} U_i$, taking $g_{1_1} = \text{id}$. Clearly, then

$$a_1 \in \bigcap \{g_{1_i} U_i \mid i \in I(a_1)\}.$$

Also, if $j \notin I(a_1)$, then

$$a_1 \notin \bigcup \{g \bar{U}_j \mid g \in G\};$$

for, otherwise,

$$a_1 \in \left[\bigcap \{g_{1_i} U_i \mid i \in I(a_1)\} \right] \cap g \bar{U}_j$$

for some $g \in G$ and $j \notin I(a_1)$. Hence, there exists

$$z \in \left[\bigcap \{g_{1_i} U_i \mid i \in I(a_1)\} \right] \cap g U_j,$$

implying $\text{Card } I(z) > \text{Card } I(a_1)$, which contradicts $z \in g_{1_1} U_1 = U_1$. Thus, we may choose a nonempty open set V_1 such that

$$a_1 \in V_1 \subset \bar{V}_1 \subset \bigcap \{g_{1_i} U_i \mid i \in I(a_1)\} \setminus \bigcup \{g \bar{U}_j \mid g \in G, j \notin I(a_1)\}.$$

Note that, for $i \in I(a_1)$,

$$A_i \supset U_i = g_{1_i}^{-1}(g_{1_i} U_i) \supset g_{1_i}^{-1} \bar{V}_1.$$

Let $r_1 = 1$. Let $r_2 = \min \{i \mid 1 \leq i \leq n, i \notin I(A_1)\}$. Choose $a_2 \in U_{r_2}$ such that $\text{Card } I(a_2) \geq \text{Card } I(x)$, $x \in U_{r_2}$. For $i \in I(a_2)$ choose $g_{2_i} \in G$ such that $a_2 \in g_{2_i} U_i$, taking $g_{2_{r_2}} = \text{id}$. Choose V_2 open in P such that

$$a_2 \in V_2 \subset \bar{V}_2 \subset \{g_{2_i} U_i \mid i \in I(a_2)\} \setminus \cup \{g \bar{U}_j \mid g \in G, j \notin I(a_2)\}.$$

Note that, for $i \in I(a_2)$,

$$A_i \supset U_i \supset g_{2_i}^{-1}(g_{2_i} U_i) \supset g_{2_i}^{-1} \bar{V}_2.$$

Also,

$$[\cup \{g \bar{V}_1 \mid g \in G\}] \cap [\cup \{g \bar{V}_2 \mid g \in G\}] = \emptyset,$$

for otherwise $\bar{V}_1 \cap g \bar{V}_2 \neq \emptyset$ for some $g \in G$. But this cannot be since

$$g \bar{V}_2 \subset g U_{r_2} (g_{2_{r_2}} = \text{id}) \subset \cup \{g U_j \mid g \in G, j \notin I(a_1)\} \subset G \bar{V}_1.$$

If $I(a_1) \cup I(a_2) = \{1, \dots, n\}$, we are done. Otherwise, letting

$$r_3 = \min \{i \mid 1 \leq i \leq n, i \notin I(a_1) \cup I(a_2)\},$$

we may continue the above process until, at step s , we obtain

$$I(a_1) \cup \dots \cup I(a_s) = \{1, \dots, n\}.$$

Then the collection $\{\bar{V}_1, \dots, \bar{V}_s\}$ will have the desired properties.

LEMMA 4. *Let $A_1, \dots, A_n \in 2^P$ be such that $\text{Int } A_i \neq \emptyset$, $i = 1, \dots, n$. Let $\varepsilon > 0$ be given. Then there is a G -equivariant map*

$$f: 2^P \rightarrow 2^P \setminus \cup \{2_{A_i}^P \mid i = 1, \dots, n\}$$

such that $d^*(f, \text{id}) < \varepsilon$.

Proof. By Lemma 3 there are closed sets D_1, \dots, D_s such that each A_k contains some $g D_j$, $(g, j) \in G \times \{1, \dots, s\}$ and such that

$$[\cup \{g D_i \mid g \in G\}] \cap [\cup \{g D_j \mid g \in G\}] = \emptyset \quad \text{if } i \neq j.$$

By Lemma 2 there is a continuous G -equivariant map

$$f_i: 2^P \rightarrow 2^P \setminus \cup \{2_{g D_i}^P \mid g \in G\}, \quad i = 1, \dots, s,$$

such that

$$d^*(f_i, \text{id}) < \varepsilon,$$

$$f_i(B) = [B \setminus \cup \{g D_i \mid g \in G\}] \cup [\cup \{f_i(B \cap g D_i)\}]$$

and

$$f_i(B \cap g D_i) \not\subseteq g D_i.$$

We may assume

$$\varepsilon < \min \{d(\cup \{gD_i \mid g \in G\}, \cup \{gD_j \mid g \in G\}) \mid i \neq j\}.$$

The map

$$f = f_s \circ \dots \circ f_1: 2^P \rightarrow 2^P \setminus \cup \{2^P_{A_i} \mid i = 1, \dots, n\}$$

has the desired properties.

THEOREM 1. *Let $\alpha: G \times P \rightarrow P$ be a group action with G a finite group and P a nondegenerate Peano continuum. Let $\beta: G \times 2^P \rightarrow 2^P$ be the induced group action on the hyperspace 2^P . Let d be a G -equivariant metric on P and d^* the induced metric on 2^P . Then for every $\varepsilon > 0$ there is a G -equivariant map*

$$f: 2^P \rightarrow 2^P \setminus \{A \mid \text{Int } A \neq \emptyset\}$$

such that $d^*(f, \text{id}) < \varepsilon$.

Proof. Let $\{x_i \mid i = 1, 2, 3, \dots\}$ be a countable dense set in P . Let

$$\eta(x_i, \delta) = \{x \in P \mid d(x_i, x) \leq \delta\}.$$

(The idea of looking at the sets $\eta(x_i, \delta)$ came from the proof of Corollary 2.11 in [4].) The set $C(2^P) = \{h: 2^P \rightarrow 2^P \mid h \text{ continuous}\}$ with the compact open topology is a compact metric space. Let

$$C_G(2^P) = \{h \in C(2^P) \mid h \text{ is } G\text{-equivariant}\}.$$

Then $C_G(2^P)$ is closed in $C(2^P)$. Hence $C_G(2^P)$ is a Baire space. Let

$$W_k = \{h \in C_G(2^P) \mid h(2^P) \subset 2^P \setminus \cup \{2^P_{\eta(x_i, 1/k)} \mid i = 1, \dots, k\}\}.$$

Then W_k is an open subset of $C_G(2^P)$. By Lemma 4, W_k is dense in $C(2^P)$, hence, also, in $C_G(2^P)$. Thus $\cap \{W_k \mid i = 1, 2, 3, \dots\}$ is dense in $C_G(2^P)$. Choose $f \in \cap \{W_k \mid i = 1, 2, 3, \dots\}$ such that $d^*(f, \text{id}) < \varepsilon$. Then f is the desired map into $2^P \setminus \{A \mid \text{Int } A \neq \emptyset\}$.

COROLLARY 1. *With $\alpha: G \times P \rightarrow P$, $\beta: G \times 2^P \rightarrow 2^P$ as above and $\varepsilon > 0$ given, there is a map*

$$f': 2^P/\beta \rightarrow (2^P \setminus \{[A] \mid \text{Int } A \neq \emptyset\})/\beta$$

such that $d'(f', \text{id}) < \varepsilon$.

Proof. The desired map is defined by $f'([B]) = [f(B)]$, where f is the map of the preceding theorem.

COROLLARY 2. *Let $\alpha: G \times P \rightarrow P$ be a group action with G a finite group and P a nondegenerate Peano continuum. Let $\beta: G \times 2^P \rightarrow 2^P$ be the induced action. Then $2^P/\beta$ is homeomorphic to the Hilbert cube if and only if $2^P/\beta$ is an AR.*

Proof. Only the "if" part requires the proof. Let $\beta: G \times 2^P \rightarrow 2^P$ be as

in the corollary, and assume $2^P/\beta$ is an AR. By Toruńczyk's characterization of the Hilbert cube [7], $2^P/\beta$ is a Hilbert cube if the identity can be approximated by maps with disjoint images. But the identity on 2^P can be approximated by maps of the form h , where

$$h([A]) = [\{x \in P \mid d(x, A) \leq t\}]$$

and by maps of the form f' as in Corollary 1. These maps have disjoint images as required.

3. Induced finite group actions on the hyperspace of a finite graph. In this section we restrict our attention to group actions of the form $\beta: G \times 2^{\Gamma} \rightarrow 2^{\Gamma}$ induced by a group action $\alpha: G \times \Gamma \rightarrow \Gamma$, where G is a finite group and Γ is a finite, nondegenerate, connected graph. Note that Γ is a nondegenerate Peano continuum, so the results of Section 2 apply. Thus, we know that $2^{\Gamma}/\beta$ is a Hilbert cube if it is an AR. The principal result of this section, Theorem 2, is that $2^{\Gamma}/\beta$ is an AR.

The proof of Theorem 2 is patterned after the proof of Lemma 4 in [8]. It uses Hanner's theorem [2] that a space ε -dominated by ANR's for all $\varepsilon > 0$ is an ANR (absolute neighborhood retract for metric spaces) and Haver's theorem [3] that a locally contractible countable union of finite-dimensional compact metric spaces is an ANR.

Given $\alpha: G \times \Gamma \rightarrow \Gamma$, let d be a G -equivariant metric on Γ with respect to α . Let ϱ be the minimum path length (with respect to d) metric on Γ . Note that ϱ is also G -equivariant with respect to α . Define $e: 2^{\Gamma} \times I \rightarrow 2^{\Gamma}$, an expansion homotopy, by

$$e(A, t) = \{x \in \Gamma \mid \varrho(x, A) \leq t\}, \quad A \in 2^{\Gamma}, t \in I.$$

Then e is continuous and G -equivariant with respect to β . Thus, e induces the expansion homotopy $e': 2^{\Gamma}/\beta \times I \rightarrow 2^{\Gamma}/\beta$ defined by

$$e'([A], t) = [e(A, t)].$$

For convenience we denote $e(A, t)$ and $e'([A], t)$ by $e_t(A)$ and $e'_t([A])$, respectively. We call a subset of Γ in $e(2^{\Gamma} \times (0, 1])$ an *expanded set*.

LEMMA 5. Let $A \in 2^{\Gamma}$. Then

- (a) $e_{s+t}(A) = e_s(e_t(A))$ for every $s, t \in I$;
- (b) A is an expanded set if and only if it has finitely many components, each with positive diameter;
- (c) if A is an expanded set, then $\text{Bd } A$, the topological boundary of A in Γ , is finite;
- (d) if B is connected, $B \subset A$, A an expanded set, and

$$t < \delta = \min \{\varrho(x, y) \mid x, y \in \text{Bd } A\},$$

then $e_t(B) \cap A$ is connected.

Proof. (a) The proof is elementary and is omitted.

(b) The “only if” part follows easily since the intersection of the expanded set with each edge of Γ can have only finitely many components. For the converse, assume A has only finitely many components, each with positive diameter. Let $\{\gamma_i\}$ be the edges of Γ , and let $\{\gamma_i^j\}$ be the finite collection of components of $A \cap \gamma_i$ that are not vertices of γ_i . (Note that if a vertex is a component of $A \cap \gamma_i$, then it is contained in a non-vertex component of $A \cap \gamma_j$ for some $j \neq i$.) Since $A = \bigcup \{\gamma_i^j\}$ is a finite union of closed intervals with positive length, it follows that A is an expanded set.

(c) If A is connected, then $(\text{Bd } A) \cap \gamma_i$, γ_i any edge of Γ , can have at most two points. The result now follows since Γ has only finitely many edges and any expanded set, by (b), has only finitely many components.

(d) Let $y \in e_t(B) \cap A$. Choose a path $\tau: I \rightarrow \Gamma$ of minimum length such that $\tau(0) = x \in B$ and $\tau(1) = y$. Note that $|\tau| \leq t$. (Here $|\tau|$ denotes the length of τ .) Clearly, $\tau(I) \subset e_t(B)$. Also, since $|\tau| < \delta$ and since $x, y \in A$ it follows that $\tau(I) \subset e_t(B) \cap A$. Thus, y is joined in $e_t(B) \cap A$ to a point x in connected $B \subset e_t(B) \cap A$. The result follows.

We are now ready to state and prove the main result of this section.

THEOREM 2. *Let $\alpha: G \times \Gamma \rightarrow \Gamma$ be a group action with G a finite group and Γ a finite, nondegenerate, connected graph. Let $\beta: G \times \Gamma \rightarrow \Gamma$ be the induced group action on the hyperspace 2^Γ . Then the orbit space $2^\Gamma/\beta$ is an AR, and hence (by Corollary 2) a Hilbert cube.*

Proof. Let $\pi: 2^\Gamma \rightarrow 2^\Gamma/\beta$ be the projection. We use the minimum path length metric ρ on Γ . Without loss of generality we assume that the diameter of Γ with respect to ρ is no more than one. We let ρ^* and ρ' denote the induced metrics on 2^Γ and $2^\Gamma/\beta$, respectively (see Section 1). Since $2^\Gamma/\beta$ is contractible, via the homotopy e' , we need only to show that $2^\Gamma/\beta$ is an ANR.

Let

$$E_k = e(2^\Gamma \times \{1/k\}), \quad E'_k = e(2^\Gamma/\beta \times \{1/k\}),$$

$$E = \bigcup_{k=1}^{\infty} E_k \quad \text{and} \quad E' = \bigcup_{k=1}^{\infty} E'_k.$$

From (a) of Lemma 5 it is clear that $E = e(2^\Gamma \times (0, 1])$, the collection of expanded sets in 2^Γ . It is also clear that $E' = \pi(E)$.

Let U be any open cover of $2^\Gamma/\beta$ and let λ be a Lebesgue number for U . Consider $e'_\lambda: 2^\Gamma/\beta \rightarrow E'$, where e'_λ is the composite

$$[A] \rightarrow ([A], \lambda) \rightarrow e'([A], \lambda).$$

Let $i: E' \rightarrow 2^\Gamma/\beta$ be the inclusion. The map $([A], t) \rightarrow e'_{t\lambda}([A])$ defines a homotopy from the identity on $2^\Gamma/\beta$ to $i \circ e'_\lambda$ which is limited by U . Thus, $E' \xrightarrow{i} 2^\Gamma/\beta$ is a homotopy domination. By a theorem of Hanner [2], $2^\Gamma/\beta$ is an ANR if E' is an ANR.

Consider

$$E' = \bigcup_{k=1}^{\infty} E'_k.$$

Each E'_k , being a closed subspace of $2^{\Gamma}/\beta$, is a compact metric space. It is also finite-dimensional by the arguments of Section 5 in [5]. Thus, E' is a countable union of finite-dimensional compact metric spaces. By a theorem of Haver [3], E' is an ANR if it is locally contractible.

Let $[A] \in U$, an open set in E' . If $[A] = [\Gamma]$, choose $\varepsilon > 0$ such that $\varepsilon < 1$ and

$$N = \{[B] \in 2^{\Gamma}/\beta \mid \varrho'([A], [B]) < \varepsilon\} \subset U.$$

Then $e'(N \times I) \subset U$, so that $e'|(N \times I)$ contracts N inside U .

If $[A] \neq [\Gamma]$, choose $g_0 = 1, g_1, \dots, g_r \in G$ such that if $A_i = g_i A$,

$$\{gA \mid g \in G\} = \{g_i A \mid i = 0, 1, \dots, r\}$$

and such that $A_i \neq A_j$ if $i \neq j$. Choose $\eta > 0$ such that

$$\{B \in 2^{\Gamma} \mid \varrho^*(A_i, B) < \eta\} \cap E \subset \pi^{-1}(U), \quad i = 0, 1, \dots, r.$$

By (c) of Lemma 5, $\bigcup_{i=1}^r \text{Bd } A_i$ is finite. Thus, we may choose

$$\gamma < \min \{1, \eta, (1/2)\varrho^*(A_i, A_j), (1/5)\varrho(x, y) \mid i \neq j,$$

$$x \text{ and } y \text{ two distinct points in } \bigcup_{i=1}^r \text{Bd } A_i\}.$$

Note that 3γ is less than $\varrho(x, y)$ for any distinct points x and y in $\bigcup_{i=1}^r \text{Bd } e_{\gamma}(A_i)$.

Let $W_i = \{B \in E \mid \varrho^*(B, A_i) < \gamma\}$. Then W_i is an open subset of E , and $W_i \cap W_j = \emptyset$ if $i \neq j$. Define $\phi_i: W_i \times I \rightarrow 2^{\Gamma}$ by

$$\phi_i(B, t) = e_{2\gamma t}(B) \cap e_{\gamma}(A_i).$$

If B is connected, then, by (d) of Lemma 5, $\phi_i(B, t)$ is also connected. It follows, by (b) of Lemma 5, that $\phi_i(B, t) \subset E$. This, together with the choice of $\gamma < \eta$, insures that, in fact,

$$\phi_i: W_i \times I \rightarrow \pi^{-1}(U).$$

We claim that ϕ_i is continuous. To see this let $(C, s) \in W_i \times I$ and let $\varepsilon > 0$ be given. Choose positive $\delta < \min \{\gamma, \varepsilon/3\}$. Then, if $\varrho^*(B, C) < \delta$ and $|t - s| < \delta$, it is easy to see that

$$e_{2\gamma t}(B) \subset e_{2\gamma t + \delta}(C) \subset e_{2\gamma s + \varepsilon}(C).$$

Now, let $x \in \phi_i(B, t)$. Choose $c \in C$ so that $\rho(x, c)$ is a minimum, and let τ be a path with $\tau(0) = x$ and $\tau(1) = c$ having minimum length. Since

$$x \in e_{2\gamma t}(B) \subset e_{2\gamma t + \delta}(C),$$

$|\tau|$, the length of τ , is no more than $2\gamma + \delta < 3\gamma$, which, in turn, is less than the distance between distinct boundary points of $e_\gamma(A_i)$. Therefore, $\tau(I) \subset e_\gamma(A_i)$. If $|\tau| \leq \varepsilon$, then

$$\rho(x, c) \leq \varepsilon, \quad \text{where } c \in C \cap e_\gamma(A_i) \subset \phi_i(C, s).$$

If $|\tau| > \varepsilon$, choosing $z \in \tau(I)$ such that $\rho(x, z) = \varepsilon$, we have

$$\rho(z, c) \leq (2\gamma s + \varepsilon) - \varepsilon = 2\gamma s,$$

so that $z \in \phi_i(C, s)$. In either case $x \in e_\varepsilon(\phi_i(C, s))$. It follows that

$$\phi_i(B, t) \subset e_\varepsilon(\phi_i(C, s)).$$

By symmetry,

$$\phi_i(C, s) \subset e_\varepsilon(\phi_i(B, t)),$$

so $\rho^*(\phi_i(B, t), \phi_i(C, s)) < \varepsilon$. Thus, ϕ_i is continuous.

Let

$$W = \bigcup_{i=1}^r W_i$$

and define $\phi: W \times I \rightarrow \pi^{-1}(U)$ by $\phi|(W_i \times I) = \phi_i$. Then ϕ_i is well-defined, continuous, and, as is easily checked, G -equivariant with respect to β . Thus, ϕ induces a continuous map $\phi': \pi(W) \times I \rightarrow U$ defined by

$$\phi'([B], t) = [\phi(B, t)].$$

Since π is an open map, $\pi(W)$ is an open neighborhood of $[A]$. Clearly,

$$\phi'([B], 0) = [B] \quad \text{and} \quad \phi'([B], 1) = [e_\gamma(A)].$$

Thus ϕ' contracts $\pi(W)$ inside U .

Thus, E' is locally contractible as required.

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Reçu par la Rédaction le 20.6.1986
