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QUADRATIC FORM SCHEMES WITH NON-TRIVIAL RADICAL

BY

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Introduction. Quadratic form schemes were introduced by Cordes in [4]. Cordes considers a group g with $-1 \in g$ and a set of subgroups $\{X_a\}_{a \in g}$, which satisfy the following two conditions: $a \in X_a$ and $b \in X_a$ iff $-1 \cdot a \in X_{-1 \cdot b}$ for any $a, b \in g$. If k is a field, then the group g corresponds to k^*/k^{*2} and X_a can be thought of as a value set for the quadratic form $(1, a)$. However, this definition admits schemes for which there exists no corresponding field (cf. the Remark after Definition 1.1). In this paper we consider schemes satisfying a third additional condition: $D(a, b, c) = D(b, a, c)$ for any $a, b, c \in g$ (where $D(a, b, c) = \bigcup_{x \in b \cdot X_{bc}} a \cdot X_{ax}$). We do not know whether all these schemes correspond to some fields (**P 1283**). In [14] it was shown that all schemes with $[g:R] \leqslant 16$ (where $R = \{a \in g : D(1, -a) = g\}$ denotes the radical of a scheme) are realized by fields. Here we prove this for any scheme with $u \geqslant \frac{1}{2}[g:R]$ and $[g:R] < \infty$.

In Sections 1-3 we give the basic information about quadratic form schemes, sets of elements represented by forms, and equivalent forms. We introduce the notion of equivalent schemes and invariants of schemes similarly as in case of fields. In turns out that many theorems holding in the theory of quadratic forms over fields are also true in case of quadratic form schemes (in the sense of our definition).

The main part of this paper is Section 4 where we consider a scheme with a non-trivial radical and we prove that each such scheme can be split into the product of two schemes: S_1 having a trivial radical (i.e., $R(S_1) = \{1\}$) and S_2 with $R(S_2) = g(S_2)$. The exact statement of the result is given in Theorem 4.7.

In the last section we classify all schemes with $u \geqslant q/2$ and $q < \infty$. In case of fields this is done by Cordes in [5], but he does not prove the existence of the corresponding fields with $s \leqslant 2$.

1. Quadratic form schemes. Let g be an elementary 2-group with distinguished element $-1 \in g$. For every $a \in g$ the product $-1 \cdot a$ will be written as $-a$. Let d be any mapping from g into the set G of all subgroups of g. The triplet $\langle g, -1, d\rangle$ will be denoted by S. An n-tuple $\varphi = (a_1, \ldots, a_n)$, $a_i \in g$, is said to be a *form* (*over* S) of dimension $\dim\varphi = n$ and with determinant $\det\varphi = a_1 \ldots a_n$. For forms (a), (a, b), and $(a_1, \ldots, a_n)$ we write

$$D_S(a) = \{a\}, \quad D_S(a, b) = a\cdot d(ab),$$
$$D_S(a_1, \ldots, a_n) = \bigcup_{x\in D_S(a_2,\ldots,a_n)} D_S(a_1, x) \quad \text{for } n > 2.$$

The set $D_S(\varphi)$ (or, simply, $D(\varphi)$) is said to be the *set of elements of* g *represented by the form* φ (*over* S). If $a \in D_S(\varphi)$, we also write $\varphi \underset{S}{\approx} a$ (or $\varphi \approx a$).

1.1. Definition. $S = \langle g, -1, d\rangle$ is said to be a *quadratic form scheme* (or, simply, a *scheme*) if it satisfies the following conditions:

C_1: $a \in D(1, a)$ for any $a \in g$.

C_2: $a \in D(1, b) \Leftrightarrow -b \in D(1, -a)$ for any $a, b \in g$.

C_3: $D(a, b, c) = D(b, a, c)$ for any $a, b, c \in g$.

Remark. We observe that C_1 and C_2 are independent. We show that C_1 and C_2 do not imply C_3. Let g be an elementary 2-group with F_2-basis $\{a, b, c\}$ and define

$$d(1) = \{1, -1\}, \quad d(a) = \{1, a, b, ab\}, \quad d(b) = \{1, b\},$$
$$d(ab) = \{1, ab\}, \quad d(-1) = g,$$
$$d(-a) = \{1, -a\}, \quad d(-b) = \{1, -a, -b, ab\},$$
$$d(-ab) = \{1, -a, b, -ab\}.$$

Clearly, $S = \langle g, -1, d\rangle$ satisfies C_1 and C_2. We show that S does not satisfy C_3. We have

$$D(1, b, b) = \bigcup_{x\in D(b,b)} D(1, x) = \{1, b, -b, -a, ab\}.$$

Analogously,

$$D(b, 1, b) = \bigcup_{x\in D(1,b)} D(b, x) = \{1, b, -b\}.$$

Hence $D(1, b, b) \neq D(b, 1, b)$ and C_3 does not hold for S.

1.2. Example. Let k be a field of characteristic different from 2 and let $d_k(a)$ denote the subgroup of $g(k)$ consisting of elements represented by the form $(1, a)$. Then $S(k) = \langle g(k), -k^{*2}, d_k\rangle$ is a quadratic form scheme. It will be called the *scheme of the field k*.

1.3. Corollary. *For any* $a, b, a_i, b_j \in g$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, *we have*

(i) $D(a, b) = D(b, a)$;

(ii) $D(a, -a) = g$;

(iii) $D(a_1, \ldots, a_n) \subset D(a_1, \ldots, a_n, b_1, \ldots, b_m)$;

(iv) $b \in D(a_1, \ldots, a_n) \Rightarrow D(b_1, \ldots, b_m, b) \subset D(b_1, \ldots, b_m, a_1, \ldots, a_n)$;
(v) $D(aa_1, \ldots, aa_n) = a \cdot D(a_1, \ldots, a_n)$;
(vi) $D(1, -a) \cap D(1, -b) \subset D(1, -ab)$.

The proof is trivial.

1.4. LEMMA. *If S_n is the symmetric group on $\{1, \ldots, n\}$, $\sigma \in S_n$, and $a_1, \ldots, a_n \in g$, then $D(a_1, \ldots, a_n) = D(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$.*

Proof. For $n = 2, 3$ we get the equality by C_3 and Corollary 1.3 (i). Let $n \geqslant 4$ and suppose the lemma is true for all forms of dimension $n-1$. Now, since S_n is generated by transpositions $(i, i+1)$, we may assume that $\sigma \in S_n$ is a transposition $(i, i+1)$. We have two cases.

If $\sigma(1) = 1$, then by induction we get

$$D(a_1, \ldots, a_n) = \bigcup_{x \in D(a_2, \ldots, a_n)} D(a_1, x) = \bigcup_{x \in D(a_{\sigma(2)}, \ldots, a_{\sigma(n)})} D(a_{\sigma(1)}, x)$$
$$= D(a_{\sigma(1)}, \ldots, a_{\sigma(n)}).$$

If $\sigma(1) \neq 1$, then $\sigma(1) = 2$, $\sigma(2) = 1$, and $\sigma(k) = k$ for $k = 3, \ldots, n$. It is sufficient to prove that

$$D(a_1, \ldots, a_n) \subset D(a_2, a_1, a_3, \ldots, a_n).$$

Let $c \in D(a_1, \ldots, a_n)$. Then there exists an element $x \in D(a_2, \ldots, a_n)$ such that $c \in D(a_1, x)$. Similarly, there exists $y = D(a_3, \ldots, a_n)$ such that $x \in D(a_2, y)$. Hence

$$c \in D(a_1, x) \subset \bigcup_{z \in D(a_2, y)} D(a_1, z) = D(a_1, a_2, y) = D(a_2, a_1, y)$$
$$= \bigcup_{w \in D(a_1, y)} D(a_2, w).$$

Now, $D(a_1, y) \subset D(a_1, a_3, \ldots, a_n)$ (by Corollary 1.3 (iv)); hence

$$c \in \bigcup_{w \in D(a_1, a_3, \ldots, a_n)} D(a_2, w) = D(a_2, a_1, a_3, \ldots, a_n)$$

and the lemma is proved.

For any forms $\varphi = (a_1, \ldots, a_n)$ and $\psi = (b_1, \ldots, b_m)$ over S we define the (orthogonal) sum

$$\varphi \perp \psi = (a_1, \ldots, a_n, b_1, \ldots, b_m)$$

and the (tensor) product

$$\varphi \otimes \psi = b_1 \varphi \perp \ldots \perp b_m \varphi,$$

where $b\varphi = (ba_1, \ldots, ba_n)$. If $\psi = (1, \ldots, 1)$ is a form of dimension n, then the form $\varphi \otimes \psi = \varphi \perp \ldots \perp \varphi$ will be denoted by $n \times \varphi$.

1.5. COROLLARY. *If $a \in D(\varphi)$ and $b \in D(\psi)$, then*

$$D(a, b) \subset D(\varphi \perp \psi) \quad \textit{and} \quad ab \in D(\varphi \otimes \psi).$$

Proof. Write $\varphi = (a_1, \ldots, a_n)$ and $\psi = (b_1, \ldots, b_m)$. By Corollary 1.3 (iv) and Lemma 1.4 we get

$$D(a, b) \subset D(a, b_1, \ldots, b_m)$$
$$= D(b_1, \ldots, b_m, a) \subset D(b_1, \ldots, b_m, a_1, \ldots, a_n) = D(\varphi \perp \psi).$$

Next we use the induction. If $m = 1$, then $ab \in D(\varphi \otimes \psi)$ by Corollary 1.3 (v). Let $m \geq 2$ and $b \in D(b_1, \ldots, b_m)$. Then there exists $x \in D(b_2, \ldots, b_m)$ such that $b \in D(b_1, x)$; thus $ax \in D(\varphi \otimes (b_2, \ldots, b_m))$. But $ab_1 \in D(a_1 b_1, \ldots, a_n b_1)$; hence $ab \in aD(b_1, x) \subset D(\varphi \otimes \psi)$.

1.6. THEOREM. *For any forms φ and ψ over S we have*

$$D(\varphi \perp \psi) = \bigcup \{D(x, y) \colon x \in D(\varphi),\ y \in D(\psi)\}.$$

Proof. If $c \in D(a_1, \ldots, a_n, b_1, \ldots, b_m)$, then there exists $x \in D(a_2, \ldots, a_n, b_1, \ldots, b_m)$ such that $c \in D(a_1, x)$. Using induction on n, we get $x \in D(w, z)$ for some $w \in D(a_2, \ldots, a_n)$, $z \in D(b_1, \ldots, b_m)$. We have now

$$c \in D(a_1, x) \subset D(a_1, w, z) \subset \bigcup_{y \in D(a_1, w)} D(z, y) \subset \bigcup_{y \in D(a_1, \ldots, a_n)} D(z, y),$$

and hence $c \in D(y', z)$ for some $y' \in D(\varphi)$, $z \in D(\psi)$. The converse is trivial.

To simplify the notation we write

$$D_S(n) = D_S(n \times (1)) \quad \text{and} \quad D_S(\infty) = \bigcup_{n=1}^{\infty} D_S(n)$$

and, motivated by the case of schemes of fields, we classify the schemes as follows:

1.7. Definition. The scheme S is said to be *non-real* if $-1 \in D(\infty)$, and S is said to be *formally real* otherwise.

1.8. Definition. A subgroup P of the group g is said to be an *ordering of the scheme* $S = \langle g, -1, d \rangle$ if it satisfies the following conditions:

(1) $[g \colon P] = 2$,

(2) $D(a, b) \subset P$ for any $a, b \in P$.

We denote by $r(S)$ the cardinality of the set of orderings of the scheme S.

1.9. COROLLARY. *For any ordering P we have*

(i) $-1 \notin P$,

(ii) $a_1, \ldots, a_n \in P \Rightarrow D(a_1, \ldots, a_n) \subset P$,

(iii) $D(\infty) \subset P$.

1.10. THEOREM. $r(S) > 0$ *if and only if S is a formally real scheme.*

Proof. If there is an ordering P, then S is formally real by Corollary 1.9. To prove the converse observe first that if S is a formally real scheme, then $D(\infty) \subsetneqq g$. Let R be the family of all non-trivial subgroups P of the group g such that $D(a, b) \subset P$ for any $a, b \in P$. Obviously, $D(\infty) \in R$ and

R is partially ordered by inclusion. If L is a chain in R, then $\bigcup_{P \in L} P$ belongs to R. By Zorn's lemma, there exists a maximal element P_0 of the family R. We prove that P_0 is an ordering of the scheme S. Since $-1 \notin P_0$, it is sufficient to show that for any $c \in g$ we have $c \in P_0$ or $-c \in P_0$. Let

$$P' = \bigcup_{x, y \in P_0} D(x, cy).$$

Then $P_0 \subset P'$ and $c \in P'$.

Case 1. $-1 \in P'$. Then there exist $x, y \in P_0$ such that $(x, cy) \approx -1$. Hence $(1, x) \approx -yc$, and so $-yc \in P_0$. It follows that $-c \in P_0$.

Case 2. $-1 \notin P'$. We shall prove that $P' \in R$. Let $a, b \in P'$, $(x, cy) \approx a$, and $(x', cy') \approx b$ for some $x, x', y, y' \in P_0$. Then xx', yy', xy' and $x'y$ belong to P_0. Hence

$$D(xx', yy') \cup D(xy', x'y) \subset P_0.$$

We have

$$\begin{aligned} ab \in D\big((x, cy) \otimes (x', cy')\big) &= D(xx', cyx', cxy', yy') \\ &= \bigcup \{D(w, cz) \colon w \in D(xx', yy'),\ z \in D(xy', yx')\} \\ &\subset \bigcup \{D(w, cz) \colon w, z \in P_0\} \subset P' \end{aligned}$$

and we conclude that P' is a non-trivial subgroup of g. It remains to show that $D(w, z) \subset P'$ for any $w, z \in P'$. We have

$$D(w, z) \subset D(x, cy, x', cy') \subset \bigcup \{D(t, cu) \colon t, u \in P_0\} \subset P'.$$

Thus we have proved that $P' \in R$. Since $P_0 \subset P'$ and P_0 is maximal, we get $P' = P_0$. Hence $c \in P_0$ and the theorem is proved.

1.11. THEOREM. *If $r(S) > 0$, then the intersection $\sigma(S)$ of all orderings of the scheme S is equal to $D(\infty)$.*

Proof. It is clear that $D(\infty) \subset \sigma(S)$. Let $a \in g$ and suppose that $a \notin D(\infty)$. We shall prove that $a \notin \sigma(S)$. Let R be a family of non-trivial subgroups of g satisfying the following conditions:

(i) $-a \in P$ for any $P \in R$,

(ii) $D(c, d) \subset P$ for any $P \in R$ and $c, d \in P$.

Similarly as in the proof of the preceding theorem we show that $P' = \bigcup \{D(x, -ay) \colon x, y \in D(\infty)\}$ belongs to R. Hence R is non-empty. Moreover, if P_0 denotes a maximal element of R, then P_0 is an ordering of S and $-a \in P_0$. Hence $a \notin P_0$, so $a \notin \sigma(S)$, and the theorem is proved.

Now, let $S = \langle g, -1, d \rangle$ be any formally real scheme, $a \in g$, and let P be an ordering of the scheme S. We define $\mathrm{sgn}_P(a) = 1$ if $a \in P$, and $\mathrm{sgn}_P(a) = -1$ otherwise. It is clear that $\mathrm{sgn}_P \colon g \to \{\pm 1\}$ is the group homomorphism. Further, we define

$$\mathrm{Sgn}_P(a_1, \ldots, a_n) = \sum_{i=1}^{n} \mathrm{sgn}_P(a_i)$$

for any form $(a_1, \ldots, a_n)$ over S and observe that

$$\dim \varphi \equiv \mathrm{Sgn}_P \varphi \pmod 2$$

and

$$\mathrm{Sgn}_P \varphi \perp \psi = \mathrm{Sgn}_P \varphi + \mathrm{Sgn}_P \psi$$

for any forms φ and ψ over S.

2. Equivalent forms.

2.1. Definition. For any forms $\varphi = (a_1, \ldots, a_n)$ and $\psi = (b_1, \ldots, b_n)$ (of the same dimension) over S we say that φ and ψ are *simply equivalent* if

1. $a_1 = b_1$ for $n = 1$,
2. $a_1 a_2 = b_1 b_2$ and $b_1 \in D(a_1, a_2)$ for $n = 2$,
3. there exist $i, j \in \{1, \ldots, n\}$, $i \neq j$, such that the forms (a_i, a_j) and (b_i, b_j) are simply equivalent and $a_k = b_k$, $k \neq i, j$, for $n \geqslant 3$.

We say that φ and ψ are *equivalent* ($\varphi \cong \psi$) if there exists a finite sequence of forms $\varphi = \varphi_0, \varphi_1, \ldots, \varphi_k = \psi$ such that φ_i and φ_{i+1} are simply equivalent for $i = 0, 1, \ldots, k-1$.

The relation $\cong$ is an equivalence relation. The equivalence class of the form $\varphi = (a_1, \ldots, a_n)$ will be denoted by $\langle a_1, \ldots, a_n \rangle$.

2.2. Corollary. *For any $a, a_1, \ldots, a_n \in g$ and for any forms φ, φ' and ψ, ψ' over S we have*

(i) $(a, -a) \cong (1, -1)$;

(ii) $\sigma \in S_n \Rightarrow (a_1, \ldots, a_n) \cong (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$;

(iii) $\varphi \cong \psi$ *and* $\varphi' \cong \psi' \Rightarrow \varphi \perp \varphi' \cong \psi \perp \psi'$ *and* $\varphi \otimes \varphi' \cong \psi \otimes \psi'$;

(iv) $\varphi \cong \psi \Rightarrow \mathrm{Sgn}_P \varphi = \mathrm{Sgn}_P \psi$ *for any ordering P over S.*

2.3. Theorem. *If $\varphi \cong \psi$, then $D(\varphi) = D(\psi)$.*

Proof. This is trivial if $n =: \dim \varphi = \dim \psi = 1$. If $n > 1$, we can assume that φ and ψ are simply equivalent. Let $\varphi = (a, b)$, $\psi = (c, d)$, $c \in D(a, b)$, and $ab = cd$. We shall prove that $D(a, b) \subset D(c, d)$. If $x \in D(a, b)$, then $(1, ab) \approx ax, ac$. Hence $(1, cd) = (1, ab) \approx cx$ and $(c, d) \approx x$. By symmetry we get $D(c, d) \subset D(a, b)$. For $n > 2$, we use Lemma 1.4 and Theorem 1.6.

2.4. Lemma. *If $b \in D(a_1, \ldots, a_n)$, then there exist $b_2, \ldots, b_n \in g$ such that $(a_1, \ldots, a_n) \cong (b, b_2, \ldots, b_n)$.*

Proof. We use induction on n. For $n = 1, 2$ this is obvious. Let $b \in D(a_1, \ldots, a_n)$, $n \geqslant 3$. Then there exists an element $c \in D(a_2, \ldots, a_n)$ such that $b \in D(a_1, c)$. Now, by induction, there exist $c_3, \ldots, c_n$ such that

$(a_2, \ldots, a_n) \cong (c, c_3, \ldots, c_n)$ and, by Corollary 2.2, we get

$$(a_1, \ldots, a_n) = (b, ba_1, c, c_3, \ldots, c_n),$$

as required.

2.5. Definition. The form φ is said to be *isotropic* if either $\varphi \cong (1, -1)$ or there exists a form ψ such that $\varphi \cong (1, -1) \perp \psi$. Otherwise φ is said to be *anisotropic.*

2.6. THEOREM. *The following statements are equivalent:*

(i) *φ is isotropic;*

(ii) *if $\varphi \cong \varphi_1 \perp \varphi_2$, then there exists an element $x \in g$ such that $\varphi_1 \approx x$ and $\varphi_2 \approx -x$.*

We say that φ *satisfies* $(*)$ if for any φ_1 and φ_2 such that $\varphi \cong \varphi_1 \perp \varphi_2$ there exists $x \in g$ such that $\varphi_1 \approx x$ and $\varphi_2 \approx -x$. First we prove

LEMMA. *If ψ and τ are simply equivalent and ψ satisfies $(*)$, then τ also satisfies $(*)$.*

Proof. If $\dim\psi = \dim\tau = 2$, then $\psi \cong (a, -a)$ for $a \in g$. Hence $\psi \cong (b, -b)$, $b \in g$, and τ satisfies $(*)$.

Now assume that $\dim\psi = \dim\tau > 2$. By Lemma 1.4, it is sufficient to consider 4 cases:

1. $\psi = (a_1, \ldots, a_n, a, b)$, $\tau = (a_1, \ldots, a_n, c, d)$, $(a, b) \cong (c, d)$, $\tau_1 = (a_1, \ldots$ $\ldots, a_n)$, $\tau_2 = (c, d)$;

2. $\psi = (a_1, \ldots, a_k, \ldots, a_n, a, b)$, $\tau = (a_1, \ldots, a_k, \ldots, a_n, c, d)$, $(a, b) \cong$ $\cong (c, d)$, $\tau_1 = (a_1, \ldots, a_k)$, $\tau_2 = (a_{k+1}, \ldots, a_n, c, d)$, $1 \leqslant k \leqslant n-1$;

3. $\psi = (a_1, \ldots, a_n, a, b)$, $\tau = (a_1, \ldots, a_n, c, d)$, $(a, b) \cong (c, d)$, $\tau_1 = (a_1, \ldots$ $\ldots, a_n, c)$, $\tau_2 = (d)$;

4. $\psi = (a_1, \ldots, a_n, a, b, b_1, \ldots, b_m)$, $\tau = (a_1, \ldots, a_n, c, d, b_1, \ldots, b_m)$, $(a, b) \cong (c, d)$, $\tau_1 = (a_1, \ldots, a_n, c)$, $\tau_2 = (d, b_1, \ldots, b_m)$.

In the first case we take $\psi_1 = (a_1, \ldots, a_n)$ and $\psi_2 = (a, b)$. Since ψ satisfies $(*)$, there exists $x \in g$ such that $\psi_1 \approx x$ and $\psi_2 \approx -x$. But $D(a, b)$ $= D(c, d)$. Hence $\tau_1 \approx x$ and $\tau_2 \approx -x$. In cases 2-4 the proof is similar.

Proof of Theorem 2.6. (ii)$\Rightarrow$(i) follows from Lemma 2.4 and Corollary 2.2.

(i)$\Rightarrow$(ii). We observe that the forms $(1, -1)$ and $(1, -1, a_1, \ldots, a_n)$ satisfy $(*)$. Moreover, $\varphi_1 \perp \varphi_2 \cong (1, -1)$ or $\varphi_1 \perp \varphi_2 \cong (1, -1, a_1, \ldots, a_n)$. Using the Lemma we infer that $\varphi_1 \perp \varphi_2$ satisfies $(*)$.

2.7. THEOREM. *Let $\varphi = (a_1, \ldots, a_n)$ be any form of dimension greater than or equal to 2 and let $a, b \in g$. If $\varphi \otimes (a, b)$ is isotropic, then either*

$$\varphi \otimes (a, b) \cong (1, -1, 1, -1)$$

or there exist $c, d \in g$ such that $\varphi \cong (c, d) \perp \psi$ for some form ψ and $(c, d) \otimes (a, b) \cong (1, -1, 1, -1)$.

Proof. If $\varphi\otimes(a, b) = a\varphi \perp b\varphi$ is isotropic, then, by Theorem 2.6, there exists an element $x \in g$ such that $a\varphi \approx x$ and $b\varphi \approx -x$. For $n = 2$ we get

$$a(a_1, a_2) \cong (x, xa_1 a_2), \qquad b(a_1, a_2) \cong (-x, -xa_1 a_2),$$

and

$$\varphi\otimes(a, b) \cong (x, xa_1 a_2) \perp (-x, -xa_1 a_2) \cong (1, -1, 1, -1).$$

For $n > 2$ we have $\varphi \cong (ax, b_2, \ldots, b_n)$ and $\varphi \approx -bx$. Hence there exists $d \in D(b_2, \ldots, b_n)$ such that $(ax, d) \approx -bx$. We have

$$\varphi \cong (ax, d, d_3, \ldots, d_n) \quad \text{for some } d_3, \ldots, d_n \in g$$

and

$$\begin{aligned}(ax, d)\otimes(a, b) &\cong (x, ad) \perp (abx, bd)\\ &\cong (-abx, -bd) \perp (abx, bd) \cong (1, -1, 1, -1),\end{aligned}$$

as required.

2.8. COROLLARY. *If $\varphi = (a_1, \ldots, a_n)$, $n \geqslant 2$, $a \in g$, and $\varphi\otimes(1, a)$ is isotropic, then*

$$\varphi \cong a_1(1, -b_1) \perp \ldots \perp a_r(1, -b_r) \perp (z_1, \ldots, z_k),$$

where $b_i \in D(1, a)$, and either $k = 0$ (i.e., $\varphi \cong a_1(1, -b_1) \perp \ldots \perp a_r(1, -b_r)$) or $k = 1$ or $k \geqslant 2$ and $(z_1, \ldots, z_k)\otimes(1, a)$ is anisotropic.

Proof. Applying Theorem 2.7 we get

$$\varphi \cong (c_1, d_1) \perp \ldots \perp (c_r, d_r) \perp (z_1, \ldots, z_k),$$

where $(c_i, d_i)\otimes(1, a) = (1, -1, 1, -1)$ and $k = 0$ or $k = 1$ or $k \geqslant 2$ and $(z_1, \ldots, z_k)\otimes(1, a)$ is anisotropic. Moreover, by Theorem 2.6, $c_i(1, a) \approx x$ and $d_i(1, a) \approx -x$, and so $(1, a) \approx -c_i d_i$. Putting $b_i = -c_i d_i$ we obtain the required representation of φ.

2.9. Definition. The form $\bigotimes_{i=1}^{n} (1, a_i)$, $a_i \in g$ will be called an *n-fold Pfister form* and denoted by $((a_1, \ldots, a_n))$. The equivalence class of the form $((a_1, \ldots, a_n))$ will be denoted by $\langle\langle a_1, \ldots, a_n\rangle\rangle$.

2.10. LEMMA. *For any $a_1, a_2, y \in g$ we have*

(i) $(1, a_1) \approx y \Rightarrow \langle\langle a_1, a_2\rangle\rangle = \langle\langle a_1, a_2 y\rangle\rangle$;

(ii) $(a_1, a_2) \approx y \Rightarrow \langle\langle a_1, a_2\rangle\rangle = \langle\langle y, a_1 a_2\rangle\rangle$.

The proof is the same as in [10], Proposition 1.3, p. 276.

For any n-fold Pfister form $\varphi = ((a_1, \ldots, a_n))$, we define the form φ' inductively: if $\varphi = (1, a)$, then $\varphi' = (a)$, and $\varphi' = ((a_1, \ldots, a_{n-1}))' \perp \perp a_n(a_1, \ldots, a_{n-1})$ for $n > 1$. Clearly, $\varphi = (1) \perp \varphi'$.

2.11. THEOREM. *If* $((a_1, \ldots, a_n)) = (1) \perp \varphi'$ *and* $\varphi' \approx b$, *then there exist elements* $b_2, \ldots, b_n \in g$ *such that* $\langle\langle a_1, \ldots, a_n\rangle\rangle = \langle\langle b, b_2, \ldots, b_n\rangle\rangle$.

Proof. For $n = 1$ we have $\varphi' = (a)$; hence $b = a_1$. Let $n \geqslant 2$ and $\tau = ((a_1, \ldots, a_{n-1}))$. Then $\varphi' = \tau' \perp a_n\tau$ and, by Theorem 1.6, there exist elements $y \in D(\tau')$ and $x \in D(\tau)$ such that $(y, a_n x) \approx b$. Further, there exists $x_0 \in D(\tau')$ such that $(1, x_0) \approx x$ and we continue similarly as in [10], Proposition 1.5, Case 2, p. 279.

For any form φ we write $G(\varphi) = \{a \in g\colon a\varphi \cong \varphi\}$. We observe that $G(\varphi)$ is a group and if $\varphi \approx 1$, then $G(\varphi) \subset D(\varphi)$.

2.12. COROLLARY. *For any n-fold Pfister form,*

(i) *if* φ *is isotropic, then* $\varphi \cong 2^{n-1} \times (1, -1)$;

(ii) $D(\varphi) = G(\varphi)$ *and* $D(\varphi)$ *is a subgroup of* g;

(iii) *the sets* $D(2^k)$ *and* $D(\infty)$ *are groups.*

Proofs of (i) and (ii) are analogous as for Corollaries 16 and 17 in [10], p. 279-280 ($-1 \in D(\varphi')$ by Theorem 2.6); (iii) follows from (ii).

3. Equivalent schemes and their invariants. Let $S = \langle g, -1, d\rangle$ be a quadratic form scheme. We introduce now the notation used in the case of fields:

$q(S) = |g|$, $q_2(S) = |D_S(1, 1)|$,

$m(S)$ is the number of equivalence classes of 2-fold Pfister forms,

$$s(S) = \begin{cases} \min\{k\colon -1 \in D_S(k)\} & \text{if } S \text{ is a non-real scheme,} \\ \infty & \text{if } S \text{ is a formally real scheme.} \end{cases}$$

If S is the scheme of a field F, then $m(S) = m(F)$ is the number of quaternion algebras over F and $s(S) = s(F)$ is the "Stufe" of F. If φ is a form over S and $n \times \varphi \cong k \times (1, -1)$ for some natural n and k, then φ is said to be a *torsion form*. We have also the scheme counterpart of the u-invariant:

$$u(S) = \max\{\dim\varphi\colon \varphi \text{ is an anisotropic and torsion form over } S\}.$$

3.1. KNESER'S LEMMA. *If* S *is non-real,* $a \in g$, *and* φ *is an anisotropic form over* S, *then* $D(\varphi) \subsetneqq D(\varphi \perp (a))$.

This can be proved similarly as Lemma 4.5 in [10], p. 317, using $c \in D(i)$ instead of $e_1^2 + \ldots + e_i^2$.

3.2. LEMMA. *If* S *is a non-real scheme, then any form* φ *over* S *is a torsion form. If* S *is a formally real scheme and* φ *is a torsion form over* S, *then* $\dim\varphi$ *is even.*

Proof. If $s = s(S) < \infty$, then $2s \times (1) \cong s \times (1, -1)$ by Corollary 2.12. Hence

$$2s \times (a) \cong s \times (a, -a) \cong s \times (1, -1) \quad \text{for any } a \in g.$$

We have

$$2s\times(a_1, \ldots, a_n) \cong \mathop{\perp}_{i=1}^{n} 2s\times(a_i) \cong ns\times(1, -1)$$

and $(a_1, \ldots, a_n)$ is a torsion form. If S is formally real, then $\mathrm{Sgn}_P\,\varphi = 0$ for any ordering P over S (by Corollary 2.2), and so $\dim\varphi$ is even.

3.3. Corollary. *If S is non-real and $q(S) < \infty$, then $s(S) \leqslant u(S) \leqslant q(S)$. If S is a formally real scheme, then $u(S)$ is even.*

3.4. Theorem. *If S is a non-real scheme, then $s(S)$ is a power of 2.*

Proof. Let $s = s(S)$ and $2^k \leqslant s < 2^{k+1}$. Then the form $2^k\times(1)\perp 2^k\times(1) = 2^{k+1}\times(1)$ is isotropic. Thus there exists $c\in g$ such that $2^k\times(1) \approx \pm c$ (by Theorem 2.6). Now from Corollary 2.12 we get $2^k\times(1) \approx -1$ and, consequently, $s = 2^k$.

3.5. Theorem. *If $s(S) = 2^{s_0} < \infty$, then*

$$[D_S(2^{i+1})\colon D_S(2^i)] \geqslant 2^{s_0-i}, \qquad i = 0, \ldots, s_0.$$

3.6. Corollary. *If $s(S) = 2^{s_0} < \infty$, $q_2 = q_2(S)$, and $q = q(S)$, then*

$$2^{s_0(s_0-1)/2}q_2 \leqslant q \quad \text{and} \quad 2^{s_0(s_0+1)/2} \leqslant q.$$

The proofs of Theorem 3.5 and Corollary 3.6 are analogous as the proof of Satz 18 in [12].

Finally, we define the radical $R(S)$ of a scheme S as

$$R(S) = \{a\in g\colon D(1, -a) = g\}.$$

From Corollary 1.3 (vi) we infer that $R(S)$ is a subgroup of g. Moreover, $R(S) = \bigcap\{D(1, a)\colon a\in g\}$.

Now, let $S = \langle g, -1, d\rangle$ and $S' = \langle g', -1', d'\rangle$ be any form schemes.

3.7. Definition. The form schemes S and S' are said to be *equivalent* ($S \cong S'$) if there exists a group isomorphism $f\colon g \to g'$ such that $f(-1) = -1'$ and $f(d(a)) = d'(f(a))$ for any $a\in g$. Such an isomorphism f will be called an *equivalence map*. If for the scheme S there is a field k such that $S \cong S(k)$, then we say that the scheme S is *realized* by the field k.

We observe that the fields k and k' are equivalent with respect to quadratic forms (or Witt rings $W(k)$ and $W(k')$ are isomorphic ([2], Theorem 2.3)) if and only if $S(k) \cong S(k')$.

3.8. Corollary. *If $S \cong S'$, then $q(S) = q(S')$, $q_2(S) = q_2(S')$, $m(S) = m(S')$, $s(S) = s(S')$, $u(S) = u(S')$, $r(S) = r(S')$, and $R(S') = f(R(S))$, where $f\colon g \to g'$ is the corresponding equivalence map.*

In [9] Kula defined the product of schemes and power schemes. For two schemes $S_1 = \langle g_1, -1_1, d_1\rangle$ and $S_2 = \langle g_2, -1_2, d_2\rangle$ the product is defined to be

$$S = S_1 \sqcap S_2 = \langle g_1 \times g_2, (-1_1, -1_2), d\rangle,$$

where $d(a, b) = d(a) \times d(b) \subset g_1 \times g_2$. Fundamental properties of the product operation on the schemes are summarized in the following two theorems (due to Kula [9]):

3.9. THEOREM. *S is a quadratic form scheme. Moreover,*

(i) $D_S((a_1, b_1), \ldots, (a_n, b_n)) = D_{S_1}(a_1, \ldots, a_n) \times D_{S_2}(b_1, \ldots, b_n)$ *for any* $a_i \in g_1$, $b_j \in g_2$;

(ii) $((a_1, b_1), \ldots, (a_n, b_n)) \cong ((c_1, d_1), \ldots, (c_n, d_n))$ *over S if and only if* $(a_1, \ldots, a_n) \cong (c_1, \ldots, c_n)$ *over* S_1 *and* $(b_1, \ldots, b_n) \cong (d_1, \ldots, d_n)$ *over* S_2 $(a_i, c_j \in g_1, b_k, d_l \in g_2)$;

(iii) $q(S) = q(S_1)q(S_2)$, $q_2(S) = q_2(S_1)q_2(S_2)$, *and* $R(S) = R(S_1) \times R(S_2)$;

(iv) *if* $m(S_i) < \infty$, $i = 1, 2$, *then* $m(S) = m(S_1)m(S_2)$;

(v) $s(S) = \max\{s(S_i), i = 1, 2\}$ *and* $r(S) = r(S_1)+r(S_2)$;

(vi) *if* $u_0 = \max\{u(S_i), i = 1, 2\}$, *then*

$$u(S) = \begin{cases} u_0 - 1 & \text{if } u_0 \text{ is odd and } S \text{ is formally real,} \\ u_0 & \text{otherwise.} \end{cases}$$

3.10. THEOREM. *If the schemes S_1 and S_2 are realized by the fields k_1 and k_2, then $S = S_1 \sqcap S_2$ is also realized by some field k.*

Now, let $S = \langle g, -1, d\rangle$ be any quadratic form scheme. We follow the example of the formal power series field in defining the power scheme of S. For a 2-element group $\{1, t\}$ we define

$$g^t = g \times \{1, t\},$$

$$d^t(a) = \begin{cases} d(a) & \text{if } a \in g,\ a \neq -1, \\ g^t & \text{if } a = -1, \end{cases}$$

$$d^t(at) = \{1, at\} \quad \text{if } a \in g.$$

Kula proved also the following theorem (cf. [9]):

3.11. THEOREM. *$S^t = \langle g^t, -1, d^t\rangle$ is a quadratic form scheme. If φ and ψ are anisotropic forms over S, then φ and ψ are anisotropic forms over S^t and*

(i) $D_{S^t}(\varphi) = D_S(\varphi)$ *and* $D_{S^t}(\varphi \perp t\psi) = D_S(\varphi) \cup t D_S(\psi)$;

(ii) *if* $\varphi \cong \psi$ *over S, then* $\varphi \cong \psi$ *over* S^t; *if* $\varphi \cong \psi$ *and* $\varphi' = \psi'$ *over S, then* $\varphi \perp t\varphi' \cong \psi \perp t\psi'$ *over* S^t.

Moreover,

(iii) $q(S^t) = 2q(S)$ *and*

$$q_2(S^t) = \begin{cases} q_2(S) & \text{if } 1 \neq -1, \\ 2q_2(S) & \text{if } 1 = -1; \end{cases}$$

(iv) $s(S^t) = s(S)$, $u(S^t) = 2u(S)$, *and* $r(S^t) = 2r(S)$;

(v) *if* $q(S) < \infty$, *then*

$$m(S^t) = m(S) - 1 + \sum_{x \in g} [g\colon D_S(1, x)].$$

3.12. Definition. The scheme S' is said to be a *power scheme* if there exists a scheme S such that $S' \cong S^t$.

3.13. Theorem. *Let* $S = \langle g, -1, d\rangle$ *and* $q(S) > 2$. *The following statements are equivalent*:

(i) S *is a power scheme*;

(ii) *there exists* $a \in g$, $a \neq \pm 1$, *such that* $|D_S(1, a)| = |D_S(1, -a)| = 2$;

(iii) *there exists a subgroup* $h \subset g$ *such that* $|D_S(1, a)| = 2$ *for any* $a \notin h$;

(iv) *there exists a subgroup* $h \subset g$ *such that* $[g\colon h] = 2$ *and* $|D(1, a)| = 2$ *for any* $a \neq h$.

Proof. (i)$\Rightarrow$(ii) and (iii)$\Rightarrow$(iv) are trivial. (ii)$\Rightarrow$(iii) can be proved analogously as Theorem 1 in [1].

(iv)$\Rightarrow$(i). Let $g = h \times \{1, t\}$ and $|D_S(1, a)| = 2$ for any $a \notin h$. Since $q > 2$, we get $-1 \in h$. Observe that if $a \in h$, $a \neq -1$, then $D_S(1, a) \subset h$. Hence we have $D_S(a, b) \subset h$ for any $a, b \in h$, $a \neq -b$. By induction we obtain $D_S(a_1, \ldots, a_n) \subset h$ for any $a_1, \ldots, a_n \in h$ such that $(a_1, \ldots, a_n)$ is an anisotropic form over S. It follows that $S_0 = \langle h, -1, d_0\rangle$ is a quadratic form scheme, where $d_0(a) = D_S(1, a)$, $a \in h$. It is trivial that $S \cong S_0^t$ and the theorem is proved.

3.14. Corollary. *If* S *is non-real and* $q(S) > 2$, *then* S *is a power scheme if and only if there exists* $a \in g$, $a \neq \pm 1$, *such that* $|D_S(1, a)| = 2$.

Proof. Let $|D(1, a)| = 2$, $a \neq \pm 1$. First suppose that the form $s \times (1)$ is universal. We write $k = \min\{i\colon -a \in D(i \times (1))\}$ and observe that $k \leqslant s$. Since $-a \in D(k \times (1))$, there exists $b \in D((k-1) \times (1))$ such that $-a \in D(1, b)$. Hence $-b \in D(1, a)$. Now $k - 1 < s$ implies $b \neq -1$. Moreover, $-a \notin D((k-1) \times (1))$. Consequently, $b \neq -a$, and so $|D(1, a)| > 3$, a contradiction. Thus $s \times (1)$ is not universal and $-1 \in D(s \times (1))$, and we continue the proof similarly as for Theorem 1 in [5].

From Theorem 3.13 and [9], [13] we get

3.15. Corollary. *If* $S \cong S(k)$ *for a field* k, *then* $S^t \cong S(k((t)))$. *If* S^t *is realized by a field, then* S *is also realized by a field.*

4. Schemes with non-trivial radical. Let g be an elementary 2-group. We define $d(a) = g$ for any $a \in g$. Obviously, for any fixed element of g denoted by -1, the triple $S = \langle g, -1, d\rangle$ is a quadratic form scheme and $R(S) = g$. Such a scheme will be called *radical*. Observe that for a radical scheme S we have $s(S) = 1$ if $-1 = 1$ and $s(S) = 2$ otherwise. In this section we shall show that any scheme S with a non-trivial radical can be split into the product of two schemes $S = S_1 \sqcap S_2$ with S_1 having a trivial radical (i.e., $|R(S)| = 1$) and with S_2 radical in the above-mentioned sense.

4.1. LEMMA. *Let β be a power of 2 or an infinite cardinal number. There exist fields k_1^β and k_2^β such that*

(i) $q(k_1^\beta) = q(k_2^\beta) = \beta$;

(ii) $R(k_i^\beta) = g(k_i^\beta)$, $i = 1, 2$;

(iii) $s(k_1^\beta) = 1$ *and* $s(k_2^\beta) = 2$.

Proof. For $\beta = 2^n$, $n \geqslant 1$, such fields can be found in [2] and [15]. Let β be an infinite cardinal number, and k any field with $|k| = \beta$ (cf. [7]). We denote by $\bar{k}$ the algebraic closure of k and put $k_1^\beta = \bar{k}(x)$, the rational function field. It is obvious that $s(k_1^\beta) = s(\bar{k}) = 1$ and, by the Tsen-Lang theorem ([10], p. 296), $R(k_1^\beta) = g(k_1^\beta)$. We prove that $q(k_1^\beta) = \beta$. Since the set $B = \{(x-c)k_1^{*2}\colon c \in \bar{k}\}$ is a basis for the group $g(k_1^\beta)$, we have $|g(k_1^\beta)| = |B| = |k| = \beta$ and k_1^β satisfies (i)-(iii). Now, by Theorem 3.10, there exists k_2^β such that $S(k_2^\beta) = S(k_1^\beta) \sqcap S(F_3)$ and k_2^β satisfies (i)-(iii) by Theorem 3.9.

4.2. Definition. The scheme $S_i^\beta = S(k_i^\beta)$, $i = 1, 2$, will be called a *radical scheme of cardinality β*.

4.3. COROLLARY. *Let $S = \langle g, -1, d\rangle$, $|g| > 1$ be a radical scheme and let $\beta = q(S)$. Then $S \cong S_1^\beta$ or $S \cong S_2^\beta$ according as $s(S) = 1$ or $s(S) = 2$.*

4.4. LEMMA. *For any scheme $S = \langle g, -1, d\rangle$, if $a_1, \ldots, a_n \in g$ and $r_1, \ldots, r_n \in R(S)$ for $n \geqslant 2$, then $D_S(a_1 r_1, \ldots, a_n r_n) = D_S(a_1, \ldots, a_n)$.*

This can be proved as in [3] by using Corollary 1.3 (vi).

Now, let $S = \langle g, -1, d\rangle$ be any quadratic form scheme, R its radical, and $f\colon g \to g/R$ the canonical homomorphism. We define a mapping d_R on the group g/R with values in the family of subgroups of g/R by putting $d_R(aR) = f(d(a))$. By Lemma 4.4, the equality $f(a) = f(b)$ implies $d(a) = d(b)$ for any $a, b \in g$. Hence d_R is well defined.

4.5. THEOREM. *For any scheme $S = \langle g, -1, d\rangle$, $S/R = \langle g/R, -R, d_R\rangle$ is a quadratic form scheme. Moreover,*

(i) $D_{S/R}(a_1 R, \ldots, a_n R) = f(D_S(a_1, \ldots, a_n))$ *for any* $a_1, \ldots, a_n \in g$;

(ii) $|R(S/R)| = 1$ *and*

$$s(S/R) = \begin{cases} 1 & \text{if } -1 \in R(S), \\ s(S) & \text{if } -1 \notin R(S). \end{cases}$$

Proof. It is sufficient to show (i). For $n = 1, 2$ this is trivial. If $n > 2$, then (i) can be proved by using induction.

4.6. LEMMA. *For any scheme $S = \langle g, -1, d\rangle$ with $s(S) \geqslant 2$, the schemes $S \sqcap S_1^\beta$ and $S \sqcap S_2^\beta$ are equivalent.*

Proof. Suppose $S_1^\beta = \langle g_1, -1_1, d_1\rangle$, $S_2^\beta = \langle g_2, -1_2, d_2\rangle$, and assume that B_1 is any basis of the group g_1, $B_2 = \{-1_2\} \cup \{c_i\}_{i \in I}$ is a basis of g_2, and $B = \{-1\} \cup \{a_j\}_{j \in J}$ is a basis of the group g. Since $|g_1| = |g_2| = \beta$, we have $|B_1| = |B_2|$. We choose f: $B_2 \to B_1$ to be any bijective mapping and let $b_0 = f(-1_2)$, $b_i = f(c_i)$, $i \in I$. Then the sets

$$B_1' = \{(1, b_0), (-1, 1_1)\} \cup \{(1, b_i)\}_{i \in I} \cup \{(a_j, 1_1)\}_{j \in J}$$

and

$$B_2' = \{(1, -1_2), (-1, 1_2)\} \cup \{(1, c_i)\}_{i \in I} \cup \{(a_j, 1_2)\}_{j \in J}$$

are bases of the groups $g \times g_1$ and $g \times g_2$, respectively. We observe that the set

$$B_1'' = \{(1, b_0), (-1, b_0)\} \cup \{(1, b_i)\}_{i \in I} \cup \{(a_j, 1_1)\}_{j \in J}$$

is also a basis of the group $g \times g_1$. We define the group isomorphism $\bar{f}$: $g \times g_2 \to g \times g_1$ such that

$$\bar{f}(1, -1_2) = (1, b_0), \quad \bar{f}(-1, 1_2) = (-1, b_0),$$

$$\bar{f}(1, c_i) = (1, b_i), \quad \bar{f}(a_j, 1_2) = (a_j, 1_1), \quad i \in I,\ j \in J.$$

It is easy to see that $\bar{f}$ is an equivalence map. Hence $S \sqcap S_2$ and $S \sqcap S_1$ are equivalent.

4.7. THEOREM. *Let $S = \langle g, -1, d \rangle$ be a quadratic form scheme with radical $R(S)$ of cardinality $\beta \neq 1$. Let $S/R = \langle g/R, -R, d_R \rangle$ and let S_i^β be radical schemes of cardinality β. Then the scheme S decomposes into the product schemes as follows*:

(i) $S \cong S/R \sqcap S_1^\beta \cong S/R \sqcap S_2^\beta$ *if* $-1 \notin R(S)$;

(ii) $S \cong S/R \sqcap S_2^\beta$ *if* $-1 \in R(S)$ *and* $s(S) \geq 2$;

(iii) $S \cong S/R \sqcap S_1^\beta$ *if* $s(S) = 1$.

Proof. First observe that $S_0 = \langle R, -1_0, d_0 \rangle$, where $R = R(S)$, $d_0(r) = R$ for any $r \in R$ and the scheme

$$-1_0 = \begin{cases} -1 & \text{if } -1 \in R(S), \\ 1 & \text{if } -1 \notin R(S) \end{cases}$$

is a quadratic form scheme. Moreover, $S_0 \cong S_2^\beta$ for $-1 \in R$, $s > 1$, and $S_0 \cong S_1^\beta$ otherwise.

Let g' be a subgroup of g such that $g = g' \times R$. If $-1 \notin R$, we assume that $-1 \in g'$. If f: $g \to g/R$ is the canonical homomorphism, then $\bar{f} = f_{g'}$: $g' \to g/R$ is a group isomorphism. We define a mapping h: $g/R \times R \to g$ by $h(aR, r) = \bar{f}^{-1}(aR) \cdot r$, $a \in g$, $r \in R$. Clearly, h is also an isomorphism. Moreover, we can show that

$$d(h(aR, r)) = h(d_R(aR) \times d_0(r))$$

for any $a \in g$, $r \in R$, and $h(-R, -1_0) = -1$.

Hence h is an equivalence map for the schemes $S/R \sqcap S_0$ and S.

4.8. THEOREM. *For any schemes S and S' with radicals R and R', respectively, $S \cong S'$ if and only if $S/R \cong S'/R'$, $|R| = |R'|$, and $s(S) = 1 \Leftrightarrow s(S') = 1$.*

Proof. Sufficiency. From Theorem 4.5 (i) we have

$$-1 \in R \Leftrightarrow D_{S/R}(R, R) = g/R \Leftrightarrow D_{S'/R'}(R', R') = g'/R' \Leftrightarrow -1' \in R'.$$

This and the equivalence $s(S) = 1 \Leftrightarrow s(S') = 1$ imply that $S_0 \cong S'_0$, where S_0 and S'_0 have been defined in the proof of Theorem 4.7. We have

$$S \cong S/R \sqcap S_0 \cong S'/R' \sqcap S'_0 \cong S'.$$

Necessity. If $h\colon g \to g'$ is an equivalence map for the schemes S and S', then $h(R) = R'$. This implies that the mapping $\bar{h}\colon S/R \to S'/R'$ defined by $\bar{h}(aR) = h(a) \cdot R'$ is an equivalence map establishing the equivalence of schemes S/R and S'/R' and the theorem is proved.

4.9. Remark. Theorems 4.7 and 4.8 can be applied in the classification of schemes with a radical of fixed cardinality. For example, let X be the set of all schemes with $q = \alpha$ and $R = \{1\}$. We put

$$X_1 = \{S \in X\colon s(S) = 1\}, \qquad X_2 = \{S \in X\colon s(S) > 1\}.$$

Then the set of all schemes such that $|R| = \beta$ and $[g\colon R] = \alpha$ is

$$X' = X'_1 \cup X''_1 \cup X'_2,$$

where

$$X'_1 = \{S \sqcap S_1^\beta\colon S \in X_1\}, \qquad X''_1 = \{S \sqcap S_2^\beta\colon S \in X_1\},$$
$$X'_2 = \{S \sqcap S_1^\beta\colon S \in X_2\}.$$

By Theorem 4.8 we see that all schemes of X' are pairwise non-equivalent. Moreover, it follows from the results of Kula [9] that if all schemes of X are realized by fields, then all schemes of X' are also realized by fields. (For $\alpha \leqslant 16$ this can be found in [14]. Other applications of Theorems 4.7 and 4.8 will be given in the next section.)

5. Non-real schemes with $u \geqslant q/2$. In this section we give the classification of non-real schemes with $q < \infty$ and $u \geqslant q/2$ and we prove that all these schemes are realized by fields. First we prove some auxiliary properties.

Let φ be any form with $\dim \varphi \geqslant 2$ and such that $\varphi \otimes (1, 1)$ is isotropic. Then from Corollary 2.8 we get

$$\varphi \cong a_1(1, -b_1) \perp \ldots \perp a_n(1, -b_n) \perp (z_1, \ldots, z_k), \qquad a_i, b_i, z_j \in g,$$

where $b_i \in D(1, 1)$ and either $k = 0$ or $k = 1$ or $k \geqslant 2$ and $(z_1, \ldots, z_k) \otimes (1, 1)$ is anisotropic. We refer to such an equivalence of forms as to a β-decomposition of φ.

5.1. Theorem. *Let φ be an anisotropic form with* $\dim\varphi \geqslant 2$ *and let* $\varphi\otimes(1, 1)$ *be isotropic. If* $\varphi \cong a_1(1, -b_1)\perp \ldots \perp a_n(1, -b_n)\perp(z_1, \ldots, z_k)$ *is the β-decomposition of φ, then the following sets are mutually disjoint*:

1. $D\big(a_1(1, -b_1)\perp \ldots \perp a_n(1, -b_n)\big)$, $D(z_1, \ldots, z_k)$, $-D(z_1, \ldots, z_k)$;
2. $D\big(a_{i_1}(1, -b_{i_1})\perp \ldots \perp a_{i_r}(1, -b_{i_r})\big)$, $D\underset{j\neq i_1,\ldots,i_r}{\perp} a_j(1, -b_j)$, $1 \leqslant r < n$.

The proof of Theorem 5.1 is the same as in [6], p. 290.

5.2. Theorem. *If $S = \langle g, -1, d\rangle$ is a non-real and non-power scheme with $q \geqslant 4$ (i.e., $D(a, b) \geqslant 4$ for any $a, b\in g$) and φ is an anisotropic form of dimension $n = \dim\varphi \geqslant 2$, then $|D(\varphi)| \geqslant 2n$.*

Proof. For $q = 4$ this is trivial and we assume that $q \geqslant 8$.

Step 1. If $s \geqslant 4$, then $|D(1, 1, 1)| \geqslant |D(1, 1)|+2 \geqslant s+2$ (cf. [6], Lemma 3.2). By Kneser's lemma, there exists $x\in D(1, 1, 1)\backslash D(1, 1)$, i.e., $(1, y) \approx x$ for some $y\in D(1, 1)\backslash\{1\}$. We have $x \neq xy$, $xy\notin D(1, 1)$, and $xy\in D(1, y) \subset D(1, 1, 1)$. We get the second inequality by Theorem 3.5.

Step 2. $|D(\varphi)| \geqslant 6$ for $\varphi = (1, a, b)$. We observe that it is sufficient to prove

(*) there exist $x, y\in D(a, b)$ such that $D(1, x) \neq D(1, y)$.

If $a = b = 1$, then $s \geqslant 4$ and $|D(1, 1, 1)| \geqslant 6$. Let $a = b$ and $(1, 1) \not\approx a$. For $s = 2$ we have $D(1, a) \neq D(1, -a)$, and if $s \geqslant 4$, then $D(1, a) \neq D(1, c)$ for some $c\in D(1, 1)$, $c \neq 1$. Finally, if $1, a, b$ are three different elements of g, then $D(1, a) \neq D(1, b)$ or $(1, a, b) \cong (1, a, a)$ and (*) holds.

Step 3. Using Theorem 5.1, the previous steps, and Lemma 8 in [14], we can prove Theorem 5.2 (for any $n \geqslant 4$) similarly as Theorem 3.5 in [6].

5.3. Corollary. *Let S be a non-real and non-power scheme with $4 \leqslant q < \infty$ and $u \geqslant q/2$. If φ is a u-dimensional anisotropic form and $\varphi \cong a_1(1, -b_1)\perp \ldots \perp a_n(1, -b_n)\perp(z_1, \ldots, z_k)$ is the β-decomposition of φ, then $k = 0$ and $n = q/4$ (i.e., $u = q/2$).*

5.4. Theorem. *Let S be a non-real scheme with $4 \leqslant q = 2^n < \infty$. Then $u = q$ if and only if*

$$S \cong S\big(F_3((t_1))\ldots((t_{n-1}))\big) \quad \text{or} \quad S \cong S\big(F_5((t_1))\ldots((t_{n-1}))\big).$$

Proof. If $n > 2$ and $u = q$, then S is a power scheme and we use induction on n. The converse is trivial.

5.5. Lemma. *Let S be a non-real scheme with $s \leqslant 2$ and $u = q/2$, $q < \infty$. Then $|R| = 2$ or S is a power scheme.*

The proof of the lemma is the same as the proof of Theorems 4-6 in [5].

5.6. Theorem. *Let S be a non-real scheme with $u = q/2 < \infty$. If S is a non-power scheme, then it is equivalent to one of the schemes*

$$S(F_5)\sqcap S(F_5),\ S(F_5)\sqcap S(F_3),\ S(F_5)\sqcap S\big(F_5((t_1))\dots((t_n))\big),$$
$$S(F_3)\sqcap S\big(F_5((t_1))\dots((t_n))\big),\ S(F_3)\sqcap S\big(F_3((t_1))\dots((t_n))\big),\ S(Q_2)$$

and S is realized by a field. If S is a power scheme, then it is realized by an iterated power series extension of a field k, where $S(k)$ is one of the above schemes.

Proof. For $q(S) \leqslant 4$ the result is trivial. Let $q(S) \geqslant 8$ and let S be a non-power scheme. If $s(S) \leqslant 2$, then $|R(S)| = 2$, $u(S/R) \cong q(S/R)$ and we use Theorems 4.7 and 5.4. Let $s \geqslant 4$ and let φ be a u-dimensional anisotropic form and, by scaling and Corollary 5.3, we may assume that $\varphi \cong (1, -b_1) \perp \dots \perp a_n(1, -b_n)$, $n = q/4 \geqslant 4$, is the β-decomposition of φ. From Theorem 5.1 we get $b_i = b_1$, $i = 2, \dots, n$. Hence $n = 2$ and $S \cong S(Q_2)$. If S is a power scheme, we use induction since $S \cong S_0^t$ and $u(S_0) = \frac{1}{2} q(S_0)$.

5.7. Remark. Theorems 4.7, 5.4, and 5.6 can be used to the classification of all non-real schemes with $|R| > 1$ and $u \geqslant \frac{1}{2}[g\colon R]$, $[g\colon R] < \infty$. Then $S \cong S/R\sqcap S_i^\beta$, $i = 1, 2$, and S/R is one of the schemes in Theorems 5.4 and 5.6. Clearly, all these schemes are realized by fields.

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