

ON ALGEBRAIC OPERATIONS IN IDEMPOTENT ALGEBRAS

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I. Introduction. In this paper we adopt the definitions and notation given by Marczewski in [2] and [3]. Let $\mathfrak{A} = (A; \mathbf{F})$ be an algebra, i. e. a set A of elements and a class \mathbf{F} of fundamental operations consisting of A -valued functions of several variables running over A . If $A = \{a, b, \dots\}$ and $\mathbf{F} = \{f, g, \dots\}$, we shall sometimes write $(a, b, \dots; f, g, \dots)$ or $(A; f, g, \dots)$ instead of $(A; \mathbf{F})$. The n -ary operations

$$e_k^{(n)}(x_1, x_2, \dots, x_n) = x_k \quad (k = 1, 2, \dots, n; n = 1, 2, \dots)$$

will be called *trivial*. We denote by \mathbf{A} the class of all algebraic operations, i. e. the smallest class containing trivial operations and closed under the composition with fundamental operations. The subclass of all n -ary algebraic operations will be denoted by $\mathbf{A}^{(n)}$. If $1 \leq k \leq n$, then $\mathbf{A}^{(n,k)}$ will denote the subclass of $\mathbf{A}^{(n)}$ consisting of all operations depending on at most k variables. Thus $f \in \mathbf{A}^{(n,k)}$ if there is an operation $g \in \mathbf{A}^{(k)}$ such that $f(x_1, x_2, \dots, x_n) = g(x_{i_1}, x_{i_2}, \dots, x_{i_k})$ for a system of indices i_1, i_2, \dots, i_k . Two algebras $(A; \mathbf{F}_1)$ and $(A; \mathbf{F}_2)$ having the same class of algebraic operations will be treated here as identical. In particular, we have the equation $(A; \mathbf{F}) = (A; \mathbf{A})$.

The algebra $(A; \mathbf{F})$ is called *idempotent* if $\mathbf{A}^{(1)}$ consists of trivial operations only. In other words, the algebra is idempotent if and only if for every algebraic operation f the equation $f(x, x, \dots, x) = x$ holds.

Let $\mathcal{S}(\mathfrak{A})$ be the set of all non-negative integers n for which there exists an algebraic non-trivial n -ary operation in \mathfrak{A} depending on every variable. The investigation of the sets $\mathcal{S}(\mathfrak{A})$ was suggested by Marczewski. In particular, he proved in [5] that if there is no constant operation in the algebra \mathfrak{A} and there is an n -ary symmetrical (or even quasi-symmetrical) operation, then the set $\mathcal{S}(\mathfrak{A})$ contains the arithmetical progression $n + (n-1)k$ ($k = 0, 1, \dots$). This result for $n = 2$ was previously obtained by Płonka in [7]. The aim of the present paper is to give a complete description of all possible sets $\mathcal{S}(\mathfrak{A})$.

II. Examples. First of all we shall prove that for any subset E of the set $\{0, 1, \dots\}$ satisfying the condition $E \cap \{0, 1\} \neq \emptyset$ there exists an algebra \mathfrak{A}_E for which $\mathcal{S}(\mathfrak{A}_E) = E$.

Let A be the set of all non-negative integers. We define an n -ary operation t_n ($n \geq 1$) on A as follows: $t_n(x_1, x_2, \dots, x_n) = 2$ if the integers x_1, x_2, \dots, x_n are all different and odd and $t_n(x_1, x_2, \dots, x_n) = 0$ in the opposite case. Moreover, we define a constant operation $t_0(x) = 0$ for all $x \in A$. Of course, the operations t_n depend on every variable.

Suppose that $0 \in E$. Put $F_E = \{t_n : n \in E\}$ and $\mathfrak{A}_E = (A; F_E)$. From the definition of fundamental operations t_n it follows that each non-trivial algebraic operation in \mathfrak{A}_E is of the form

$$f(x_1, x_2, \dots, x_k) = t_n(x_{j_1}, x_{j_2}, \dots, x_{j_n}),$$

where $n \in E$, $1 \leq j_i \leq k$ ($i = 1, 2, \dots, n$) and all indices j_1, j_2, \dots, j_n are different. Thus the equation $\mathcal{S}(\mathfrak{A}_E) = E$ is true.

Now for any positive integer n we define an n -ary operation w_n as follows: $w_n(x_1, x_2, \dots, x_n) = 2x_n$ if the integers x_1, x_2, \dots, x_n are different and odd and $w_n(x_1, x_2, \dots, x_n) = 2x_1$ in the opposite case. Of course, each operation w_n is non-trivial and depends on every variable.

Suppose that $0 \notin E$ and $1 \in E$. Put $F_E = \{w_n : n \in E\}$ and $\mathfrak{A}_E = (A; F_E)$. It is very easy to prove that each algebraic non-trivial operation in the algebra \mathfrak{A}_E is of the form

$$f(x_1, x_2, \dots, x_k) = 2^r w_n(x_{j_1}, x_{j_2}, \dots, x_{j_n}),$$

where $n \in E$, $r \geq 0$, $1 \leq j_i \leq k$ ($i = 1, 2, \dots, n$) and the indices j_1, j_2, \dots, j_n are all different. Hence the equation $\mathcal{S}(\mathfrak{A}_E) = E$ follows, which completes the proof.

Consequently, it remains the question of a characterization of all subsets $E \subset \{2, 3, \dots\}$ for which there exists an algebra \mathfrak{A} with $\mathcal{S}(\mathfrak{A}) = E$. Obviously, this question is simply the question of a description of all possible sets $\mathcal{S}(\mathfrak{A})$ for idempotent algebras \mathfrak{A} .

Now we shall give some examples of the sets $\mathcal{S}(\mathfrak{A})$ for idempotent algebras \mathfrak{A} .

1. Let \mathfrak{A} be a *trivial algebra*, i. e. an algebra in which all algebraic operations are trivial. In this case the set $\mathcal{S}(\mathfrak{A})$ is empty. Of course, this property characterizes the trivial algebras among idempotent ones.

2. Let G be a Boolean group with the addition as a group operation, i. e. a group in which all elements different from the zero element are of order 2. Put $g(x, y, z) = x + y + z$ for $x, y, z \in G$ and $\mathfrak{A} = (G; g)$. It is easy to see that each algebraic operation in \mathfrak{A} is a sum of an odd number of different variables. If G contains at least two elements, then each

operation $x_1 + x_2 + \dots + x_{2k+1}$ depends on every variable. Consequently, in this case the set $\mathcal{S}(\mathfrak{A})$ consists of all odd integers greater than 1.

3. Consider a diagonal algebra, i. e. an algebra of the form $\mathfrak{A} = (A; d)$, where the set A is a Cartesian product $A_1 \times A_2 \times \dots \times A_n$ and the fundamental n -ary operation d is defined by the formula

$$d(\langle a_1^1, a_2^1, \dots, a_n^1 \rangle, \langle a_1^2, a_2^2, \dots, a_n^2 \rangle, \dots, \langle a_1^n, a_2^n, \dots, a_n^n \rangle) = \langle a_1^1, a_2^2, \dots, a_n^n \rangle,$$

where $a_j^i \in A_j$ ($i, j = 1, 2, \dots, n$). Diagonal algebras were introduced by Płonka in [7]. If the n -ary operation d depends on every variable, then the algebra \mathfrak{A} is called *n -dimensional*. Obviously, one-dimensional diagonal algebras are trivial. If \mathfrak{A} is an n -dimensional diagonal algebra and $n \geq 2$, then, according to [7], the set $\mathcal{S}(\mathfrak{A})$ consists of the integers $2, 3, \dots, n$.

We note that diagonal algebras can be defined in terms of binary fundamental operations. Namely, we have the equation $(A; d) = (A; d_1, d_2, \dots, d_n)$, where

$$(1) \quad \begin{aligned} d_j(\langle a_1, a_2, \dots, a_n \rangle, \langle b_1, b_2, \dots, b_n \rangle) \\ = \langle b_1, b_2, \dots, b_{j-1}, a_j, b_{j+1}, \dots, b_n \rangle \quad (j = 1, 2, \dots, n) \end{aligned}$$

for $a_k, b_k \in A_k$ ($k = 1, 2, \dots, n$).

4. Let A be an arbitrary set containing at least m elements, where $m \geq 3$. Let l_m be an m -ary operation on A defined as follows: $l_m(x_1, \dots, x_m) = x_1$ if x_1, \dots, x_m are all different and $l_m(x_1, \dots, x_m) = x_m$ in the opposite case (see [4], p. 2). Put $\mathfrak{A} = (A; l_m)$. Taking into account that the operation l_m depends on every variable and applying Lemma 1 proved in Chapter IV, we infer that the set $\mathcal{S}(\mathfrak{A})$ consists of all integers $\geq m$.

Further, let A be a three-element set and let the binary operation r_3 on A be defined by the conditions: if $x \neq y$, then $r_3(x, y) \notin \{x, y\}$ and $r_3(x, x) = x$ (see [4], p. 2). The operation r_3 is symmetrical. Thus, by previously cited Płonka's result [7], the set $\mathcal{S}(\mathfrak{A})$ for the algebra $\mathfrak{A} = (A; r_3)$ contains all integers ≥ 2 .

5. Let G be an infinite Boolean group and $(G; g)$ the algebra defined in Example 2. Given an integer $m \geq 5$ we define an m -ary operation p_m on G as follows: $p_m(x_1, x_2, \dots, x_m) = x_m$ if all elements x_1, x_2, \dots, x_m belong to a subalgebra of the algebra $(G; g)$ generated by less than m elements and $p_m(x_1, x_2, \dots, x_m) = x_1$ in the opposite case. Put $\mathfrak{A} = (G; g, p_m)$. It is very easy to verify that for $k < m$ every k -ary algebraic operation in \mathfrak{A} is also algebraic in $(G; g)$. Thus the intersection $\mathcal{S}(\mathfrak{A}) \cap \{2, 3, \dots, m-1\}$ consists of all odd integers less than m . Hence and from the inclusion $\mathcal{S}(\mathfrak{A}) \supset \mathcal{S}((G; g)) \cup \mathcal{S}((G; p_m))$ it follows, by

Lemma 1, that the set $\mathcal{S}(\mathfrak{A})$ consists of all integers $\geq m$ and all odd integers greater than 1.

6. Consider an infinite n -dimensional diagonal algebra $(A; d)$, where $n \geq 2$. Given an integer $m > n$ we define an m -ary operation q_m on A by the conditions: $q_m(x_1, x_2, \dots, x_m) = x_m$ if the elements x_1, x_2, \dots, x_m belong to a subalgebra of the diagonal algebra $(A; d)$ generated by less than m elements and $q_m(x_1, x_2, \dots, x_m) = x_1$ in the opposite case. Put $\mathfrak{A} = (A; d, q_m)$. Since the set A is infinite, the operation q_m depends on every variable. Moreover, $\mathcal{S}(\mathfrak{A}) \cap \{2, 3, \dots, m-1\} = \{2, 3, \dots, n\}$. Hence and from the inclusion $\mathcal{S}(\mathfrak{A}) \supset \mathcal{S}((A; d)) \cup \mathcal{S}((A; q_m))$ it follows, by Lemma 1, that the set $\mathcal{S}(\mathfrak{A})$ consists of all integers s satisfying one of the inequalities $2 \leq s \leq n$, $s \geq m$.

III. Theorems. The examples give six types of the sets $\mathcal{S}(\mathfrak{A})$ for idempotent algebras \mathfrak{A} . The main result of the present paper is that these six types give a complete description of all possible sets $\mathcal{S}(\mathfrak{A})$ for idempotent algebras \mathfrak{A} .

THEOREM 1. *For each idempotent algebra \mathfrak{A} one of the following cases holds:*

- (i) $\mathcal{S}(\mathfrak{A})$ is an empty set,
- (ii) $\mathcal{S}(\mathfrak{A})$ consists of all odd integers greater than 1,
- (iii) $\mathcal{S}(\mathfrak{A})$ consists of all integers s satisfying the inequality $2 \leq s \leq n$, where $n \geq 2$,
- (iv) $\mathcal{S}(\mathfrak{A})$ consists of all integers $\geq m$, where $m \geq 2$,
- (v) $\mathcal{S}(\mathfrak{A})$ consists of all odd integers greater than 1 and all integers $\geq m$, where $m \geq 5$.
- (vi) $\mathcal{S}(\mathfrak{A})$ consists of all integers s satisfying one of the inequalities $2 \leq s \leq n$, $s \geq m$, where $m > n \geq 2$.

Sometimes the set $\mathcal{S}(\mathfrak{A})$ completely determines the algebraic structure of an idempotent algebra \mathfrak{A} . For instance, the set $\mathcal{S}(\mathfrak{A})$ for an idempotent algebra is empty if and only if the algebra is trivial. We shall prove that except two cases $\mathcal{S}(\mathfrak{A}) = \{2, 3, \dots\}$ and $\mathcal{S}(\mathfrak{A}) = \{3, 4, \dots\}$ the set $\mathcal{S}(\mathfrak{A})$ determines the algebraic structure of the idempotent algebra \mathfrak{A} .

THEOREM 2. *Let \mathfrak{A} be an idempotent algebra.*

- 1. $\mathcal{S}(\mathfrak{A}) = \{s: 2 \leq s \leq n\}$, where $n \geq 1$ if and only if \mathfrak{A} is a diagonal algebra.
- 2. $\mathcal{S}(\mathfrak{A}) = \{s: 2 \leq s \leq n\} \cup \{s: s \geq m\}$, where $m > n \geq 1$, $m \geq 4$ if and only if $\mathfrak{A} = (A; \{d\} \cup \mathbf{F})$, where $(A; d)$ is an n -dimensional diagonal algebra, the class \mathbf{F} contains an m -ary operation depending on every

variable, all operations f from \mathbf{F} depend on at least m variables and satisfy the equation $f(x_1, x_2, \dots, x_k) = x_1$ whenever the elements x_1, x_2, \dots, x_k belong to a subalgebra of the algebra $(A; d)$ generated by less than m elements.

3. $\mathcal{S}(\mathfrak{A})$ consists of all odd integers greater than 1 if and only if $\mathfrak{A} = (G; g)$, where G is an at least two-element Boolean group and $g(x, y, z) = x + y + z$.

4. $\mathcal{S}(\mathfrak{A})$ consists of all odd integers greater than 1 and all integers $\geq m$, where $m \geq 5$ if and only if $\mathfrak{A} = (G; \{g\} \cup \mathbf{F})$, where $(G; g)$ is the algebra defined in the preceding assertion, \mathbf{F} contains an m -ary operation depending on every variable, all operations f from \mathbf{F} depend on at least m variables and satisfy the equation $f(x_1, x_2, \dots, x_k) = x_1$ whenever the elements x_1, x_2, \dots, x_k belong to a subalgebra of the algebra $(G; g)$ generated by less than m elements.

As a simple consequence of Theorems 1 and 2 we obtain the following results which are a generalization of a characterization theorem for diagonal algebras presented in [7].

THEOREM 3. *Let \mathfrak{A} be an idempotent algebra. The set $\mathcal{S}(\mathfrak{A})$ is finite if and only if \mathfrak{A} is a diagonal algebra.*

THEOREM 4. *Let \mathfrak{A} be an idempotent algebra. The set $\mathcal{S}(\mathfrak{A})$ and its complement are both infinite if and only if $\mathfrak{A} = (G; g)$, where G is an at least two-element Boolean group and $g(x, y, z) = x + y + z$.*

THEOREM 5. *Let \mathfrak{A} be an idempotent algebra. If $n \notin \mathcal{S}(\mathfrak{A})$ and all fundamental operations in \mathfrak{A} depend on at most $n-1$ variables, then either \mathfrak{A} is a diagonal algebra or $\mathfrak{A} = (G; g)$, where G is a Boolean group and $g(x, y, z) = x + y + z$.*

THEOREM 6. *Let \mathfrak{A} be an idempotent algebra with binary fundamental operations. If $\mathcal{S}(\mathfrak{A}) \neq \{2, 3, \dots\}$, then \mathfrak{A} is a diagonal algebra.*

The proof of Theorems 1 and 2 will be given in the next section. Before proving the Theorems we shall prove some Lemmas.

IV. Lemmas and proof of the Theorems. In this section we assume that considered algebras are idempotent.

LEMMA 1. *Let $m \geq 3$ and let $\mathfrak{A} = (A; f)$, where f is an m -ary operation in A satisfying the equation*

$$(2) \quad f(x_1, x_2, \dots, x_m) = x_1$$

whenever at least two elements among x_1, x_2, \dots, x_m are equal. If the operation f depends on every variable, then $\mathcal{S}(\mathfrak{A}) = \{s: s \geq m\}$.

Proof. From the assumption (2) it follows that the composition $f(e_{j_1}^{(m-1)}, e_{j_2}^{(m-1)}, \dots, e_{j_m}^{(m-1)})$ ($1 \leq j_k \leq m-1; k = 1, 2, \dots, m$) is the trivial

operation $e_{j_1}^{(m-1)}$. Thus the class of $(m-1)$ -ary trivial operations is closed under the composition with the fundamental operation f . Hence it follows that $A^{(m-1)}$ consists of trivial operations. Consequently, to prove the Lemma it suffices to prove that for each integer $k \geq m$ there exists an algebraic k -ary operation f_k depending on every variable. We shall prove slightly stronger statement by induction with respect to k . Namely, we shall prove that the operation f_k satisfies the additional condition

$$(3) \quad f_k(x, y, y, \dots, y) = x.$$

Set $f_m = f$. By assumption (2) the operation f_m satisfies equation (3) and depends on every variable. Given an algebraic k -ary operation f_k ($k \geq m$) depending on every variable and satisfying equation (3), we put

$$(4) \quad f_{k+1}(x_1, x_2, \dots, x_{k+1}) = f(f_k(x_1, x_2, \dots, x_k), x_{k+1}, x_2, x_3, \dots, x_{m-1}).$$

Of course, the operation f_{k+1} is algebraic and the equation

$$f_{k+1}(x, y, y, \dots, y) = f(f_k(x, y, y, \dots, y), y, y, \dots, y) = x$$

holds. It remains to prove that f_{k+1} depends on every variable. Taking into account the inequality $m-1 \geq 2$ and setting $x_{k+1} = x_2$ into (4) we get, in view of (2), the equation

$$f_{k+1}(x_1, x_2, \dots, x_k, x_2) = f_k(x_1, x_2, \dots, x_k),$$

which shows that the operation f_{k+1} depends on the variables $x_1, x_3, x_4, \dots, x_k$.

Since all $(m-1)$ -ary algebraic operations are trivial, the operation $f_k(x_1, x_2, \dots, x_{m-2}, x_k, x_k, \dots, x_k)$ is trivial too. Thus, by (3), we have the equation

$$(5) \quad f_k(x_1, x_2, \dots, x_{m-2}, x_k, x_k, \dots, x_k) = x_1.$$

Setting $x_j = x_k$ ($j = m-1, m, \dots, k$) into (4) we get, in view of (5), the equation

$$\begin{aligned} f_{k+1}(x_1, x_2, \dots, x_{m-2}, x_k, x_k, \dots, x_k, x_{k+1}) \\ = f(x_1, x_{k+1}, x_2, x_3, \dots, x_{m-2}, x_k), \end{aligned}$$

which, in view of the inequality $k \geq m$, shows that the operation f_{k+1} depends on the variables x_2 and x_{k+1} . Consequently, the operation f_{k+1} depends on every variable, which completes the proof of the Lemma.

LEMMA 2. Suppose that there exists a ternary algebraic operation f in the algebra \mathcal{A} depending on every variable and satisfying the condition

$$(6) \quad f(x, y, y) = x.$$

If $s \in \mathcal{S}(\mathcal{A})$, then $s+2 \in \mathcal{S}(\mathcal{A})$.

Proof. Suppose that $s \in \mathcal{S}(\mathcal{A})$. Let g be an s -ary algebraic operation depending on every variable. Put

$$h(x_1, x_2, \dots, x_{s+2}) = f(g(x_1, x_2, \dots, x_s), x_{s+1}, x_{s+2}).$$

Since

$$h(x, x, \dots, x, x_{s+1}, x_{s+2}) = f(g(x, x, \dots, x), x_{s+1}, x_{s+2}) = f(x, x_{s+1}, x_{s+2})$$

and, by (6),

$$h(x_1, x_2, \dots, x_s, y, y) = f(g(x_1, x_2, \dots, x_s), y, y) = g(x_1, x_2, \dots, x_s),$$

we infer that the $(s+2)$ -ary algebraic operation h depends on every variable. Thus, $s+2 \in \mathcal{S}(\mathcal{A})$.

LEMMA 3. Suppose that there exists a ternary algebraic operation g in the algebra \mathcal{A} depending on every variable and satisfying the condition

$$(7) \quad g(x, y, y) = y.$$

If $2 \notin \mathcal{S}(\mathcal{A})$, then $\mathcal{S}(\mathcal{A}) = \{s : s \geq 3\}$.

Proof. Since the only binary algebraic operations in \mathcal{A} are trivial ones, we have one of the following cases:

$$(8) \quad g(y, x, y) = y, \quad g(y, y, x) = x,$$

$$(9) \quad g(y, x, y) = x, \quad g(y, y, x) = y,$$

$$(10) \quad g(y, x, y) = y, \quad g(y, y, x) = y,$$

$$(11) \quad g(y, x, y) = x, \quad g(y, y, x) = x.$$

Setting

$$f(x, y, z) = \begin{cases} g(z, x, y) & \text{in the case (8),} \\ g(y, x, z) & \text{in the case (9),} \end{cases}$$

we get a ternary algebraic operation depending on every variable and satisfying the equation $f(x, y, z) = x$ whenever at least two elements among x, y, z are equal. Hence, by Lemma 1, we get the assertion of the Lemma in both cases (8) and (9).

Put

$$p^*(x, y, z) = \begin{cases} g(x, y, z) & \text{in the case (10),} \\ g(g(x, y, z), y, z) & \text{in the case (11).} \end{cases}$$

It is very easy to verify that the operation p^* satisfies the equations

$$(12) \quad p^*(x, x, y) = p^*(x, y, x) = p^*(y, x, x) = x.$$

The algebra in question is non-trivial. Consequently, it contains at least two elements. Denoting by 0 and 1 a pair of elements of the algebra \mathcal{A} we infer, in virtue of (12), that the set $\{0, 1\}$ is closed under the operation p^* . Moreover, the algebra $(0, 1; p^*)$ is the Post algebra \mathfrak{P}^* (see [6], p. 200). Thus, $\mathcal{S}(\mathfrak{P}^*) \subset \mathcal{S}(\mathcal{A})$. But $\mathcal{S}(\mathfrak{P}^*) = \{s : s \geq 3\}$ (see [6], p. 202), which completes the proof of the Lemma in the cases (10) and (11).

LEMMA 4. *If $2 \notin \mathcal{S}(\mathcal{A})$, $3 \in \mathcal{S}(\mathcal{A})$ and $4 \in \mathcal{S}(\mathcal{A})$, then $\mathcal{S}(\mathcal{A}) = \{s : s \geq 3\}$.*

Proof. Since $3 \in \mathcal{S}(\mathcal{A})$, there exists a ternary algebraic operation f depending on every variable. Further, taking into account the assumption $2 \notin \mathcal{S}(\mathcal{A})$, we have either

$$(13) \quad f(x, y, y) = y$$

or

$$(14) \quad f(x, y, y) = x.$$

In the case (13) our statement is a consequence of Lemma 3. In the case (14) from the relations $3 \in \mathcal{S}(\mathcal{A})$ and $4 \in \mathcal{S}(\mathcal{A})$ we obtain, by Lemma 2, the assertion of the Lemma.

LEMMA 5. *If $2 \notin \mathcal{S}(\mathcal{A})$, $3 \in \mathcal{S}(\mathcal{A})$ and $4 \notin \mathcal{S}(\mathcal{A})$, then $\mathcal{A} = (G; \{g\} \cup \mathbf{F})$, where G is an at least two-element Boolean group, $g(x, y, z) = x + y + z$ and all operations from \mathbf{F} depend on at least five variables. Moreover, g is the only algebraic ternary operation depending on every variable.*

Proof. Let f and g be ternary algebraic operations in the algebra \mathcal{A} depending on every variable. Since $2 \notin \mathcal{S}(\mathcal{A})$ and $4 \notin \mathcal{S}(\mathcal{A})$, we have, by Lemma 3, the equations

$$(15) \quad f(x, y, y) = f(y, x, y) = f(y, y, x) = x,$$

and

$$(16) \quad g(x, y, y) = g(y, x, y) = g(y, y, x) = x.$$

Put

$$(17) \quad h(x_1, x_2, x_3, x_4) = f(g(x_1, x_2, x_4), x_4, x_3).$$

Since, by (15),

$$(18) \quad h(x_1, x_2, x, x) = f(g(x_1, x_2, x), x, x) = g(x_1, x_2, x)$$

and, by (16),

$$(19) \quad h(x_1, y, x_3, y) = f(g(x_1, y, y), y, x_3) = f(x_1, y, x_3),$$

we infer that the operation h depends on the variables x_1, x_2 and x_3 . Therefore, by the assumption $4 \notin \mathcal{S}(\mathcal{A})$, it does not depend on the variable x_4 . Thus

$$h(x_1, x_2, x_3, x_4) = h(x_1, x_2, x_3, x_3) = h(x_1, x_2, x_3, x_2).$$

Consequently, by (17), (18) and (19), we have the equations

$$f(g(x_1, x_2, x_4), x_4, x_3) = g(x_1, x_2, x_3) = f(x_1, x_2, x_3),$$

which show that there is exactly one ternary algebraic operation g in the algebra \mathfrak{A} depending on every variable. Moreover, this operation is symmetric and fulfils the equation

$$(20) \quad g(g(x_1, x_2, x_4), x_4, x_3) = g(x_1, x_2, x_3).$$

Since $4 \notin \mathcal{S}(\mathfrak{A})$, the algebra \mathfrak{A} can be written in the form $\mathfrak{A} = (G; \{g\} \cup F)$, where all operations from F depends on at least five variables.

Let 0 be an element of G . Put

$$(21) \quad x + y = g(x, y, 0)$$

for all elements x and y from G . From the symmetry of the operation g the equation $x + y = y + x$ follows. Further, we have, according to (20),

$$(22) \quad (x + y) + z = g(g(x, y, 0), z, 0) = g(g(x, y, 0), 0, z) = g(x, y, z)$$

and

$$x + (y + z) = g(x, g(y, z, 0), 0) = g(g(y, z, 0), 0, x) = g(y, z, x) = g(x, y, z),$$

which proves the associativity law. Since, by (16), $x + 0 = g(x, 0, 0) = x$, the element 0 is the zero-element. Further, by (16), $x + x = g(x, x, 0) = 0$, which shows that the set G is a Boolean group under the addition (21). Since $3 \in \mathcal{S}(\mathfrak{A})$, the set G contains at least two elements. Finally, from (22) we get the formula $g(x, y, z) = x + y + z$, which completes the proof of the Lemma.

LEMMA 6. *Let G be a Boolean group and $g(x, y, z) = x + y + z$ ($x, y, z \in G$). Let f be an m -ary operation in G , where $m \geq 4$. If for each system i_1, i_2, \dots, i_m of indices satisfying the condition $1 \leq i_j \leq m-1$ ($j = 1, 2, \dots, m$) the operation $f(x_{i_1}, x_{i_2}, \dots, x_{i_m})$ is algebraic in the algebra $(G; g)$, then there exists an m -ary algebraic operation f_0 in $(G; g)$ such that*

$$f(x_1, x_2, \dots, x_m) = f_0(x_1, x_2, \dots, x_m)$$

whenever at least two variables among x_1, x_2, \dots, x_m are equal.

Proof. We note that each algebraic operation in the algebra $(G; g)$ is of the form

$$h(x_1, x_2, \dots, x_n) = x_{i_1} + x_{i_2} + \dots + x_{i_k},$$

where the indices i_1, i_2, \dots, i_k are all different, $1 \leq i_j \leq n$ ($j = 1, 2, \dots, k$) and k is an odd integer.

Setting $x_i = x_j$ ($i \neq j$) into $f(x_1, x_2, \dots, x_m)$ we obtain an algebraic operation h_{ij} which, of course, does not depend on the variable x_i . Consequently,

$$(23) \quad h_{ij}(x_1, x_2, \dots, x_m) = \sum_{s \in M(i,j)} x_s,$$

where $M(i, j)$ is a subset of the set $\{1, 2, \dots, m\}$ having an odd number of elements and satisfying the condition $i \notin M(i, j)$.

Set $u_s^{(s)} = x$ and $u_k^{(s)} = y$ if $k \neq s$ ($k = 1, 2, \dots, m$). For any index s , $f(u_1^{(s)}, u_2^{(s)}, \dots, u_m^{(s)})$ is equal either to x or to y . Let M be the set of all indices s for which the equation

$$f(u_1^{(s)}, u_2^{(s)}, \dots, u_m^{(s)}) = x$$

holds. It is very easy to see that for $s \neq i, j$ ($i \neq j$) the relations $s \in M(i, j)$ and $s \in M$ are equivalent. Consequently,

$$(24) \quad M(i, j) \setminus \{i\} = M \setminus (\{i\} \cup \{j\}) \quad (i \neq j; i, j = 1, 2, \dots, m).$$

Suppose that $i \neq j$. Since $m \geq 4$, we can find a pair of indices k, s ($k \neq s$) different from i and j . Put $v_i = v_j = x$ and $v_r = y$ if $r \neq i, j$ ($r = 1, 2, \dots, m$). Then the equation

$$(25) \quad h_{ij}(v_1, v_2, \dots, v_m) = h_{ks}(v_1, v_2, \dots, v_m)$$

is true. Moreover, by (23),

$$h_{ij}(v_1, v_2, \dots, v_m) = \begin{cases} x & \text{if } j \in M(i, j), \\ y & \text{if } j \notin M(i, j) \end{cases}$$

and

$$h_{ks}(v_1, v_2, \dots, v_m) = \begin{cases} y & \text{if both } i, j \in M(k, s) \text{ or both } i, j \notin M(k, s), \\ x & \text{in the opposite case.} \end{cases}$$

Hence and from equation (25) it follows that $j \in M(i, j)$ if and only if either $i \in M(k, s)$ and $j \notin M(k, s)$ or $i \notin M(k, s)$ and $j \in M(k, s)$. Now taking into account that all indices i, j, k, s are different we infer, in view of (24), that $i \in M(k, s)$ if and only if $i \in M$ and $j \in M(k, s)$ if and only if $j \in M$. Consequently, $j \in M(i, j)$ if and only if either $i \in M$ and $i \notin M$ or $i \notin M$ and $j \in M$. Hence and from (24) we obtain the equation

$$(26) \quad M(i, j) = \begin{cases} M \setminus (\{i\} \cup \{j\}) & \text{if } i, j \in M \text{ or } i, j \notin M, \\ (M \setminus (\{i\} \cup \{j\})) \cup \{j\} & \text{in the opposite case.} \end{cases}$$

In particular, from this equation it follows that the set M consists of an odd number of elements. Consequently, the operation

$$f_0(x_1, x_2, \dots, x_m) = \sum_{s \in M} x_s$$

is algebraic in the algebra $(G; g)$. Further, by (23) and (26), we have the equation $f_0(x_1, x_2, \dots, x_m) = h_{ij}(x_1, x_2, \dots, x_m)$ whenever $x_i = x_j$. Thus the operation f_0 satisfies the assertion of the Lemma.

LEMMA 7. *Let G be a Boolean group and $g(x, y, z) = x + y + z$ ($x, y, z \in G$). Let f be an m -ary operation on G and $m \geq 4$. If for every system i_1, i_2, \dots, i_m of indices less than m the operation $f(x_{i_1}, x_{i_2}, \dots, x_{i_m})$ is algebraic in the algebra $(G; g)$, then there exists an m -ary operation h such that $(G; g, f) = (G; g, h)$ and $h(x_1, x_2, \dots, x_m) = x_1$ whenever at least two variables among x_1, x_2, \dots, x_m are equal.*

Proof. By Lemma 6 there exists an m -ary algebraic operation f_0 in the algebra $(G; g)$ satisfying the equation

$$f(x_1, x_2, \dots, x_m) = f_0(x_1, x_2, \dots, x_m)$$

whenever at least two elements among x_1, x_2, \dots, x_m are identical. Without loss of generality we may assume that

$$(27) \quad f_0(x_1, x_2, \dots, x_m) = x_1 + x_2 + \dots + x_k,$$

where k is an odd integer satisfying the inequality $1 \leq k \leq m$. Setting

$$h(x_1, x_2, \dots, x_m) = f(x_1, x_2, \dots, x_m) + x_2 + x_3 + \dots + x_{k+2}$$

we have the equation

$$f(x_1, x_2, \dots, x_m) = h(x_1, x_2, \dots, x_m) + x_2 + x_3 + \dots + x_k$$

which shows that $(G; g, f) = (G; g, h)$. Moreover,

$$h(x_1, x_2, \dots, x_m) = f_0(x_1, x_2, \dots, x_m) + x_2 + x_3 + \dots + x_k$$

whenever at least two elements among x_1, x_2, \dots, x_m are equal. But, according to (27), the right-hand side of the last equation is equal to x_1 , which completes the proof.

LEMMA 8. *Let G be a Boolean group and $g(x, y, z) = x + y + z$ ($x, y, z \in G$). Let f be an m -ary operation in G and $m \geq 4$. If for every system h_1, h_2, \dots, h_m of $(m-1)$ -ary algebraic operations in the algebra $(G; g)$ the composition*

$$f(h_1(x_1, x_2, \dots, x_{m-1}), h_2(x_1, x_2, \dots, x_{m-1}), \dots, h_m(x_1, x_2, \dots, x_{m-1}))$$

is an algebraic operation in the algebra $(G; g)$ and $f(x_1, x_2, \dots, x_m) = x_1$ whenever at least two variables among x_1, x_2, \dots, x_m are identical, then $f(x_1, x_2, \dots, x_m) = x_1$ whenever x_1, x_2, \dots, x_m belong to a subalgebra of the algebra $(G; g)$ generated by less than m elements.

Proof. Given a system h_1, h_2, \dots, h_m of $(m-1)$ -ary algebraic operations in the algebra $(G; g)$ we put

$$(28) \quad \begin{aligned} & h_0(x_1, x_2, \dots, x_{m-1}) \\ &= f(h_1(x_1, x_2, \dots, x_{m-1}), h_2(x_1, x_2, \dots, x_{m-1}), \dots, h_m(x_1, x_2, \dots, x_{m-1})). \end{aligned}$$

Of course, the operation h_0 is algebraic in the algebra $(G; g)$. Since each algebraic operation in $(G; g)$ is a sum of an odd number of variables, we have the equations

$$(29) \quad h_j(x_1, x_2, \dots, x_{m-1}) = \sum_{s \in M_j} x_s \quad (j = 0, 1, \dots, m),$$

where M_0, M_1, \dots, M_m are subsets of the set $\{1, 2, \dots, m-1\}$ consisting of an odd number of elements.

Put $u_s^{(s)} = x$ and $u_k^{(s)} = y$ if $k \neq s$ ($k, s = 1, 2, \dots, m-1$). Since all binary algebraic operations in the algebra $(G; g)$ are trivial, we infer that, for every index s and every index j , $h_j(u_1^{(s)}, u_2^{(s)}, \dots, u_{m-1}^{(s)})$ is equal either to x or to y . Consequently, the system $h_1(u_1^{(s)}, u_2^{(s)}, \dots, u_{m-1}^{(s)}), h_2(u_1^{(s)}, u_2^{(s)}, \dots, u_{m-1}^{(s)}), \dots, h_m(u_1^{(s)}, u_2^{(s)}, \dots, u_{m-1}^{(s)})$ contains at least two identical elements. Thus, by the assumption, we have the equation

$$\begin{aligned} & h_1(u_1^{(s)}, u_2^{(s)}, \dots, u_{m-1}^{(s)}) \\ &= f(h_1(u_1^{(s)}, u_2^{(s)}, \dots, u_{m-1}^{(s)}), h_2(u_1^{(s)}, u_2^{(s)}, \dots, u_{m-1}^{(s)}), \dots, h_m(u_1^{(s)}, u_2^{(s)}, \dots, u_{m-1}^{(s)})). \end{aligned}$$

Hence and from (28) the equation

$$(30) \quad \begin{aligned} h_0(u_1^{(s)}, u_2^{(s)}, \dots, u_{m-1}^{(s)}) &= h_1(u_1^{(s)}, u_2^{(s)}, \dots, u_{m-1}^{(s)}) \\ &\quad (s = 1, 2, \dots, m-1) \end{aligned}$$

follows. Formula (29) implies the equivalence of the relation $s \in M_j$ and the equation $h_j(u_1^{(s)}, u_2^{(s)}, \dots, u_{m-1}^{(s)}) = x$. Thus, by (30), we have the equation $M_0 = M_1$, which, by (29), implies the equation $h_0(x_1, x_2, \dots, x_{m-1}) = h_1(x_1, x_2, \dots, x_{m-1})$. Now the assertion of the Lemma is a simple consequence of formula (28).

LEMMA 9. If $2 \notin \mathcal{S}(\mathfrak{A})$, $3 \in \mathcal{S}(\mathfrak{A})$ and $4 \notin \mathcal{S}(\mathfrak{A})$, then either $\mathcal{S}(\mathfrak{A})$ consists of all odd integers greater than 1 or $\mathcal{S}(\mathfrak{A})$ consists of all odd integers greater than 1 and all integers $\geq m$, where $m \geq 5$. Moreover, the set $\mathcal{S}(\mathfrak{A})$ consists of all odd integers greater than 1 if and only if $\mathfrak{A} = (G; g)$, where G is an at least two-element Boolean group and $g(x, y, z) = x + y + z$. The set $\mathcal{S}(\mathfrak{A})$ consists of all odd integers greater than 1 and all integers $\geq m$, where $m \geq 5$ if and only if $\mathfrak{A} = (G; \{g\} \cup \mathbf{F})$, where $(G; g)$ is the algebra

defined in the preceding assertion, \mathbf{F} contains an m -ary operation depending on every variable, all operations f from \mathbf{F} depend on at least m variables and satisfy the equation $f(x_1, x_2, \dots, x_k) = x_1$ whenever the elements x_1, x_2, \dots, x_k belong to a subalgebra of the algebra $(G; g)$ generated by less than m elements.

Proof. Suppose that $2 \notin \mathcal{S}(\mathfrak{A})$, $3 \in \mathcal{S}(\mathfrak{A})$ and $4 \notin \mathcal{S}(\mathfrak{A})$. By Lemma 5 $\mathfrak{A} = (G; \{g\} \cup \mathbf{F}_0)$, where G is an at least two-element Boolean group, $g(x, y, z) = x + y + z$ and all operations from \mathbf{F}_0 depend on at least five variables. Moreover, g is the only ternary algebraic operation in \mathfrak{A} . If all operations from \mathbf{F}_0 are algebraic in the algebra $(G; g)$, then, of course, $\mathfrak{A} = (G; g)$ and, consequently, $\mathcal{S}(\mathfrak{A})$ consists of all odd integers greater than 1.

Consider the remaining case and denote by m the smallest integer for which there exists an m -ary algebraic operation in \mathfrak{A} depending on every variable and which is not algebraic in the algebra $(G; g)$. Obviously, $m \geq 5$ and, by Lemmas 6, 7 and 8, $\mathfrak{A} = (G; \{g\} \cup \mathbf{F})$, where \mathbf{F} contains an m -ary operation depending on every variable, all operations f from \mathbf{F} depend on at least m variables and satisfy the equation $f(x_1, x_2, \dots, x_k) = x_1$ whenever the elements x_1, x_2, \dots, x_k belong to a subalgebra of the algebra $(G; g)$ generated by less than m elements. To prove the Lemma it suffices to prove that for each such algebra $(G; \{g\} \cup \mathbf{F})$ the set $\mathcal{S}(\mathfrak{A})$ consists of all odd integers greater than 1 and all integers $\geq m$.

The equation

$$(31) \quad \mathcal{S}(\mathfrak{A}) \cap \{2, 3, \dots, m-1\} = \mathcal{S}((G; g)) \cap \{2, 3, \dots, m-1\}$$

is obvious. Consequently, to prove our statement it suffices to prove that $\mathcal{S}(\mathfrak{A})$ contains all integers $\geq m$. But this is a simple consequence of the inclusion $\mathcal{S}(\mathfrak{A}) \supset \mathcal{S}(G; \mathbf{F})$ and Lemma 1. The Lemma is thus proved.

LEMMA 10. *Let $r \geq 2$, $r \in \mathcal{S}(\mathfrak{A})$ and $r+1 \notin \mathcal{S}(\mathfrak{A})$. Let q be an r -ary algebraic operation in the algebra \mathfrak{A} depending on every variable. For each binary algebraic operation f there exists a subset Q_f of the set $\{1, 2, \dots, r\}$ such that*

$$(32) \quad q(f(x_1, y_1), f(x_2, y_2), \dots, f(x_r, y_r)) = q(u_1, u_2, \dots, u_r),$$

where

$$(33) \quad u_j = \begin{cases} x_j & \text{if } j \in Q_f, \\ y_j & \text{if } j \notin Q_f. \end{cases}$$

The correspondence between binary operations f and subsets Q_f is one-to-one. Moreover, for each pair f, g of binary algebraic operations we have the equation

$$(34) \quad q(f(g(x_1, y_1), z_1), f(g(x_2, y_2), z_2), \dots, f(g(x_r, y_r), z_r)) = q(v_1, v_2, \dots, v_r)$$

where

$$(35) \quad v_j = \begin{cases} x_j & \text{if } j \in Q_f \cap Q_g, \\ y_j & \text{if } j \in Q_f \setminus Q_g, \\ z_j & \text{if } j \notin Q_f. \end{cases}$$

Proof. Given a binary operation f we put

$$(36) \quad h_f(x_1, x_2, \dots, x_r, y_1, y_2, \dots, y_r) = q(f(x_1, y_1), f(x_2, y_2), \dots, f(x_r, y_r)).$$

From the equation

$$h_f(x_1, x_2, \dots, x_r, x_1, x_2, \dots, x_r) = q(x_1, x_2, \dots, x_r)$$

it follows that for each index j ($j = 1, 2, \dots, r$) the operation h_f depends on at least one variable x_j or y_j . If the operation h_f would depend on a pair x_j, y_j simultaneously, then the $(r+1)$ -ary operation

$$h(x_1, x_2, \dots, x_r, y_j) = q(x_1, x_2, \dots, x_{j-1}, f(x_j, y_j), x_{j+1}, \dots, x_r)$$

would depend on every variable. But this contradicts the assumption $r+1 \notin \mathcal{S}(\mathcal{A})$. Thus for each index j ($j = 1, 2, \dots, r$) the operation h_f depends on exactly one variable x_j or y_j . Let Q_f be the set of all indices j for which $h_f(x_1, x_2, \dots, x_r, y_1, y_2, \dots, y_r)$ depends on the variable x_j . If the elements u_1, u_2, \dots, u_r are defined by formula (33), then we have the equation

$$h_f(x_1, x_2, \dots, x_r, y_1, y_2, \dots, y_r) = h_f(u_1, u_2, \dots, u_r, u_1, u_2, \dots, u_r),$$

which, in view of (36), implies equation (32).

It is obvious that the set Q_f uniquely determines the operation h_f . Since, by (36),

$$h_f(x, x, \dots, x, y, y, \dots, y) = q(f(x, y), f(x, y), \dots, f(x, y)) = f(x, y),$$

we infer that the correspondence between binary operations f and the sets Q_f is one-to-one.

Further, by (32) and (33), we have the equation

$$(37) \quad q(f(g(x_1, y_1), z_1), f(g(x_2, y_2), z_2), \dots, f(g(x_r, y_r), z_r)) \\ = q(w_1, w_2, \dots, w_r),$$

where

$$w_j = \begin{cases} g(x_j, y_j) & \text{if } j \in Q_f, \\ z_j & \text{if } j \notin Q_f. \end{cases}$$

On the other hand, according to (36), we have the equation

$$(38) \quad q(w_1, w_2, \dots, w_r) = h_g(a_1, a_2, \dots, a_r, b_1, b_2, \dots, b_r),$$

where

$$(39) \quad a_j = \begin{cases} x_j & \text{if } j \in Q_f, \\ z_j & \text{if } j \notin Q_f, \end{cases} \quad b_j = \begin{cases} y_j & \text{if } j \in Q_f, \\ z_j & \text{if } j \notin Q_f. \end{cases}$$

Applying once more formula (32) we get the equation

$$(40) \quad h_g(a_1, a_2, \dots, a_r, b_1, b_2, \dots, b_r) = q(v_1, v_2, \dots, v_r),$$

where

$$v_j = \begin{cases} a_j & \text{if } j \in Q_g, \\ b_j & \text{if } j \notin Q_g, \end{cases}$$

Hence and from (39) it follows that the elements v_1, v_2, \dots, v_r satisfy condition (35). Equation (34) is now a simple consequence of equations (37), (38) and (40). The Lemma is thus proved.

LEMMA 11. *Let $2 \in \mathcal{S}(\mathfrak{A})$ and $\mathcal{S}(\mathfrak{A}) \neq \{2, 3, \dots\}$. The set of all binary algebraic operations in \mathfrak{A} is a finite Boolean algebra under the operations*

$$(41) \quad (f')(x, y) = f(y, x),$$

$$(42) \quad (f \cap g)(x, y) = f(g(x, y), y),$$

$$(43) \quad (f \cup g)(x, y) = f(x, g(x, y)).$$

The unit element 1 and the neutral element 0 are trivial operations

$$1(x, y) = x, \quad 0(x, y) = y.$$

Further, for all binary operations f in the algebra \mathfrak{A} the equation

$$(44) \quad f(f(x, y), f(z, u)) = f(x, u)$$

holds. Moreover, if $f \cap g = 0$, then

$$(45) \quad f(g(x, y), z) = f(y, z)$$

and

$$(46) \quad f(x, g(y, z)) = g(y, f(x, z)).$$

Proof. By the definitions (41), (42) and (43) we have the equation

$$(47) \quad \begin{aligned} (f \cap g)'(x, y) &= [f(g(x, y), y)]' \\ &= f(g(y, x), x) = f'(x, g'(x, y)) = (f' \cup g')(x, y). \end{aligned}$$

Since $\mathcal{S}(\mathfrak{A}) \neq \{2, 3, \dots\}$ and $2 \in \mathcal{S}(\mathfrak{A})$, there exists an integer $r \geq 2$ such that $r \in \mathcal{S}(\mathfrak{A})$ and $r+1 \notin \mathcal{S}(\mathfrak{A})$. Let q be an r -ary algebraic operation depending on every variable. By Lemma 10 the operation q induces a one-to-one correspondence between binary algebraic operations f and

subsets Q_f of the set $\{1, 2, \dots, r\}$ such that equations (32) and (34) hold. To prove that the set of all binary algebraic operations in the algebra \mathfrak{A} is a Boolean algebra it suffices, by (47), to prove the formulas

$$(48) \quad Q_{f'} = Q'_f,$$

$$(49) \quad Q_{f \cap g} = Q_f \cap Q_g,$$

where $Q'_f = \{1, 2, \dots, r\} \setminus Q_f$. Indeed, these formulas show that the correspondence between binary operations f and sets Q_f preserves Boolean operations and, consequently, is a Boolean isomorphism.

Taking into account (41) we have the equation

$$\begin{aligned} q(f(x_1, y_1), f(x_2, y_2), \dots, f(x_r, y_r)) \\ = q(f'(y_1, x_1), f'(y_2, x_2), \dots, f'(y_r, x_r)). \end{aligned}$$

By Lemma 10 the left-hand side of the last equation is equal to $q(u_1, u_2, \dots, u_r)$, where the quantities u_1, u_2, \dots, u_r are defined by formula (33) and the right-hand side is equal to $q(t_1, t_2, \dots, t_r)$, where $t_j = y_j$ if $j \in Q_{f'}$ and $t_j = x_j$ in the opposite case. Hence we get formula (48).

Further, from (32), (33) and (42) we get the equation

$$q(f(g(x_1, y_1), y_1), f(g(x_2, y_2), y_2), \dots, f(g(x_r, y_r), y_r)) = q(a_1, a_2, \dots, a_r),$$

where $a_j = x_j$ if $j \in Q_{f \cap g}$ and $a_j = y_j$ in the opposite case. On the other hand, by formulas (34) and (35), we have the equation

$$q(f(g(x_1, y_1), y_1), f(g(x_2, y_2), y_2), \dots, f(g(x_r, y_r), y_r)) = q(b_1, b_2, \dots, b_r),$$

where $b_j = x_j$ if $j \in Q_f \cap Q_g$ and $b_j = y_j$ in the opposite case. Hence we get formula (49), which completes the proof that the set of all binary algebraic operations is a Boolean algebra under the operations (41), (42) and (43). The proof that the operations $1(x, y)$ and $0(x, y)$ are a unit element and a neutral element respectively is obvious.

Now we proceed to the proof of formula (44). From (34) and (35) it follows that

$$q(f(f(x_1, y_1), z_1), f(f(x_2, y_2), z_2), \dots, f(f(x_r, y_r), z_r)) = q(v_1, v_2, \dots, v_r),$$

where $v_j = x_j$ if $j \in Q_f$ and $v_j = z_j$ in the opposite case. Hence and from (32) and (33) we get the equation

$$\begin{aligned} q(f(f(x_1, y_1), z_1), f(f(x_2, y_2), z_2), \dots, f(f(x_r, y_r), z_r)) \\ = q(f(x_1, z_1), f(x_2, z_2), \dots, f(x_r, z_r)). \end{aligned}$$

Setting $x_j = x$, $y_j = y$, $z_j = z$ ($j = 1, 2, \dots, r$) into the last equation and taking into account the formula $q(x, x, \dots, x) = x$, we get the equation

$$f(f(x, y), z) = f(x, z)$$

for all binary algebraic operations f . Thus, by (41),

$$f(f(x, y), f(z, u)) = f(x, f(z, u)) = f'(f'(u, z), x) = f'(u, x) = f(x, u).$$

Formula (44) is thus proved.

Suppose that $f \cap g = 0$. Consequently, the sets Q_f and Q_g are disjoint. Thus, by (34) and (35),

$$q(f(g(x_1, y_1), z_1), f(g(x_2, y_2), z_2), \dots, f(g(x_r, y_r), z_r)) = q(v_1, v_2, \dots, v_r),$$

where $v_j = y_j$ if $j \in Q_f$ and $v_j = z_j$ in the opposite case. Hence and from (32) and (33) we obtain the equation

$$\begin{aligned} q(f(g(x_1, y_1), z_1), f(g(x_2, y_2), z_2), \dots, f(g(x_r, y_r), z_r)) \\ = q(f(y_1, z_1), f(y_2, z_2), \dots, f(y_r, z_r)). \end{aligned}$$

Setting $x_j = x$, $y_j = y$ and $z_j = z$ ($j = 1, 2, \dots, r$) into the last equation we obtain formula (45).

Further, we have, according to (34) and (35), the equations

$$(50) \quad q(f'(g(y_1, z_1), x_1), f'(g(y_2, z_2), x_2), \dots, f'(g(y_r, z_r), x_r)) \\ = q(w_1, w_2, \dots, w_r),$$

where

$$(51) \quad w_j = \begin{cases} x_j & \text{if } j \notin Q_f \\ y_j & \text{if } j \in Q_f \cap Q_g, \\ z_j & \text{if } j \in Q_f \setminus Q_g, \end{cases}$$

and

$$(52) \quad q(g'(f(x_1, z_1), y_1), g'(f(x_2, z_2), y_2), \dots, g'(f(x_r, z_r), y_r)) \\ = q(t_1, t_2, \dots, t_r),$$

where

$$(53) \quad t_j = \begin{cases} x_j & \text{if } j \in Q_{g'} \cap Q_f, \\ y_j & \text{if } j \notin Q_{g'}, \\ z_j & \text{if } j \in Q_{g'} \setminus Q_f. \end{cases}$$

We assumed that the sets Q_f and Q_g are disjoint. Consequently,

$$Q_f = Q_{g'} \cap Q_f, \quad Q_f \cap Q_g = Q_g \quad \text{and} \quad Q_f \setminus Q_g = Q_{g'} \setminus Q_f.$$

Hence and from (51) and (53), in view of formula (48), the equation $w_j = t_j$ ($j = 1, 2, \dots, r$) follows. Consequently, from (41), (50) and (52) we get the equation

$$\begin{aligned} & q(f(x_1, g(y_1, z_1)), f(x_2, g(y_2, z_2)), \dots, f(x_r, g(y_r, z_r))) \\ &= q(g(y_1, f(x_1, z_1)), g(y_2, f(x_2, z_2)), \dots, g(y_r, f(x_r, z_r))). \end{aligned}$$

Now, setting $x_j = x$, $y_j = y$ and $z_j = z$ ($j = 1, 2, \dots, r$) into the last equation we obtain formula (46). The Lemma is thus proved.

COROLLARY. *Let $2 \in \mathcal{S}(\mathfrak{A})$ and $\mathcal{S}(\mathfrak{A}) \neq \{2, 3, \dots\}$. If f is a binary algebraic operation in \mathfrak{A} and g an n -ary not necessarily algebraic operation in \mathfrak{A} and*

$$h(x_1, x_2, \dots, x_n) = f(g(x_2, x_1, \dots, x_n), g(x_1, x_2, \dots, x_n)),$$

then

$$g(x_1, x_2, \dots, x_n) = f(h(x_2, x_1, \dots, x_n), h(x_1, x_2, \dots, x_n)).$$

In particular, the operations g and h are either both algebraic or both non-algebraic.

Indeed, from formula (44) we get the equation

$$\begin{aligned} & f(h(x_2, x_1, \dots, x_n), h(x_1, x_2, \dots, x_n)) \\ &= f(f(g(x_1, x_2, \dots, x_n), g(x_2, x_1, \dots, x_n)), f(g(x_2, x_1, \dots, x_n), g(x_1, x_2, \dots, x_n))) \\ &= f(g(x_1, x_2, \dots, x_n), g(x_1, x_2, \dots, x_n)) = g(x_1, x_2, \dots, x_n). \end{aligned}$$

LEMMA 12. *Let $2 \in \mathcal{S}(\mathfrak{A})$ and $\mathcal{S}(\mathfrak{A}) \neq \{2, 3, \dots\}$. Then each ternary algebraic operation f satisfies the equation*

$$f(f(x, y, z), x, x) = x.$$

Proof. Put

$$h(x, y, z) = f(f(x, y, z), x, x) \quad \text{and} \quad g(x, y) = f(x, y, y).$$

From Lemma 11 (formula (44)) we get the equation

$$(54) \quad h(x, y, y) = g(g(x, y), x) = g(x, x) = x.$$

Suppose that the operation h does not depend on every variable. Then formula (54) implies the equation $h(x, y, z) = x$, which gives the assertion of the Lemma.

If the operation h depends on every variable, then, of course, $3 \in \mathcal{S}(\mathfrak{A})$. Since, in addition, $2 \in \mathcal{S}(\mathfrak{A})$, we infer, in view of (54) and Lemma 2, that $\mathcal{S}(\mathfrak{A}) = \{2, 3, \dots\}$, which contradicts the assumption $\mathcal{S}(\mathfrak{A}) \neq \{2, 3, \dots\}$. The Lemma is thus proved.

LEMMA 13. *Let $2 \in \mathcal{S}(\mathfrak{A})$ and $\mathcal{S}(\mathfrak{A}) \neq \{2, 3, \dots\}$. Then each ternary algebraic operation in \mathfrak{A} is generated by binary algebraic operation in \mathfrak{A} , i. e. is a composition of binary algebraic operations.*

Proof. Contrary to this let us suppose that there exists a ternary algebraic operation g which is not generated by binary algebraic operations. Put

$$(55) \quad f_1(x, y) = g(x, y, y)$$

and

$$(56) \quad h_1(x, y, z) = f_1(g(y, x, z), g(x, y, z)).$$

By virtue of the Corollary to Lemma 11, the operation h_1 is not generated by binary algebraic operations. Moreover, according to (44), (55) and Lemma 12, we have the equation

$$(57) \quad \begin{aligned} h_1(x, y, y) &= f_1(g(y, x, y), f_1(x, y)) = f_1(g(y, x, y), y) \\ &= g(g(y, x, y), y, y) = y. \end{aligned}$$

Set

$$(58) \quad f_2(x, y) = g(y, y, x)$$

and

$$(59) \quad h_2(x, y, z) = f_2(h_1(z, x, y), h_1(z, y, x)).$$

By Corollary to Lemma 11, the operation h_2 is not generated by binary algebraic operations. Further, from (57) and (59) it follows that

$$(60) \quad h_2(y, y, x) = f_2(h_1(x, y, y), h_1(x, y, y)) = h_1(x, y, y) = y.$$

Let us introduce an auxiliary operation $h_0(x, y, z) = h_1(z, y, x)$. Since, by (56) and (58),

$$(61) \quad \begin{aligned} h_0(x, y, y) &= h_1(y, y, x) = f_1(g(y, y, x), g(y, y, x)) = g(y, y, x) \\ &= f_2(x, y), \end{aligned}$$

we have, by Lemma 12, the equation

$$f_2(h_1(y, x, y), y) = f_2(h_0(y, x, y), y) = h_0(h_0(y, x, y), y, y) = y.$$

Hence and from (44), (59) and (61) we get the equation

$$(62) \quad \begin{aligned} h_2(x, y, y) &= f_2(h_1(y, x, y), h_1(y, y, x)) = f_2(h_1(y, x, y), f_2(x, y)) \\ &= f_2(h_1(y, x, y), y) = y. \end{aligned}$$

If the operation $h_2(y, x, y)$ does not depend on both variables x and y and, consequently, is a trivial operation, then, by (60) and (62), the algebra $(A; h_2)$ contains no binary operation depending on every variable. Thus, by Lemma 3, $\mathcal{S}((A; h_2)) = \{s : s \geq 3\}$ and, consequently, $\mathcal{S}(\mathfrak{A}) = \{2, 3, \dots\}$, which contradicts the assumption $\mathcal{S}(\mathfrak{A}) \neq \{2, 3, \dots\}$. Therefore the operation $h_2(y, x, y)$ depends on both variables x and y . Put

$$(63) \quad f(x, y) = h_2(y, x, y).$$

Since $\mathcal{S}(\mathfrak{A}) \neq \{2, 3, \dots\}$, there exists an integer $r \geq 2$ such that $r \in \mathcal{S}(\mathfrak{A})$ and $r+1 \notin \mathcal{S}(\mathfrak{A})$. Let q be an r -ary algebraic operation depending on every variable. By Lemma 10 there exists a subset Q_f of the set $\{1, 2, \dots, r\}$ such that formula (32) holds. Since the operation f is non-trivial, we have the inequality $Q_f \neq \{1, 2, \dots, r\}$. Without loss of generality we may assume that $1 \notin Q_f$. From formula (32) we get the equation

$$(64) \quad q(f(x_1, y), x_2, \dots, x_r) = q(y, x_2, \dots, x_r).$$

We define an $(r+1)$ -ary algebraic operation by means of the formula

$$(65) \quad h(x_1, x_2, \dots, x_{r+1}) = q(h_2(x_1, x_{r+1}, x_2), x_2, \dots, x_r).$$

Setting $x_1 = x_{r+1} = y$ into (65) we obtain, in view of (60), the equation

$$h(y, x_2, \dots, x_r, y) = q(h_2(y, y, x_2), x_2, \dots, x_r) = q(y, x_2, \dots, x_r),$$

which shows that the operation h depends on the variables x_2, x_3, \dots, x_r and on at least one of the variables x_1, x_{r+1} .

Suppose that the operation h depends on the variable x_{r+1} and does not depend on the variable x_1 . Then, by (63) and (64), we have the equation

$$\begin{aligned} h(x_1, x_2, \dots, x_{r+1}) &= h(x_2, x_2, \dots, x_{r+1}) = q(h_2(x_2, x_{r+1}, x_2), x_2, \dots, x_r) \\ &= q(f_2(x_{r+1}, x_2), x_2, \dots, x_r) = q(x_2, x_2, \dots, x_r), \end{aligned}$$

which shows that the operation h does not depend on the variable x_{r+1} . But this contradicts the assumption.

Further, suppose that the operation h depends on the variable x_1 and does not depend on the variable x_{r+1} . In this case, according to (62), we have the equation

$$\begin{aligned} h(x_1, x_2, \dots, x_{r+1}) &= h(x_1, x_2, \dots, x_r, x_2) \\ &= q(h_2(x_1, x_2, x_2), x_2, \dots, x_r) = q(x_2, x_2, \dots, x_r), \end{aligned}$$

which shows that the operation h does not depend on the variable x_1 . But this gives a contradiction. Consequently, the $(r+1)$ -ary operation h depends on every variable, which contradicts the assumption $r+1 \notin \mathcal{S}(\mathfrak{A})$. The Lemma is thus proved.

LEMMA 14. *If $2 \in \mathcal{S}(\mathfrak{A})$ and $\mathcal{S}(\mathfrak{A}) \neq \{2, 3, \dots\}$, then $(A; A^{(2)})$ is a diagonal algebra.*

Proof. By Lemma 11 the class $A^{(2)}$ of binary algebraic operations is a finite Boolean algebra under the operations (41), (42) and (43). Denoting by d_1, d_2, \dots, d_n the atoms of this Boolean algebra we have the obvious equation $(A; A^{(2)}) = (A; d_1, d_2, \dots, d_n)$.

Each operation d_j ($j = 1, 2, \dots, n$) induces a congruence relation in the set A as follows: $a \sim_j b$ if and only if $d_j(a, x) = d_j(b, x)$ for all $x \in A$. If $a \sim_j b$ for all $j = 1, 2, \dots, n$, then $a = b$. Indeed, by the definition of the unit element,

$$\begin{aligned} a = 1(a, x) &= \left(\bigcup_{j=1}^n d_j \right)(a, x) = d_1(a, d_2(a, \dots, d_n(a, x)) \dots) \\ &= d_1(b, d_2(b, \dots, d_n(b, x)) \dots) = \left(\bigcup_{j=1}^n d_j \right)(b, x) = 1(b, x) = b. \end{aligned}$$

Now we shall prove that for every system a_1, a_2, \dots, a_n of elements of A there exists an element c of A such that $c \sim_j a_j$ ($j = 1, 2, \dots, n$). Put

$$c = d_1(a_1, d_2(a_2, \dots, d_{n-1}(a_{n-1}, d_n(a_n, a_n)) \dots)).$$

Since the atoms d_1, d_2, \dots, d_n are disjoint, we infer, by formula (46) of Lemma 11, that for each index j ($j = 1, 2, \dots, n$) there exists an element $b_j \in A$ such that $c = d_j(a_j, b_j)$. Hence and from (44) for all $x \in A$ the equation

$$d_j(c, x) = d_j(d_j(a_j, b_j), x) = d_j(a_j, x)$$

follows. Consequently, $c \sim_j a_j$ ($j = 1, 2, \dots, n$).

Taking into account these properties of the congruence relations \sim_j , we infer, in view of the factorization theorem (see [1], Theorem 4, Chapter VI, § 2), that the set A is a Cartesian product $A = A_1 \times A_2 \times \dots \times A_n$. Moreover, denoting by a_k and b_k the elements of A_k ($k = 1, 2, \dots, n$) and setting

$$a = \langle a_1, a_2, \dots, a_n \rangle, \quad b = \langle b_1, b_2, \dots, b_n \rangle,$$

we have the relation $a \sim_j b$ if and only if $a_j = b_j$. Since, by (44) and (45),

$$d_k(d_j(a, b), x) = \begin{cases} d_j(a, x) & \text{if } k = j, \\ d_k(b, x) & \text{if } k \neq j, \end{cases}$$

we have the relations $d_j(a, b) \underset{j}{\sim} a$ and $d_j(a, b) \underset{k}{\sim} b$ if $k \neq j$. Hence it follows that

$$d_j(\langle a_1, a_2, \dots, a_n \rangle, \langle b_1, b_2, \dots, b_n \rangle) = \langle b_1, b_2, \dots, b_{j-1}, a_j, b_{j+1}, \dots, b_n \rangle.$$

In other words, $(A; d_1, d_2, \dots, d_n)$ is a diagonal algebra. The Lemma is thus proved.

LEMMA 15. *Let \mathfrak{A} be a diagonal algebra, $m \geq 4$ and f an m -ary operation in \mathfrak{A} not necessarily algebraic. If for each system i_1, i_2, \dots, i_m of indices less than m the operation $f(x_{i_1}, x_{i_2}, \dots, x_{i_m})$ is algebraic in the algebra \mathfrak{A} , then there exists an m -ary algebraic operation g in \mathfrak{A} such that*

$$f(x_1, x_2, \dots, x_m) = g(x_1, x_2, \dots, x_m)$$

whenever at least two variables among x_1, x_2, \dots, x_m are equal.

Proof. Let $\mathfrak{A} = (A_1 \times A_2 \times \dots \times A_n; d_1, d_2, \dots, d_n)$, where the fundamental binary operations d_1, d_2, \dots, d_n are defined, by formula (1). It is very easy to see that each algebraic operation is of the form

$$d_1(x_{k_1}, d_2(x_{k_2}, \dots, d_n(x_{k_n}, x_{k_n}))) \dots),$$

where the system k_1, k_2, \dots, k_n of indices is uniquely determined.

Suppose that the m -ary operation f satisfies the assumptions of the Lemma. Substituting x_i by x_j ($i \neq j; i, j = 1, 2, \dots, m$) in $f(x_1, x_2, \dots, x_m)$ we obtain an algebraic operation $f_{ij}(x_1, x_2, \dots, x_m)$ which, of course, does not depend on the variable x_i . The operation f_{ij} can be written in the form

$$(66) \quad f_{ij}(x_1, x_2, \dots, x_m) = d_1(x_{k_1(i,j)}, d_2(x_{k_2(i,j)}, \dots, d_n(x_{k_n(i,j)}, x_{k_n(i,j)}))) \dots),$$

where $1 \leq k_s(i, j) \leq m$, $k_s(i, j) \neq i$ ($s = 1, 2, \dots, n$).

First we assume that for all pairs i, j ($i \neq j$) the equation $k_1(i, j) = j$ holds. Taking into account the inequality $m \geq 4$, we get, in view of (66) the equations

$$(67) \quad f(x_2, x_2, x_3, x_4, \dots, x_m) = d_1(x_2, d_2(x_{k_2(1,2)}, \dots, d_n(x_{k_n(1,2)}, x_{k_n(1,2)}))) \dots),$$

$$(68) \quad f(x_1, x_2, x_4, x_4, \dots, x_m) = d_1(x_4, d_2(x_{k_2(3,4)}, \dots, d_n(x_{k_n(3,4)}, x_{k_n(3,4)}))) \dots).$$

Setting $x_3 = x_4$ into (67) and $x_1 = x_2$ into (68) we get identical expressions. But the right-hand side of (67) is then equal to $d_1(x_2, h)$, where h is an algebraic operation. Similarly, the right-hand side of (68) is equal to $d_1(x_4, g)$, where g is an algebraic operation. But, by virtue of (1), the equation $d_1(x_2, h) = d_1(x_4, g)$ never holds for $x_2 \neq x_4$. Consequently, there exists a pair i, j ($i \neq j$) for which the inequality $k_1(i, j) \neq j$ holds.

Without loss of generality we may assume that $k_1(1, 2) = 3$. Setting $x_i = x_j$ ($i \neq j$) into f_{12} and $x_1 = x_2$ into f_{ij} we obtain the same expressions. But, by (66), the operation f_{12} is then of the form $d_1(x_3, h)$ if $i \neq 3$ and of the form $d_1(x_j, h)$ if $i = 3$, where h is an algebraic operation. Similarly, the operation f_{ij} is of the form $d_1(x_{k_1(i,j)}, g)$ if $k_1(i, j) \neq 1$ and of the form $d_1(x_2, g)$ if $k_1(i, j) = 1$. Hence we get the equalities $k_1(3, j) = j$ and $k_1(i, j) = 3$ if $i \neq 3$. Since the system d_1, d_2, \dots, d_n of fundamental operations can be arbitrarily indexed, the iteration of the previous reasoning leads to the existence of a system of indices r_1, r_2, \dots, r_n such that $1 \leq r_s \leq m$ ($s = 1, 2, \dots, n$), $k_s(r_s, j) = j$ ($s = 1, 2, \dots, n$; $j = 1, 2, \dots, m$; $j \neq r_s$) and $k_s(i, j) = r_s$ ($s = 1, 2, \dots, n$; $i \neq j$, $i \neq r_s$; $j = 1, 2, \dots, m$). Put

$$g(x_1, x_2, \dots, x_m) = d_1(x_{r_1}, d_2(x_{r_2}, \dots, d_n(x_{r_n}, x_{r_n})) \dots).$$

The operation g is algebraic and, moreover, by (66),

$$g(x_1, x_2, \dots, x_m) = f_{ij}(x_1, x_2, \dots, x_m)$$

whenever $x_i = x_j$ ($i \neq j$). The Lemma is thus proved.

LEMMA 16. Let $(A; d)$ be a diagonal algebra, $m \geq 4$ and f an m -ary operation in A not necessarily algebraic. If for each system i_1, i_2, \dots, i_m of indices less than m the operation $f(x_{i_1}, x_{i_2}, \dots, x_{i_m})$ is algebraic in the algebra $(A; d)$, then there exists an m -ary operation h for which $h(x_1, x_2, \dots, x_m) = x_1$ whenever at least two variables among x_1, x_2, \dots, x_m are equal and $(A; d, f) = (A; d, h)$.

Proof. We use the notation introduced in the proof of the previous Lemma. By Lemma 15 there exists an m -ary algebraic operation g such that $f(x_1, x_2, \dots, x_m) = g(x_1, x_2, \dots, x_m)$ whenever at least two variables among x_1, x_2, \dots, x_m are identical. Without loss of generality we may assume that

$$g(x_1, x_2, \dots, x_m) = g_0(x_1, x_2, \dots, x_k),$$

where $1 \leq k \leq m$ and the operation g_0 depends on every variable.

If $k = 1$, then setting $h = f$ we get the assertion of the Lemma. Now we shall prove that the case $k > 1$ can be reduced to the case $k = 1$.

Suppose that $k > 1$. The fundamental operations (1) can be indexed in such a way that

$$(69) \quad g_0(x_1, x_2, \dots, x_k) = d_1(x_{r_1}, d_2(x_{r_2}, \dots, d_n(x_{r_n}, x_{r_n})) \dots),$$

where

$$(70) \quad r_1 = r_2 = \dots = r_p = k, \quad r_{p+1} = r_{p+2} = \dots = r_q = k-1, \\ r_j < k-1 \quad (j = q+1, q+2, \dots, n)$$

and $1 \leq p < q \leq n$.

Put

$$(71) \quad f_1(x, y) = d_1(x, d_2(x, \dots, d_p(x, y)) \dots),$$

$$(72) \quad g_1(x_1, x_2, \dots, x_k) = f_1(g_0(x_1, x_2, \dots, x_{k-2}, x_k, x_{k-1}), g_0(x_1, x_2, \dots, x_k)), \\ h_1(x_1, x_2, \dots, x_m) \\ = f_1(f(x_1, x_2, \dots, x_{k-2}, x_k, x_{k-1}, x_{k+1}, \dots, x_m), f(x_1, x_2, \dots, x_m)).$$

By Corollary to Lemma 11 we have the equation $(A; d, f) = (A; d, h_1)$. Moreover, $h_1(x_1, x_2, \dots, x_m) = g_1(x_1, x_2, \dots, x_k)$ whenever at least two variables among x_1, x_2, \dots, x_m are equal. By (69), (70), (71) and (72) we have the equation

$$g_1(x_1, x_2, \dots, x_k) \\ = f_1\left(d_1\left(x_{k-1}, d_2\left(x_{k-1}, \dots, d_p\left(x_{k-1}, d_{p+1}\left(x_k, \dots, d_n(x_{r_n}, x_{r_n})\right) \dots\right) \dots\right) \dots\right), \\ d_1\left(x_k, d_2\left(x_k, \dots, d_p\left(x_k, d_{p+1}\left(x_{k-1}, \dots, d_n(x_{r_n}, x_{r_n})\right) \dots\right) \dots\right) \dots\right) \\ = f_1\left(f_1\left(x_{k-1}, d_{p+1}\left(x_k, d_{p+2}\left(x_k, \dots, d_n(x_{r_n}, x_{r_n})\right) \dots\right) \dots\right), \right. \\ \left. f_1\left(x_k, d_{p+1}\left(x_{k-1}, d_{p+2}\left(x_{k-1}, \dots, d_n(x_{r_n}, x_{r_n})\right) \dots\right) \dots\right)\right).$$

Hence and from (44) we get the formula

$$g_1(x_1, x_2, \dots, x_k) = f_1\left(x_{k-1}, d_{p+1}\left(x_{k-1}, d_{p+2}\left(x_{k-1}, \dots, d_n(x_{r_n}, x_{r_n})\right) \dots\right) \dots\right),$$

which, according to (70), proves that the operation g_1 does not depend on the variable x_k . Setting $g_2(x_1, x_2, \dots, x_{k-1}) = g_1(x_1, x_2, \dots, x_k)$, we infer that $h_1(x_1, x_2, \dots, x_m) = g_2(x_1, x_2, \dots, x_{k-1})$ whenever at least two variables among x_1, x_2, \dots, x_m are equal. By a consecutive iteration of this procedure we can reduce our problem to the case $k = 1$, which completes the proof of the Lemma.

LEMMA 17. *Let f and g be a pair of m -ary algebraic operations in a diagonal algebra, where $m \geq 3$. If the equation $f(x_1, x_2, \dots, x_m) = g(x_1, x_2, \dots, x_m)$ holds whenever $x_1 = x_2$ or $x_1 = x_3$, then $f = g$.*

Proof. The operations f and g are compositions of fundamental operations d_1, d_2, \dots, d_n defined by formula (1). Moreover, the representation

$$(73) \quad f(x_1, x_2, \dots, x_m) = d_1(x_{r_1}, d_2(x_{r_2}, \dots, d_n(x_{r_n}, x_{r_n}) \dots)),$$

$$(74) \quad g(x_1, x_2, \dots, x_m) = d_1(x_{s_1}, d_2(x_{s_2}, \dots, d_n(x_{s_n}, x_{s_n}) \dots)),$$

where $1 \leq r_j \leq m$, $1 \leq s_j \leq m$ ($j = 1, 2, \dots, n$), is unique. Let f_k and g_k ($k = 1, 2, \dots, m$) be the union in the sense of (43) of those operations d_j for which $r_j = k$ and $s_j = k$ respectively. For instance, if there is no index j for which $r_j = k$, then $f_k(x, y) = 0(x, y) = y$. Of course, we have the relations

$$(75) \quad \bigcup_{k=1}^m f_k = \bigcup_{k=1}^m g_k = 1, \quad f_i \cap f_j = g_i \cap g_j = 0 \quad (i \neq j).$$

Moreover, by (73) and (74),

$$\begin{aligned} f(x_1, x_2, \dots, x_m) &= f_1(x_1, f_2(x_2, \dots, f_m(x_m, x_m)) \dots), \\ g(x_1, x_2, \dots, x_m) &= g_1(x_1, g_2(x_2, \dots, g_m(x_m, x_m)) \dots). \end{aligned}$$

Consequently, to prove the equation $f = g$ it suffices to prove the equations $f_k = g_k$ ($k = 1, 2, \dots, m$).

Put $u_s^{(s)} = x$ and $u_j^{(s)} = y$ if $j \neq s$ ($s = 1, 2, \dots, m$). Since $u_1^{(s)} = u_2^{(s)}$ if $s = 3, 4, \dots, m$ and $u_1^{(2)} = u_3^{(2)}$, we have, by the hypothesis, the equations

$$(76) \quad f(u_1^{(s)}, u_2^{(s)}, \dots, u_m^{(s)}) = g(u_1^{(s)}, u_2^{(s)}, \dots, u_m^{(s)}) \quad (s = 2, 3, \dots, m).$$

Applying Lemma 11 (formula 46) to disjoint operations f_1, f_2, \dots, f_m we obtain the equation

$$\begin{aligned} &f(u_1^{(s)}, u_2^{(s)}, \dots, u_m^{(s)}) \\ &= f_1\left(y, f_2\left(y, \dots, f_{s-1}\left(y, f_s\left(x, f_{s+1}\left(y, \dots, f_m(y, y)\right) \dots\right)\right) \dots\right) \\ &= f_s\left(x, f_1\left(y, \dots, f_{s-1}\left(y, f_{s+1}\left(y, \dots, f_m(y, y)\right) \dots\right)\right) \dots\right) = f_s(x, y). \end{aligned}$$

Similarly we get the equation

$$g(u_1^{(s)}, u_2^{(s)}, \dots, u_m^{(s)}) = g_s(x, y).$$

Thus, by (76), $f_k = g_k$ for $k = 2, 3, \dots, m$. The equation $f_1 = g_1$ is now a simple consequence of relations (75). The Lemma is thus proved.

LEMMA 18. *Let \mathfrak{A} be a diagonal algebra, $m \geq 4$ and f an m -ary not necessarily algebraic operation in \mathfrak{A} . If for each system h_1, h_2, \dots, h_m of $(m-1)$ -ary algebraic operations in \mathfrak{A} the composition*

$$f(h_1(x_1, x_2, \dots, x_{m-1}), h_2(x_1, x_2, \dots, x_{m-1}), \dots, h_m(x_1, x_2, \dots, x_{m-1}))$$

is an algebraic operation in \mathfrak{A} and $f(x_1, x_2, \dots, x_m) = x_1$ whenever at least two elements among x_1, x_2, \dots, x_m are equal, then $f(x_1, x_2, \dots, x_m) = x_1$ whenever x_1, x_2, \dots, x_m belong to a subalgebra of \mathfrak{A} generated by less than m elements.

Proof. Let k be an integer satisfying the inequality $1 \leq k \leq m-1$ and let h_1, h_2, \dots, h_m be k -ary algebraic operations. We shall prove the formula

$$(77) \quad f(h_1(x_1, x_2, \dots, x_k), h_2(x_1, x_2, \dots, x_k), \dots, h_m(x_1, x_2, \dots, x_k)) \\ = h_1(x_1, x_2, \dots, x_k)$$

by induction with respect to k .

Since $f(x, x, \dots, x) = x$, formula (77) is obvious for $k = 1$. Consider the case $k = 2$. Put

$$(78) \quad h_0(x_1, x_2, x_3) \\ = f(h_1(x_1, x_2), h_2(x_1, x_3), h_3(x_1, x_3), h_4(x_1, x_2), \dots, h_m(x_1, x_2))).$$

The operation h_0 is algebraic. Moreover, by the hypothesis and the inequality $m \geq 4$, we have the equations

$$h_0(x_2, x_2, x_3) = f(x_2, h_2(x_2, x_3), h_3(x_2, x_3), x_2, \dots, x_2) = x_2 = h_1(x_2, x_2), \\ h_0(x_3, x_2, x_3) = f(h_1(x_3, x_2), x_3, x_3, h_4(x_3, x_2), \dots, h_m(x_3, x_2)) = h_1(x_3, x_2).$$

Consequently, $h_0(x_1, x_2, x_3) = h_1(x_1, x_2)$ whenever $x_1 = x_2$ or $x_1 = x_3$. Hence, by Lemma 17, we get the equation $h_0(x_1, x_2, x_3) = h_1(x_1, x_2)$ for all elements x_1, x_2 and x_3 . By (78) the equation $h_0(x_1, x_2, x_2) = h_1(x_1, x_2)$ gives formula (77) for $k = 2$.

Suppose now that formula (77) holds for an integer k such that $2 \leq k \leq m-2$. Put

$$(79) \quad g(x_1, x_2, \dots, x_{k+1}) \\ = f(h_1(x_1, x_2, \dots, x_{k+1}), h_2(x_1, x_2, \dots, x_{k+1}), \dots, h_m(x_1, x_2, \dots, x_{k+1})).$$

By the inductive assumption we have the equations

$$g(x_2, x_2, x_3, \dots, x_{k+1}) = h_1(x_2, x_2, x_3, \dots, x_{k+1})$$

and

$$g(x_3, x_2, x_3, x_4, \dots, x_{k+1}) = h_1(x_3, x_2, x_3, x_4, \dots, x_{k+1}).$$

Hence, by Lemma 17, we get the equation

$$g(x_1, x_2, \dots, x_{k+1}) = h_1(x_1, x_2, \dots, x_{k+1}),$$

which, according to (79), implies formula (77). The Lemma is thus proved.

LEMMA 19. *If $2 \in \mathcal{S}(\mathcal{A})$ and $\mathcal{S}(\mathcal{A}) \neq \{2, 3, \dots\}$, then either $\mathcal{S}(\mathcal{A}) = \{s : 2 \leq s \leq n\}$, where $n \geq 2$ or $\mathcal{S}(\mathcal{A}) = \{s : 2 \leq s \leq n\} \cup \{s : s \geq m\}$, where $m > n \geq 2$.*

Moreover, $\mathcal{S}(\mathfrak{A}) = \{s : 2 \leq s \leq n\}$, where $n \geq 2$ if and only if \mathfrak{A} is an n -dimensional diagonal algebra. Further, $\mathcal{S}(\mathfrak{A}) = \{s : 2 \leq s \leq n\} \cup \{s : s \geq m\}$, where $m > n \geq 2$ if and only if $\mathfrak{A} = (A; \{d\} \cup \mathbf{F})$, where $(A; d)$ is an n -dimensional diagonal algebra, the class \mathbf{F} contains an m -ary operation depending on every variable, all operations f from \mathbf{F} depend on at least m variables and satisfy the equation $f(x_1, x_2, \dots, x_k) = x_1$ whenever the elements x_1, x_2, \dots, x_k belong to a subalgebra of the algebra $(A; d)$ generated by less than m elements.

Proof. It has been shown in Example 3 (Section II) that $\mathcal{S}(\mathfrak{A}) = \{s : 2 \leq s \leq n\}$ for n -dimensional diagonal algebras \mathfrak{A} . Suppose that $\mathfrak{A} = (A; \{d\} \cup \mathbf{F})$, where $(A; d)$ is an n -dimensional diagonal algebra, the class \mathbf{F} contains an m -ary operation depending on every variable, all operations f from \mathbf{F} depend on at least m variables and satisfy the equation $f(x_1, x_2, \dots, x_k) = x_1$ whenever x_1, x_2, \dots, x_k belong to a subalgebra of the algebra $(A; d)$ generated by less than m elements. The equation

$$\mathcal{S}(\mathfrak{A}) \cap \{2, 3, \dots, m-1\} = \mathcal{S}((A; d)) \cap \{2, 3, \dots, m-1\}$$

is obvious. By Lemma 1 the set $\mathcal{S}(\mathfrak{A})$ contains all integers $\geq m$. Thus $\mathcal{S}(\mathfrak{A}) = \{s : 2 \leq s \leq n\} \cup \{s : s \geq m\}$.

Suppose that $2 \in \mathcal{S}(\mathfrak{A})$ and $\mathcal{S}(\mathfrak{A}) \neq \{2, 3, \dots\}$. By Lemma 14, $(A; \mathbf{A}^{(2)})$ is a diagonal algebra $(A; d)$. Since $2 \in \mathcal{S}(\mathfrak{A})$ and, consequently, $2 \in \mathcal{S}((A; d))$, the diagonal algebra $(A; d)$ is at least two-dimensional. Suppose that $\mathfrak{A} \neq (A; d)$. Then $\mathfrak{A} = (A; \{d\} \cup \mathbf{F}_0)$, where the class \mathbf{F}_0 consists of all algebraic operations in the algebra \mathfrak{A} which are not algebraic in the algebra $(A; d)$. By Lemma 13 each operation from \mathbf{F}_0 depends on at least four variables. Let m be the integer such that the class \mathbf{F}_0 contains an m -ary operation depending on every variable and each operation from \mathbf{F}_0 depends on at least m variables. Of course, $m \geq 4$. By Lemmas 15, 16 and 18 there exists a subclass \mathbf{F} of the class \mathbf{F}_0 containing an m -ary operation, consisting of operations f satisfying the equation $f(x_1, x_2, \dots, x_k) = x_1$ whenever the elements x_1, x_2, \dots, x_k belong to a subalgebra of $(A; d)$ generated by less than m elements and, finally, satisfying the condition $(A; \{d\} \cup \mathbf{F}_0) = (A; \{d\} \cup \mathbf{F})$. Hence and from the first part of the proof it follows that $\mathcal{S}(\mathfrak{A}) = \{s : 2 \leq s \leq n\} \cup \{s : s \geq m\}$, where n is the dimension of $(A; d)$. Since $\mathcal{S}(\mathfrak{A}) \neq \{2, 3, \dots\}$, we have the inequality $m > n \geq 2$, which completes the proof of the Lemma.

LEMMA 20. *If $2 \notin \mathcal{S}(\mathfrak{A})$, $3 \notin \mathcal{S}(\mathfrak{A})$ and $\mathcal{S}(\mathfrak{A}) \neq \emptyset$, then $\mathcal{S}(\mathfrak{A}) = \{s : s \geq m\}$, where $m \geq 4$. Moreover, $\mathcal{S}(\mathfrak{A}) = \{s : s \geq m\}$, where $m \geq 4$ if and only if $\mathfrak{A} = (A; \mathbf{F})$, where the class \mathbf{F} contains an m -ary operation depending on every variable, all operations f from \mathbf{F} depend on at least m variables*

and satisfy the equation $f(x_1, x_2, \dots, x_k) = x_1$ whenever the system x_1, x_2, \dots, x_k contains at most $m-1$ different elements.

Proof. Suppose that $2 \notin \mathcal{S}(\mathfrak{A})$, $3 \notin \mathcal{S}(\mathfrak{A})$ and $\mathcal{S}(\mathfrak{A}) \neq \emptyset$. Let m be the integer such that the algebra \mathfrak{A} contains an m -ary algebraic non-trivial operation depending on every variable and each non-trivial algebraic operation in \mathfrak{A} depends on at least m variables. Of course, $m \geq 4$ and $\mathfrak{A} = (A; F_0)$, where F_0 denotes the class of all non-trivial algebraic operations in \mathfrak{A} . Applying Lemmas 15, 16 and 18 to one-dimensional diagonal algebras, i.e. to trivial algebras, we obtain the equation $\mathfrak{A} = (A; F)$, where F is a subset of F_0 containing an m -ary operation and consisting of operations f satisfying the equation $f(x_1, x_2, \dots, x_k) = x_1$ whenever the elements x_1, x_2, \dots, x_k belong to a subalgebra of a trivial algebra generated by less than m elements or, in other words, whenever the system x_1, x_2, \dots, x_k contains at most $m-1$ different elements. To prove the Lemma it suffices to prove the formula $\mathcal{S}(\mathfrak{A}) = \{s : s \geq m\}$. But this formula is a simple consequence of Lemma 1, which completes the proof.

Proof of Theorem 1. We consider four cases

$$(80) \quad 2 \in \mathcal{S}(\mathfrak{A}),$$

$$(81) \quad 2 \notin \mathcal{S}(\mathfrak{A}), \quad 3 \in \mathcal{S}(\mathfrak{A}), \quad 4 \in \mathcal{S}(\mathfrak{A}),$$

$$(82) \quad 2 \notin \mathcal{S}(\mathfrak{A}), \quad 3 \in \mathcal{S}(\mathfrak{A}), \quad 4 \notin \mathcal{S}(\mathfrak{A}),$$

$$(83) \quad 2 \notin \mathcal{S}(\mathfrak{A}), \quad 3 \notin \mathcal{S}(\mathfrak{A}).$$

In the case (80) we have, by Lemma 19 one of the following equations: $\mathcal{S}(\mathfrak{A}) = \{s : s \geq 2\}$, $\mathcal{S}(\mathfrak{A}) = \{s : 2 \leq s \leq n\}$, where $n \geq 2$ and $\mathcal{S}(\mathfrak{A}) = \{s : 2 \leq s \leq n\} \cup \{s : s \geq m\}$, where $m > n \geq 2$.

In the case (81) from Lemma 4 it follows that $\mathcal{S}(\mathfrak{A}) = \{s : s \geq 3\}$.

Further, in the case (82) we infer, according to Lemma 9, that either $\mathcal{S}(\mathfrak{A})$ consists of all odd integers greater than 1 or $\mathcal{S}(\mathfrak{A})$ consists of all odd integers greater than 1 and all integers $\geq m$, where $m \geq 5$.

Finally, in the case (83), either the set $\mathcal{S}(\mathfrak{A})$ is empty or, by Lemma 20, $\mathcal{S}(\mathfrak{A}) = \{s : s \geq m\}$, where $m \geq 4$. The Theorem is thus proved.

Proof of Theorem 2. The first and the second statements are a consequence of Lemma 19. The third and the fourth statements are a consequence of Lemma 9.

REFERENCES

- [1] G. Birkhoff, *Lattice theory*, American Mathematical Society Colloquium Publications 25, New York 1948.
- [2] E. Marczewski, *A general scheme of the notions of independence in mathematics*, Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Mathématiques, Astronomiques et Physiques, 6 (1958), p. 731-736.
- [3] — *Independence and homomorphisms in abstract algebras*, Fundamenta Mathematicae 50 (1961), p. 45-61.
- [4] — *Nombre d'éléments indépendants et nombre d'éléments générateurs dans les algèbres abstraites finies*, Annali di Matematica Pura ed Applicata 59 (1962), p. 1-9.
- [5] — *Remarks on symmetrical and quasisymmetrical operations*, Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Mathématiques, Astronomiques et Physiques, 12 (1964), p. 735-737.
- [6] — and K. Urbanik, *Abstract algebras in which all elements are independent*, Colloquium Mathematicum 9 (1962), p. 199-207.
- [7] J. Płonka, *Diagonal algebras and algebraic independence*, Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Mathématiques, Astronomiques et Physiques, 12 (1964), p. 729-733.

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