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FREE PRODUCTS OF COMPACT GENERAL ALGEBRAS

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In this paper we show that the well-known construction of free products of general algebras with respect to classes which are 1° closed under the operation of direct product and 2° hereditary (i. e., they include all subalgebras of an algebra belonging to them) can be modified so as to obtain topological free products with respect to a class of compact algebras which are 1° closed under the operation of direct product and 2° hareditary with respect to closed subalgebras. This result seems to be new even for the class of compact Abelian groups. It is related to a problem of Semadeni (1).

By a *topological algebra* we mean a general algebra $\mathfrak{A} = \langle A, \{g_\gamma\}_{\gamma\in\Gamma}\rangle$ with a Hausdorff topology in the set A such that all functions g_γ are continuous. $\mathfrak{A}$ is called *compact* if the space A is compact. All algebras considered here have an arbitrary but fixed similarity type.

THEOREM. *If $\mathscr{K}$ is a class of compact algebras closed under direct products and hereditary with respect to closed subalgebras and $\mathfrak{A}_t$ $(t\in T)$ is a system of algebras from $\mathscr{K}$, then there exists a free product $\prod_{t\in T}^{(\mathscr{K})}\mathfrak{A}_t$ with a system of canonical maps σ_t $(t\in T)$, i. e., a system $\langle\prod_{t\in T}^{(\mathscr{K})}\mathfrak{A}_t, \{\sigma_t\}_{t\in T}\rangle$ satisfying the following conditions:*

(i) $\prod_{t\in T}^{(\mathscr{K})}\mathfrak{A}_t\in\mathscr{K}$;

(ii) $\sigma_t\colon \mathfrak{A}_t \to \prod_{t\in T}^{(\mathscr{K})}\mathfrak{A}_t$ *is a continuous homomorphism*;

(iii) $\prod_{t\in T}^{(\mathscr{K})}\mathfrak{A}_t$ *is the closure of its subalgebra generated by* $\bigcup_{t\in T}\sigma_t(\mathfrak{A}_t)$;

(iv) *for every system of continuous homomorphisms* $f_t\colon \mathfrak{A}_t \to \mathfrak{B}$, *where* $\mathfrak{B}\in\mathscr{K}$, *there exists a homomorphism* $\varphi\colon \prod_{t\in T}^{(\mathscr{K})}\mathfrak{A}_t \to \mathfrak{B}$ *such that* $\varphi\sigma_t = f_t$ *for every* $t\in T$.

(1) Z. Semadeni, *P 490*, Colloquium Mathematicum 13 (1964), p. 127.

Moreover, the system $\langle \prod_{t\in T}^{(\mathscr{K})} \mathfrak{A}_t, \{\sigma_t\}_{t\in T}\rangle$ *is determined up to an isomorphism and equivalence of topology by* $\mathscr{K}$ *and* $\mathfrak{A}_t$ $(t \in T)$.

Before we proceed to the proof we wish to mention the following

COROLLARY. *If for every* $t_0 \in T$ *there exists a system of continuous homomorphisms* $f_t : \mathfrak{A}_t \to \mathfrak{B}$, *where* $\mathfrak{B} \in \mathscr{K}$, *such that* f_{t_0} *is an isomorphism, then all* σ_t *are isomorphisms* (*this is the case e. g. if each* $\mathfrak{A}_t$ *has a continuous homomorphism into any other* $\mathfrak{A}_t$).

Proof of the Theorem. Let $(\mathscr{G}_a)_{a\in M}$ be the set of all non-isomorphic algebras in $\mathscr{K}$ such that $|\mathscr{G}_a| \leqslant 2^{2^{\Sigma|\mathfrak{A}_t|}}$. We denote by β a system of continuous homomorphisms $f_t^{a\beta}$ $(t \in T)$ of the algebras $\mathfrak{A}_t$ into a fixed $\mathscr{G}_a$. The set of all such systems may be denoted by N_a. Let $P = \underset{a,\beta}{P} \mathscr{G}_{a\beta}$, with $\mathscr{G}_{a\beta} = \mathscr{G}_a$ for $\beta \in N_a$, be the Cartesian product of $\mathscr{G}_{a\beta}$'s with Tihonov topology. For each $t \in T$ the map

$$\sigma_t(x) = \{f_t^{a\beta}(x)\}, \qquad x \in \mathfrak{A}_t,\ a \in M,\ \beta \in N_a,$$

is a continuous homomorphism of $\mathfrak{A}_t$ onto $\sigma_t(\mathfrak{A}_t) \subset P$, and under the additional assumption formulated in the corollary even a continuous isomorphism. The required topological free product $F = \prod_{t\in T}^{(\mathscr{K})} \mathfrak{A}_t$ is now obtained as the closure in P of the algebra algebraically generated by the set $\bigcup_{t\in T} \sigma(\mathfrak{A}_t)$. To show this it is enough to verify condition (iv). Without loss of generality we may assume the algebra $\mathfrak{B}$ to be equal to some $\mathscr{G}_a$, for $a = a_0$, say. If f_t $(t \in T)$ is a system of continuous homomorphisms $\mathfrak{A}_t \to \mathscr{G}_a$, then for some $\beta \in N_{a_0}$ we have $f_t = f_t^{a_0\beta}$. Hence, if we define φ to be the projection of F on the "axis" corresponding to the indices a_0, β, we get $\varphi\sigma_t = f_t$ for all $t \in T$, as required.

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