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WEAK ISOMORPHISMS OF BOOLEAN AND POST ALGEBRAS

BY

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1. Preliminaries. Let us consider algebras $(A_1; F_1)$, $(A_2; F_2)$ and sets $A_1^{(n)}, A_2^{(n)}$ of all n-ary algebraic operations in $(A_1; F_1)$ and $(A_2; F_2)$ respectively (for details see Marczewski [2]). A. Goetz and E. Marczewski have recently introduced the notion of weak isomorphism of $(A_1; F_1)$ onto $(A_2; F_2)$. It is – roughly speaking – a one-to-one mapping of A_1 onto A_2 which is a one-to-one mapping of $A_1^{(n)}$ onto $A_2^{(n)}$ for every n.

Precisely to say, a one-to-one mapping of A_1 onto A_2 is said to be a *weak isomorphism* if and only if for every $f \epsilon A_1^{(n)}$ there exists $f^* \epsilon A_2^{(n)}$ such that

$$f^*[\varphi(x_1), \ldots, \varphi(x_n)] = \varphi f(x_1, \ldots, x_n)$$

and $f_1 \neq f_2$ implies $f_1^* \neq f_2^*$.

A weak isomorphism of $(A_1; F_1)$ onto itself is said to be a *weak automorphism.*

A weak isomorphism is not necessarily an isomorphism. In the case of Boolean algebra $(B; \cup, \cap, -)$ a one-to-one mapping h of B onto B defined by the formula $h(a) = -a$ $(a \epsilon B)$ is a weak automorphism but not an automorphism.

In the sequel two weak automorphisms, just defined one and the identity, will be called *natural.*

Suppose $\mathfrak{B}_1 = (B_1; \cup, \cap, -)$ and $\mathfrak{B}_2 = (B_2; \cup, \cap, -)$ be two Boolean algebras and s an isomorphism of B_1 onto B_2. If h is a natural weak automorphism on B_2, then the superposition hs is a weak isomorphism. E. Marczewski raised the following problem: Is the superposition hs the only form of weak isomorphisms of Boolean algebras?

In section 2 this question will be answered in affirmative and in section 4 an analogical problem for Post algebras will be examined. A notion of natural weak automorphism on a Post algebra is introduced in section 3.

2. Weak isomorphisms of Boolean algebras. We now prove

THEOREM I. *If there exists a weak isomorphism* φ *of a Boolean algebra* $\mathfrak{B}_1 = (B_1; \cup, \cap, -)$ *onto a Boolean algebra* $\mathfrak{B}_2 = (B_2; \cup, \cap, -)$, *then the algebras in question are isomorphic and* $\varphi = hs$, *where* h *is a natural weak automorphism on* $\mathfrak{B}_2$, *and* s *is an isomorphism of* $\mathfrak{B}_1$ *onto* $\mathfrak{B}_2$.

Proof. Let us see first that φ maps trivial ([1]) algebraic operations onto trivial; it is an obvious consequence of the definition of the weak isomorphism.

Now, since φ maps $A_1^{(0)}$ onto $A_2^{(0)}$ (constants onto constants), only two possibilities are to be taken into consideration:

(1) $$\varphi(0) = 0 \quad \text{and} \quad \varphi(1) = 1,$$

(2) $$\varphi(0) = 1 \quad \text{and} \quad \varphi(1) = 0.$$

(Constants in both algebras are denoted by the same symbols in this paper.)

There is only one unary non-trivial and non-constant algebraic operation in B_1 (in B_2): the complementation $f(x) = -x$. Therefore

(3) $$\varphi(-x) = -\varphi(x)$$

for every $x \in B_1$ and for every weak isomorphism φ.

Let us consider now the case of algebraic operations of two variables which are neither unary nor trivial. There are only 6 of them:

$$x_1 \cup x_2, \quad x_1 \cup -x_2, \quad -x_1 \cup -x_2,$$
$$x_1 \cap x_2, \quad x_1 \cap -x_2, \quad -x_1 \cap -x_2.$$

One can easily verify that each of the formulas

$$\varphi(x_1 \cup x_2) = \varphi(x_1) \cup -\varphi(x_2), \qquad \varphi(x_1 \cup x_2) = \varphi(x_1) \cap -\varphi(x_2),$$
$$\varphi(x_1 \cup x_2) = -\varphi(x_1) \cup -\varphi(x_2), \qquad \varphi(x_1 \cup x_2) = -\varphi(x_1) \cap -\varphi(x_2)$$

contradicts (1) and (2).

Therefore the two following possibilities remain to be considered:

$$\varphi(x_1 \cup x_2) = \varphi(x_1) \cup \varphi(x_2) \text{ corresponding to (1),}$$
$$\varphi(x_1 \cup x_2) = \varphi(x_1) \cap \varphi(x_2) \text{ corresponding to (2).}$$

In the former we recognize an isomorphism (formula (3) should be remembered). In the latter the weak isomorphism φ is of the form $\varphi = hs$, where $s = h\varphi$ is an isomorphism and h is a natural weak automorphism but not an identity.

([1]) $f \in B_1^{(n)}$ is said to be *trivial* if there exists $k \leqslant n$ such that $f(x_1, \ldots, x_n) = x_k$.

In fact:

$$h\varphi(x_1 \cup x_2) = h[\varphi(x_1) \cap \varphi(x_2)] = -\varphi(x_1) \cup -\varphi(x_2)$$
$$= h\varphi(x_1) \cup h\varphi(x_2).$$

On the other hand,

$$h\varphi(-x) = -\varphi(-x) = \varphi(x) \quad \text{and} \quad h\varphi(x) = -\varphi(x),$$

so that

$$h\varphi(-x) = -h\varphi(x).$$

In consequence the one-to-one mapping $s = h\varphi$ is an isomorphism. This completes the proof of the theorem.

COROLLARY. *If φ is a weak isomorphism of a Boolean algebra $\mathfrak{B}_1$ onto a Boolean algebra $\mathfrak{B}_2$ and $\varphi(0) = 0$, then φ is an isomorphism.*

3. Natural weak automorphisms on Post algebras. Let

$$\mathfrak{P} = (P; \cup, \cap, e_0, e_1, \ldots, e_{n-1}; C_0, C_1, \ldots, C_{n-1})$$

be a Post algebra. This means that $(P; \cup, \cap)$ is a distributive lattice with a chain

$$0 = e_0 < e_1 < \ldots < e_{n-1} = 1$$

of constants ($n \geqslant 2$), in which unary algebraic operations $C_0, C_1, \ldots, C_{n-1}$ are defined in such a way that

1° for every $x \in P$

$$(4) \qquad x = \bigcup_{i=0}^{n-1} C_i(x) \cap e_i, \quad \bigcup_{i=0}^{n-1} C_i(x) = 1, \quad C_i(x) \cap C_j(x) = 0$$

for $i \neq j$ and

2° if $x = \bigcup_{i=0}^{n-1} c_i \cap e_i$ for some $x \in P$, where $\bigcup_{i=0}^{n-1} c_i = 1$ and $c_i \cap c_j = 0$ for $i \neq j$, then $c_i = C_i(x)$ (see Traczyk [3], compare also Epstein [1]).

A representation like (4) is called a *disjoint representation* of x.

Now let $\{i_j\}$, $j = 0, 1, \ldots, n-1$, be an arbitrary permutation of the set of integers $0, 1, \ldots, n-1$.

THEOREM II. *The algebraic operation h defined on $\mathfrak{P}$ by the formula*

$$(+) \qquad h(x) = C_{i_0}(x) \cap e_0 \cup \ldots \cup C_{i_{n-1}}(x) \cap e_{n-1}$$

is a weak automorphism.

Proof. The inequality $x_1 \neq x_2$ implies $C_{i_j}(x_1) \neq C_{i_j}(x_2)$ for some $i_j \neq 0$, by (4). Hence $h(x_1) \neq h(x_2)$ for $x_1 \neq x_2$.

On the other hand, let $\{k_j\}$, $j = 0, 1, \ldots, n-1$, be the inverse permutation of $\{i_j\}$, and let us put

$$(++)\qquad y = \bigcup_{i=0}^{n-1} C_{k_j}(x) \frown e_j \quad \text{for arbitrary} \quad x \in P.$$

It is a disjoint representation of y. Hence $C_j(y) = C_{k_j}(x)$ and this implies

$$C_{i_j}(y) = C_{k_{i_j}}(x) = C_j(x).$$

Consequently,

$$x = \bigcup_{i=0}^{n-1} C_{i_j}(y) \frown e_j = h(y).$$

Thus we proved that h maps P onto P in a one-to-one manner. In particular, h maps $P^{(0)}$ onto $P^{(0)}$ in a one-to-one manner, because $h(e_{i_j}) = C_{i_j}(e_{i_j}) \frown e_j = e_j$ by (4).

For every $f \in P^{(n)}$ the superposition hf also belongs to $P^{(n)}$, and the formula

$$f^*(y_1, \ldots, y_n) = hf[h^{-1}(y_1), \ldots, h^{-1}(y_n)]$$

defines an algebraic operation $f^* \in P^{(n)}$, which corresponds to f. One can easily see that this correspondence is a one-to-one correspondence of $P^{(n)}$ onto itself.

Definition. For any permutation $\{i_j\}$, $j = 0, 1, \ldots, n-1$, the weak automorphism h defined by the formula $(+)$ will be called *natural*.

COROLLARY. *It follows from* $(++)$ *that if* h *is a natural weak automorphism, then so is* h^{-1}.

4. Weak isomorphisms of Post algebras. Now let us consider two Post algebras

$$\mathfrak{P}_1 = (P_1; \cup, \frown, e_0, e_1, \ldots, e_{n-1}; C_0, C_1, \ldots, C_{n-1}),$$
$$\mathfrak{P}_2 = (P_2; \cup, \frown, e_0, e_1, \ldots, e_{n-1}; C_0, C_1, \ldots, C_{n-1}).$$

For Post algebras the following theorem is a generalization of theorem I:

THEOREM III. *If there exists a weak isomorphism* φ *of* $\mathfrak{P}_1$ *onto* $\mathfrak{P}_2$, *then the algebras in question are isomorphic, and, moreover, there exists an isomorphism* s *of* $\mathfrak{P}_1$ *onto* $\mathfrak{P}_2$ *and a natural weak automorphism on* $\mathfrak{P}_2$ *such that* $\varphi = hs$.

Proof. Let B_1 be the set of all elements x of P_1 of the following disjoint representation:

$$x = C_0(x) \frown e_0 \cup C_{n-1}(x) \frown e_{n-1} = C_{n-1}(x).$$

It is well known that $(B_1; \cup, \cap)$ is a Boolean algebra (of complemented elements of the lattice $(P_1; \cup, \cap)$. We are going to prove that $(\varphi(B_1); \cup, \cap)$ is a Boolean algebra, too.

If $y_1, y_2 \in \varphi(B_1)$, then there exists an algebraic operation $f_1 \in P_1^{(2)}$ $(f_2 \in P_1^{(2)})$ such that

$$y_1 \cup y_2 = \varphi f_1[\varphi^{-1}(y_1), \varphi^{-1}(y_2)] \quad (y_1 \cap y_2 = \varphi f_2[\varphi^{-1}(y_1), \varphi^{-1}(y_2)]).$$

It is known (see, e. g., Traczyk [4]) that

$$C_i(f_1[\varphi^{-1}(y_1), \varphi^{-1}(y_2)]) \quad \big(C_i(f_2[\varphi^{-1}(y_1), \varphi^{-1}(y_2)])\big), \qquad i = 0, \ldots, n-1,$$

is a join of a subset of the set

$$C = \big(C_j(\varphi^{-1}(y_1)) \cap C_k(\varphi^{-1}(y_2))\big), \qquad j, k = 0, 1, \ldots, n-1.$$

Since $\varphi^{-1}(y_1) \in B_1$ and $\varphi^{-1}(y_2) \in B_1$, we have $C_j(\varphi^{-1}(y_1)) \cap C_k(\varphi^{-1}(y_2)) = 0$ if at least one of the indices j, k differs from 0 and $n-1$, and $C_{n-1}(\varphi^{-1}(y_i)) = \varphi^{-1}(y_i)$ for $i = 1, 2$. Let us put $C_0(\varphi^{-1}(y_i)) = -\varphi^{-1}(y_i)$, $i = 1, 2$.

Only four elements of the set C need to be taken into consideration (those do not equal 0):

$$-\varphi^{-1}(y_1) \cap -\varphi^{-1}(y_2), \qquad -\varphi^{-1}(y_1) \cap \varphi^{-1}(y_2),$$
$$\varphi^{-1}(y_1) \cap -\varphi^{-1}(y_2), \qquad \varphi^{-1}(y_1) \cap \varphi^{-1}(y_2).$$

If $C_i(f_j[\varphi^{-1}(y_1), \varphi^{-1}(y_2)])$, $j = 1, 2$, were a join of some of them, $i = 1, 2, \ldots, n-2$, then — putting $y_k = \varphi(0) = e_{i_0}$ or $y_k = \varphi(1) = e_{i_{n-1}}$ $(k = 1, 2)$ — we would obtain $e_i \in B_1$. Contradiction.

Hence we easily infer that f_j belongs to $B_1^{(2)}$, $j = 1, 2$, and therefore $y_1 \cup y_2 \in \varphi(B_1)$, $(y_1 \cap y_2 \in \varphi(B_1))$. Thus $(\varphi(B_1); \cup, \cap)$ is an algebra. Putting $y_k = e_{i_0}$ or $y_k = e_{i_{n-1}}$ $(k = 1, 2)$ one can easily see that only two following cases are to be considered:

(*) $\quad y_1 \cup y_2 = \varphi[\varphi^{-1}(y_1) \cup \varphi^{-1}(y_2)]$ and $y_1 \cap y_2 = \varphi[\varphi^{-1}(y_1) \cap \varphi^{-1}(y_2)]$

or

(**) $\quad y_1 \cup y_2 = \varphi[\varphi^{-1}(y_1) \cap \varphi^{-1}(y_2)]$ and $y_1 \cap y_2 = \varphi[\varphi^{-1}(y_1) \cup \varphi^{-1}(y_2)]$.

From these formulas it follows that the fundamental operations $\cup$ and $\cap$ of the algebra $(\varphi(B_1); \cup, \cap)$ are commutative, associative and distributive.

In the case (*) for every $y \in \varphi(B_1)$

$$y \cup e_{i_0} = \varphi[\varphi^{-1}(y) \cup e_0] = \varphi[\varphi^{-1}(y)] = y,$$
$$y \cap e_{i_{n-1}} = \varphi[\varphi^{-1}(y) \cap e_{n-1}] = \varphi[\varphi^{-1}(y)] = y,$$
$$y \cup \varphi[-\varphi^{-1}(y)] = \varphi[\varphi^{-1}(y) \cup -\varphi^{-1}(y)] = \varphi(1) = e_{n-1},$$
$$y \cap \varphi[-\varphi^{-1}(y)] = \varphi[\varphi^{-1}(y) \cap -\varphi^{-1}(y)] = \varphi(0) = e_{i_0}.$$

Thus $(\varphi(B_1); \cup, \cap)$ is a Boolean algebra, e_{i_0} is its zero-element, $e_{i_{n-1}}$ is its unit-element; $\varphi[-\varphi^{-1}(y)]$ is the complement of y.

In a similar way one can prove in the case (**) that $(\varphi(B_1); \cup, \cap)$ is a Boolean algebra, too.

Now let $\{i_j\}$, $j = 0, 1, \ldots, n-1$, be a permutation of integers $0, 1, \ldots, n-1$ defined by the formula

$$\varphi(e_j) = e_{i_j}, \quad j = 0, 1, \ldots, n-1,$$

and let h be the natural weak automorphism on $\mathfrak{P}_2$ corresponding to this permutation, i. e.

$$h(x) = \bigcup_{j=0}^{n-1} C_{i_j}(x) \cap e_j.$$

Hence

$$h(e_{i_j}) = C_{i_j}(e_{i_j}) \cap e_j = e_j \quad \text{and} \quad h\varphi(e_j) = e_j, \quad j = 0, \ldots, n-1.$$

By the above part of the proof, $(h\varphi(B_1); \cup, \cap)$ is a Boolean algebra, and $e_0 = 0$ and $e_{n-1} = 1$ are the zero-element and the unit-element, respectively, of this algebra.

Let $\mathfrak{B}_2 = (B_2; \cup, \cap)$ be the Boolean algebra of complemented elements of the lattice $(P_2; \cup, \cap)$. The Boolean algebra $(h\varphi(B_1); \cup, \cap)$ is a subalgebra of B_2.

On the other hand, $(\varphi^{-1}h^{-1}(B_2); \cup, \cap)$ is a subalgebra of the Boolean algebra $(B_1; \cup, \cap)$. Consequently, $h\varphi(B_1) = B_2$.

Since $h\varphi$ is a weak isomorphism of a Boolean algebra onto a Boolean algebra and $h\varphi(0) = 0$, it follows, by the corollary of theorem I, that $h\varphi$ is an isomorphism.

Since $h\varphi$ is defined all over P_1 and, in addition,

$$h\varphi(e_i) = e_i, \quad i = 0, \ldots, n-1,$$

$s = h\varphi$ is an isomorphism (see Traczyk [3], p. 202) of the Post algebra $\mathfrak{P}_1$ onto the Post algebra $\mathfrak{P}_2$. Hence $\varphi = h^{-1}s$, where h^{-1} is, by the corollary of theorem II, a natural weak automorphism on $\mathfrak{P}_2$. The proof of theorem III is complete.

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