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ON COHEN'S THEOREM

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Introduction. The theorem of the title is the well-known result due to Cohen ([2], Theorem 2, p. 29), which states that each ideal of a commutative ring R with identity is finitely generated (i.e. R is Noetherian) if each prime ideal of R is finitely generated (see also [3], p. 5, and [5], p. 8). This theorem rests on the facts that the set of ideals of a ring that are not finitely generated, ordered under the inclusion relation, is an inductive set and that an ideal maximal with respect to the property of being not finitely generated is prime. The main purpose of the present note is to strengthen Cohen's theorem by restricting the ambit of its statement to a proper subset of the prime spectrum of R. The method is to impose a condition of finite character on those prime ideals which are possible candidates for being maximal elements in the set of ideals of R having no finite basis. As might be expected, this family of prime ideals includes each maximal ideal of R but does not, in general, include every prime ideal of R. By characterizing this family of prime ideals explicitly, information is given of where to seek for ideals which have no finite basis, since each such ideal is contained in an ideal which is maximal with respect to this property ([6], p. 72). In addition, as an application of Theorem 1, it is shown, when R has only a finite number of maximal ideals, that several simple equivalent Noetherian conditions can be formulated on R; in particular, the hypotheses of Theorem (31.8) in [5], p. 110, can be weakened and a simpler proof given (i.e. without using the notion of completion) of the stronger result. Throughout this paper R will denote a commutative ring with identity having maximal ideals $\{\boldsymbol{m}_i\}_{i\in X}$.

1. If $\boldsymbol{a}$ is an ideal of R, then the ideal $\bigcap_{n=1}^{\infty} (\boldsymbol{a}+\boldsymbol{m}_i^n)$ with $i \in X$ is called the *closure* of $\boldsymbol{a}$ with respect to the $\boldsymbol{m}_i$-adic topology; $\boldsymbol{a}$ is said to have a *prime closure* in case its closure is a prime ideal of R. If $\boldsymbol{a}$ is equal to its closure with respect to the $\boldsymbol{m}_i$-adic topology, then it is said to be a *closed ideal* with respect to the $\boldsymbol{m}_i$-adic topology. This terminology is used below.

THEOREM 1. *R is a Noetherian ring if, for all $i \in X$, the following prime ideals of R have finite bases:*

(1) m_i;

(2) *prime closures of finitely generated ideals with respect to the m_i-adic topology.*

Proof. Let us suppose that R is not a Noetherian ring and that P is a maximal (prime) element in the set of ideals of R having no finite basis. Let h denote the natural homomorphism of R onto $\bar{R} = R/P$ and let $\bar{A}$ be a non-zero ideal of $\bar{R}$. Then $h^{-1}(\bar{A}) = A$ is an ideal of R properly containing P and so, by the maximal character of P, A has a finite basis. But $h(h^{-1}(\bar{A})) = \bar{A} = h(A)$, which means that $\bar{A}$ has a finite basis since it is the homomorphic image of a finitely generated ideal. We deduce that every non-zero ideal of $\bar{R}$ has a finite basis, i.e. $\bar{R}$ is a Noetherian integral domain. Since R has an identity element, there is a maximal ideal m_i containing the prime ideal P. Also, from the observation that m_i/Pm_i is a finitely generated (R/P)-module we see that m_i/Pm_i itself is a Noetherian module ([5], Theorem (3.5), p. 8). Therefore, P/Pm_i, being a submodule of m_i/Pm_i, is also Noetherian. In that case P can be finitely generated modulo Pm_i, and so $P = (p_1, \ldots, p_r) + Pm_i$ for suitable elements $p_1, \ldots, p_r$ of P. Accordingly, on using an inductive argument, it follows that $P = (p_1, \ldots, p_r) + Pm_i^n$ for each positive integer n: thus

$$P \subseteq \bigcap_{n=1}^{\infty}\big((p_1, \ldots, p_r) + m_i^n\big).$$

On the other hand, since $\bar{R}$ is a Noetherian integral domain, we have

$$\bigcap_{n=1}^{\infty} \overline{m}_i^n = (\bar{0}),$$

on using Krull's intersection theorem ([6], p. 206), where $\overline{m}_i$ corresponds to m_i under the natural homomorphism h, and we therefore infer that P is a closed ideal with respect to the m_i-adic topology. Accordingly,

$$\bigcap_{n=1}^{\infty}\big((p_1, \ldots, p_r) + m_i^n\big) \subseteq \bigcap_{n=1}^{\infty}(P + m_i^n) = P$$

with the result that

$$P = \bigcap_{n=1}^{\infty}\big((p_1, \ldots, p_{\,}) + m_i^n\big),$$

which contradicts the fact that P has no finite basis. We deduce that our original assumption that R is not Noetherian is false and Theorem 1 is proved.

In general, not every prime ideal of R is a member of the set of prime ideals characterized in the statement of Theorem 1. For instance, if A

denotes the ring of germs of indefinitely differentiable functions of a real variable x in the neighbourhood of 0 ([1], p. 110), then A is a quasi-local ring whose maximal ideal $\boldsymbol{m}$ is principal, being generated by the identity map $i: x \to x$. The intersection of all powers $\boldsymbol{m}^n$ $(n = 1, 2, \ldots)$ is the ideal of germs all of whose derivatives vanish at 0. The ring A is non-Noetherian because the ideal

$$Q = \bigcap_{n=1}^{\infty} \boldsymbol{m}^n$$

is not zero, since it contains the non-zero germ of the function $\exp(-1/x^2)$. Since this germ is not nilpotent, let P be a prime ideal of A not containing it. Then P must be strictly contained in Q and clearly cannot take the form $\bigcap_{n=1}^{\infty} (\boldsymbol{a}+\boldsymbol{m}^n)$ for any ideal $\boldsymbol{a}$ of A. We note that in this case, i.e. where the maximal ideal $\boldsymbol{m}$ is principal, Q is the unique maximal element in the set of ideals of A having no finite basis. For if S is such a maximal element, then S is contained in all powers of $\boldsymbol{m} = (m_1)$ say; since $z \in S \subset \boldsymbol{m}^t$ with t a positive integer implies that $z = rm_1^t$ with $r \in A$ and since m_1^t cannot be contained in S, we have $r \in S \subset \boldsymbol{m}$, and so $z \in \boldsymbol{m}^{t+1}$. Consequently, the conclusion follows by induction. On the other hand, as was seen in the proof of Theorem 1, S is a closed ideal with respect to the $\boldsymbol{m}$-adic topology, and so S contains the ideal Q: hence $S = Q$.

2. Hereafter we assume that the set X is finite, i.e. that R has only a finite number of maximal ideals, and we let J be the Jacobson radical of R. In Theorem (31.8) of [5], p. 110 (see also [4], p. 135), it is proved, using a structure theorem of Cohen, that if $\bigcap_{n=1}^{\infty} J^n = (0)$ and if each maximal ideal $\boldsymbol{m}_i$ has a finite basis, then R is Noetherian if every finitely generated ideal of R is closed with respect to the J-adic topology. The hypotheses of Theorem 1 permit us to strengthen Theorem (31.8) in [5] and to give a simpler proof of the stronger result. In addition, several simple equivalent Noetherian conditions can be formulated on R. Before stating Theorem 2 we note the following observation which is used below. Namely, that the closure $\boldsymbol{c}$ of a particular ideal $\boldsymbol{a} = (a_1, \ldots, a_k)$ of R be finitely generated does not alone suffice, in general, to ensure that $\boldsymbol{a}$ be a closed ideal with respect to the J-adic topology: some additional restriction is needed. For example, if R is quasi-local with the unique maximal ideal $\boldsymbol{m}$ and if it is known that the images of the elements $a_1, \ldots, a_k$ generate the vector space $\boldsymbol{c}/\boldsymbol{cm}$ over $R/\boldsymbol{m}$, then the ideal $\boldsymbol{a}$ is indeed closed. For in this case we have $\boldsymbol{c} = \boldsymbol{a}+\boldsymbol{cm}$, and so, in the residue class ring $\bar{R} = R/\boldsymbol{a}$, $\bar{\boldsymbol{c}} = \bar{\boldsymbol{c}}\bar{\boldsymbol{m}}$. Consequently, $\bar{\boldsymbol{c}} = (\bar{0})$ on using Nakayama's lemma ([1], Proposition 2.6, p. 21), since $\bar{\boldsymbol{c}}$ has a finite basis. It follows that $\boldsymbol{c} = \boldsymbol{a}$, and therefore $\boldsymbol{a}$ is a closed ideal.

THEOREM 2. *Let the ideals* m_i $(i \in X)$ *and* $\bigcap_{n=1}^{\infty} J^n$ *have finite bases. Then the following conditions on* R *are equivalent (closures are with respect to the* J*-adic topology):*

(1) *Every prime ideal of* R *has a finite basis.*

(2) R *is Noetherian.*

(3) $R/\mathfrak{a}$ *is Noetherian for every non-zero finitely generated ideal* $\mathfrak{a}$ *of* R.

(4) *Every non-zero finitely generated ideal of* R *is closed.*

(5) *The closures of non-zero finitely generated ideals of* R *are finitely generated.*

(6) *Prime closures of non-zero finitely generated ideals of* R *are finitely generated.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is straightforward. The implication (3) $\Rightarrow$(4) follows from the intersection theorem ([6], p. 208, Theorem 19, Corollary), and it is clear that (4) $\Rightarrow$ (5) $\Rightarrow$ (6). To establish that R is Noetherian from condition (6) it will be sufficient, in view of the result E1.2 in [5], p. 203, to prove that the localization of R at each maximal ideal is Noetherian. Since the finite character of the closures, with respect to the J-adic topology, of finitely generated ideals contained in a maximal ideal of R is preserved under the appropriate localization, it can be assumed that R is quasi-local. In this case (6) $\Rightarrow$ (2) follows at once by using Theorem 1. This completes the proof of Theorem 2.

Let R' denote the completion of R with respect to the J-adic filtration. For an ideal $\mathfrak{a}$ of R let $\mathfrak{a}R' \cap R$ denote the contraction in R of the ideal $\mathfrak{a}R'$ of R' under the canonical homomorphism $R \to R'$. A further simple equivalent condition can be added to those in the statement of Theorem 2. Namely:

(7) $R/(\mathfrak{a}R' \cap R)$ *is Noetherian and* $\mathfrak{a}R' \cap R$ *has a finite basis for every finitely generated non-zero ideal* $\mathfrak{a}$ *of* R.

To see that R is Noetherian under condition (7) let $q_1, \ldots, q_r$ $(q_i \in R)$ be any finite basis for the ideal $\mathfrak{a}R' \cap R$ and let P be a prime ideal of R containing it. Then $P/(\mathfrak{a}R' \cap R)$ is an ideal of $R/(\mathfrak{a}R' \cap R)$, and so has a finite basis. Therefore, we can find elements $q_{r+1}, \ldots, q_n$ of P such that

$$P = (q_{r+1}, \ldots, q_n) + \mathfrak{a}R' \cap R = (q_1, \ldots, q_n)$$

and we deduce that every prime ideal containing $\mathfrak{a}R' \cap R$ is finitely generated. But, if the element x is contained in $\mathfrak{a}R' \cap R$ and k is a positive integer, then $x \in \mathfrak{a} + J^k R'$, x being the limit in R' of a sequence of elements of $\mathfrak{a}$. Therefore, x is contained in $(\mathfrak{a} + J^k R') \cap R = \mathfrak{a} + (J^k R' \cap R) = \mathfrak{a} + J^k$ and it follows that x lies in the closure of $\mathfrak{a}$, i.e.

$$\mathfrak{a}R' \cap R \subseteq \bigcap_{n=1}^{\infty} (\mathfrak{a} + J^n).$$

Thus a prime closure of $\mathfrak{a}$ has a finite basis. Since this is true for every non-zero finitely generated ideal $\mathfrak{a}$, we deduce that (7) $\Rightarrow$ (6). On the other hand, if R is Noetherian, then conditions (3) and (7) coincide since, in this case, $\mathfrak{a}R' \cap R = \mathfrak{a}$.

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