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A NOTE ON QUASI-COMPLETE LOCAL RINGS

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Let (R, M) be a local ring (commutative with identity). If R is complete in the M-topology, then it is well known[1] that R has the following property, which we call *quasi-completeness*: given any decreasing sequence $\{A_i\}$ of ideals and any $n \geqslant 0$, $A_i \subseteq (\bigcap_j A_j)+M^n$, for some i.

Now it is easily seen by example that a local ring which is quasi-complete need not be complete. However, for purely ideal-theoretic purposes, it is interesting to note that the two properties are essentially equivalent. To show this, we introduce a rather natural topology directly on the ideals $L(R)$ of R. We then show that the ideals $L(R^*)$ of the completion R^* of R are essentially the completion space of $L(R)$, and that $L(R)$ is complete if, and only if, R is quasi-complete. It follows that if R is quasi-complete, then $L(R)$ and $L(R^*)$ are essentially the same.

Now, let A and B be any two ideals of the local ring R. If $A+M^i = B+M^i$ for all i, then $A = \bigcap_i (A+M^i) = \bigcap_i (B+M^i) = B$. Hence, if we set $S(A, B) = \sup\{i;\, A+M^i = B+M^i\}$ (we take $M^0 = R$), we have $S(A, B) \geqslant 0$ and $S(A, B) = \infty$ if and only if $A = B$. Let $d(A, B) = 1/2^{S(A,B)}$. Then d is a metric on $L(R)$.

THEOREM 1. *$L(R)$ is complete if and only if R is quasi-complete.*

Proof. Assume R is quasi-complete and let $\{A_i\}$ be any Cauchy sequence in $L(R)$. Let $\{B_i\}$ be a regular subsequence of $\{A_i\}$ ($d(B_i, B_{i+1}) \leqslant 1/2^i$ for all i) and set $C_i = B_i+M^i$ for all i. Then $\{C_i\}$ is a decreasing Cauchy sequence equivalent to $\{A_i\}$ and, further, $C_i = C_{i+1}+M^i$ for all i. Set $C_0 = \bigcap_i C_i$. Then, since R is quasi-complete, we have, given n, $C_n = C_i+M^n \subseteq C_0+M^n \subseteq C_n$ for some i, and hence $C_n = C_0+M^n$ for all n. It follows that $\lim_{n\to\infty} C_n = \lim_{n\to\infty} A_n = C_0$, so that $L(R)$ is complete.

Now, assume that $L(R)$ is complete, and let $\{A_i\}$ be any decreasing sequence of ideals of R. By the descending chain condition in R/M^n,

[1] D. G. Northcott, *Ideal theory*, Cambridge 1963.

we obtain that $\{A_i\}$ is a Cauchy sequence in $L(R)$. Set $A_0 = \lim_{n\to\infty} A_n$ and $A = \bigcap_i A_i$. Fix r. Then, given n, $A_r + M^n \supseteq A_i + M^n = A_0 + M^n$ for large i, so $A_r + M^n \supseteq A_0$ for all n, and $A_r \supseteq A_0$. Hence $A \supseteq A_0$ and $A + M^n \supseteq A_0 + M^n = A_i + M^n \supseteq A_i$ for large i. This establishes the theorem.

Now, let $L(R^*)$ denote the collection of ideals of the completion R^* of R in the M-topology, so that R^* is local with maximal ideal $M^* = MR^*$. We note that since $A + M^n = B + M^n$ is equivalent to $AR^* + M^nR^* = BR^* + M^nR^*$, the map $A \to AR^*$ of $L(R)$ into $L(R^*)$ is an isometry.

THEOREM 2. *$L(R^*)$ is isometric to the completion of $L(R)$.*

Proof. Let A^* be any element of $L(R^*)$ and, for each i, set $A_i = (A^* + M^iR^*) \cap R$. Then $\lim_{i\to\infty} A_iR^* = A^*$. Since $L(R^*)$ is complete by Theorem 1, the theorem follows.

COROLLARY. *If R is quasi-complete, then every ideal A^* of R^* is of the form AR^* for some ideal A of R.*

Proof. In this case the map $A \to AR^*$ maps $L(R)$ onto $L(R^*)$.

The degree to which a quasi-complete local ring agrees with its completion is now clear:

COROLLARY. *Let (R, M) be a quasi-complete local ring with completion (R^*, M^*). Then every element $a^* \in R^*$ is of the form $a^* = au^*$, where $a \in R$ and u^* is a unit of R^*.*

Proof. If $a^* \in R^*$, then $a^*R^* = AR^*$ for some ideal A of R. If $A = (a_1, \ldots, a_n)$, then, since R^* is local, it follows that $a^*R^* = AR^* = a_iR^*$ for some i. Hence, it may be assumed that A is principal, say $A = (a)$. Then a^* is a unit multiple of a.

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