COLLOQUIUM MATHEMATICUM

VOL. LVIII 1990 FASC. 2

SIMPLICIAL APPROXIMATION OF ANTIPODAL MAPS

BY

W. KULPA (KATOWICE)

In this note we shall prove that each continuous antipodal map $f\colon P \to \mathbf{R}^n$ defined on a symmetric polyhedron $P \subset \mathbf{R}^n$ can be approximated by a simplicial antipodal map $g\colon P \to \mathbf{R}^n$ such that $0 \in \mathbf{R}^n$ is a regular value of the map g.

The result is related to the following question of Nirenberg [4]:

Let $f\colon \mathrm{Cl}\, X \to \mathbf{R}^n$, $f(\mathrm{Bd}\, X) \subset \mathbf{R}^n \setminus \{0\}$, be a continuous antipodal map, where X is a symmetric, open and bounded subset of $\mathbf{R}^n$. Is it possible to find for each $\varepsilon > 0$ an antipodal map

$$f_\varepsilon\colon \mathrm{Cl}\, X \to \mathbf{R}^n$$

of class C^1 such that the point 0 is a regular value of the map f_ε and $\|f(x) - f_\varepsilon(x)\| < \varepsilon$ for each $x \in \mathrm{Cl}\, X$?

The purpose of the question was to obtain a simple proof, based on the degree theory, of the Borsuk antipodal theorem. The Nirenberg question was answered in the affirmative by Ivanov [3].

The main result presented here has a simple proof and, as shown, it simplifies the proof of the Borsuk theorem.

We shall use the following terminology: A set $X \subset \mathbf{R}^n$ is said to be *symmetric* if $x \in X$ implies $-x \in X$, and a map $f\colon X \to \mathbf{R}^m$ is said to be *antipodal* provided that $f(-x) = -f(x)$ for each $x \in X$. The symbols $\mathrm{Cl}\, X$, $\mathrm{Int}\, X$, $\mathrm{Bd}\, X$ mean the closure, the interior and the boundary of the set X.

1. Preliminaries. Let us recall some facts on simplicial complexes which we shall apply in this note. For details and proofs the reader is referred to [2] and [1].

A set $\{a_0, \ldots, a_k\} \subset \mathbf{R}^n$ of $k+1$ points is said to be *affinely independent* if it is not contained in any $(k-1)$-flat. This is equivalent to the fact that the points $a_1 - a_0, \ldots, a_n - a_0$ are linearly independent.

Assume that the set $\{a_0, \ldots, a_k\} \subset \mathbf{R}^n$ is affinely independent. The convex hull

$$[a_0, \ldots, a_k] := \{x \in \mathbf{R}^n\colon\ x = \sum_{i=0}^{k} \lambda_i a_i,\ 0 \leqslant \lambda_i,\ \sum_{i=0}^{k} \lambda_i = 1\}$$

is called the *k-simplex with vertices* $a_0, \ldots, a_k$. If

$$\{a_{i_0}, \ldots, a_{i_j}\} \subset \{a_0, \ldots, a_k\}, \quad j \leqslant k,$$

then the simplex $[a_{i_0}, \ldots, a_{i_j}]$ is said to be a *j-face* of $[a_0, \ldots, a_k]$.

Define a map $L\colon \boldsymbol{R}^k \to \boldsymbol{R}^n$ by

$$L(\lambda_1, \ldots, \lambda_k) := a_0 + \sum_{i=1}^{k} \lambda_i (a_i - a_0).$$

Observe that

$$[a_0, \ldots, a_k] = L(\Delta_k),$$

where

$$\Delta_k := \{(\lambda_1, \ldots, \lambda_k) \in \boldsymbol{R}^k\colon\ 0 \leqslant \lambda_i,\ \sum_{i=1}^{k} \lambda_i \leqslant 1\}.$$

If $n = k$, then the Jacobian $\det L'(x)$ is equal to

$$\det(a_1 - a_0, \ldots, a_n - a_0) \neq 0.$$

A *simplicial complex* is a finite family K of simplexes in $\boldsymbol{R}^n$ such that:

(a) If $s \in K$, then so does every face of s.

(b) If $s, \sigma \in K$, then $s \cap \sigma$ is either empty or a face common to both s and σ.

The *barycenter of a k-simplex* $s = [a_0, \ldots, a_k] \subset \boldsymbol{R}^n$ is the point

$$b(s) := \frac{1}{k+1} \sum_{i=0}^{k} a_i.$$

The *barycentric subdivision* $K^{(1)}$ of a complex K is the set of all simplexes of the form

$$[b(s_0), \ldots, b(s_j)],$$

where $s_0 \subset s_1 \subset \ldots \subset s_j$ is a strictly increasing sequence of simplexes of K. Define the $(r+1)$*-st barycentric subdivision* of a complex K by

$$K^{(r+1)} := [K^{(r)}]^{(1)}.$$

A set $P \subset \boldsymbol{R}^n$ is said to be a *polyhedron* whenever $P = |K|$ for some complex K, where

$$|K| := \bigcup \{s\colon\ s \in K\}.$$

For any vertex $a \in K$ the set

$$\mathrm{st}(a, K) := K \setminus \{s \in K\colon\ a \notin s\}$$

is called the *star* of a. Put

$$|\mathrm{st}(a, K)| := |K| \setminus \bigcup \{s \in K\colon\ a \notin s\}.$$

The set $|\mathrm{st}(a, K)|$ is an open subset of the compact space K and the following facts hold:

$$|K^{(r)}| = |K|,$$
$$\operatorname{diam}[a_0, \ldots, a_k] = \max\{\|a_i - a_j\|: i, j = k\},$$
$$\operatorname{mesh} K^{(r)} \leqslant \left(\frac{n}{n+1}\right)^r \operatorname{mesh} K,$$

where

$$|K| \subset \mathbf{R}^n \quad \text{and} \quad \operatorname{mesh} K := \max\{\operatorname{diam} s: s \in K\}.$$

Recall that if $f\colon U \to \mathbf{R}^n$, U open in $\mathbf{R}^n$, is a map of class C^1, then a point $x \in U$ is said to be *critical* whenever the Jacobian $\det f'(x) = 0$. A point $a \in \mathbf{R}^n$ is called a *regular value* of the map f if the set $f^{-1}(a)$ does not contain any critical point.

Each map $\varphi\colon V(K) \to \mathbf{R}^n$ defined on the set of all vertices of a complex K induces the so-called *simplicial map* $|\varphi|\colon |K| \to \mathbf{R}^n$ defined as follows:

$$|\varphi|(x) := \sum_{i=0}^{k} \lambda_i \varphi(a_i),$$

where

$$x = \sum_{i=0}^{k} \lambda_i a_i \in s \in K, \quad 0 \leqslant \lambda_i, \quad \sum_{i=0}^{k} \lambda_i = 1.$$

The map $|\varphi|$ is continuous and, moreover, $|\varphi|$ is of class C^∞ on the open set $U = |K| \backslash S(K)$, where $S(K)$ is the union of all k-simplexes, $k < n$. Extend the definition of simplicial map. If $X \subset \mathbf{R}^n$ is a compact set, then a continuous map $f\colon X \to \mathbf{R}^n$ is said to be *simplicial* whenever there exists a simplicial map $F\colon P \to \mathbf{R}^n$, where $P \supset X$ is a polyhedron, such that $F|X = f$.

A point $a \in \mathbf{R}^n$ is said to be a *regular value* of a simplicial map $f\colon X \to \mathbf{R}^n$ if there exists an open set $U \subset \mathbf{R}^n$, $U \subset X$, such that

(i) $a \notin f(X \backslash U)$, i.e., $f^{-1}(a) \subset U$,

(ii) $f|U$ is of class C^1,

(iii) a is a regular value of the map $f|U$ of class C^1.

For example, for a given complex K let $Z(K)$ be the union of all images $|\varphi|(s)$ of simplexes $s \in K$ such that $|\varphi|(s)$ is contained in an $(n-1)$-flat. It is clear that $|\varphi|[S(K)] \subset Z(K)$. Thus the map $|\varphi|\,|\,U$, $U = |K| \backslash S(K)$, is of class C^∞ and each point $a \in \mathbf{R}^n \backslash Z(K)$ is a regular value of the simplicial map.

2. An approximation theorem. We shall precede the main result of our note by the following

LEMMA. *For each continuous antipodal map $f\colon X \to \mathbf{R}^m$, where $X \dot{\subset} \mathbf{R}^n$ is a compact symmetric set, there exists a continuous antipodal map $F\colon \mathbf{R}^n \to \mathbf{R}^m$ such that $F|X = f$.*

Proof. Without loss of generality we may assume that $0 \in X$ because, for every antipodal map, $0 \in X$ implies $f(0) = 0$. Define for each $k = 1, \ldots, n$

$$\boldsymbol{R}_k^+ := \{(x_1, \ldots, x_n) \in \boldsymbol{R}^n\colon\ x_k \geqslant 0 \text{ and } x_i = 0 \text{ for } i > k\},$$

$$\boldsymbol{R}_k^- := \{x \in \boldsymbol{R}^n\colon\ -x \in \boldsymbol{R}_k^+\}, \qquad \boldsymbol{R}_k := \boldsymbol{R}_k^+ \cup \boldsymbol{R}_k^-.$$

We have $\boldsymbol{R}_1 \subset \boldsymbol{R}_2 \subset \ldots \subset \boldsymbol{R}_n = \boldsymbol{R}^n$. Now, we shall construct the map F in n steps.

($k = 1$) Let $f_1\colon \boldsymbol{R}_1^+ \to \boldsymbol{R}^m$ be a continuous extension of the map $f|X \cap \boldsymbol{R}_1^+$. Then, let us extend the map f_1 onto $\boldsymbol{R}_1^-$ defining

$$f_1(x) := -f(-x) \quad \text{for } x \in \boldsymbol{R}_1^-.$$

($k+1$) Assume that for $k < n$ the map $f_k\colon \boldsymbol{R}_k \to \boldsymbol{R}^m$ is defined. Since

$$f_k|X \cap \boldsymbol{R}_k = f|X \cap \boldsymbol{R}_k,$$

the map

$$g := f_k \cup f|X \cap \boldsymbol{R}_{k+1}^+\colon\ \boldsymbol{R}_k \cup X \cap \boldsymbol{R}_{k+1}^+ \to \boldsymbol{R}^m$$

is continuous. According to the Tietze–Urysohn theorem the map g is extendable to a continuous map $f_{k+1}\colon \boldsymbol{R}_{k+1}^+ \to \boldsymbol{R}^m$. Extend the map f_{k+1} onto $\boldsymbol{R}_{k+1}^-$ by the formula

$$f_{k+1}(x) := -f_{k+1}(-x) \quad \text{for } x \in \boldsymbol{R}_{k+1}^-.$$

Put $F := f_n$. This completes the proof.

THEOREM (APPROXIMATION THEOREM). *Let f: $\operatorname{Bd} X \to \boldsymbol{R}^n$ be a continuous antipodal map defined on a compact symmetric subset $X \subset \boldsymbol{R}^n$. Then for each $\varepsilon > 0$ there exists a simplicial antipodal map f_ε: $P \to \boldsymbol{R}^n$, defined on a symmetric polyhedron P, $X \subset P \subset \boldsymbol{R}^n$, such that*

(a) 0 *is a regular value of the map* f,

(b) $\|f(x) - f_\varepsilon(x)\| < \varepsilon$ *for each* $x \in \operatorname{Bd} X$.

Proof. Fix a number $M > 0$ and let $e_i \in \boldsymbol{R}^n$, $i = 1, \ldots, n$, be points of $\boldsymbol{R}^n$ defined as follows:

$$e_1 := (M, 0, \ldots, 0),\ e_2 := (0, M, 0, \ldots, 0),\ \ldots,\ e_n := (0, \ldots, 0, M).$$

Let K be a simplicial complex consisting of n-simplexes of the form

$$[0, \pm e_1, \ldots, \pm e_n]$$

and their k-faces, $k < n$. The polyhedron $|K|$ is the smallest convex set which contains the set $\{e_1, \ldots, e_n, -e_1, \ldots, -e_n\}$. Assume that the number $M > 0$ is such that $X \subset |K|$. According to the previous lemma the map f: $\operatorname{Bd} X \to \boldsymbol{R}^n$ has a continuous antipodal extension F: $|K| \to \boldsymbol{R}^n$.

Now, fix an $\varepsilon > 0$. In view of the fact that the map F is uniformly continuous there exists an r-th barycentric subdivision $K^{(r)}$ of the complex K such that

(1) $$\operatorname{mesh} K^{(r)} \leqslant \frac{\varepsilon}{36} \quad \text{and} \quad \operatorname{diam} F(s) \leqslant \frac{\varepsilon}{36} \text{ for each } s \in K^{(r)}.$$

Let $A := V(K^{(r)})$ be the set of all vertices of the complex $K^{(r)}$. Consider the following subsets of A:

$$A_1 := \{a \in A\colon\ a \in \operatorname{Bd}|\operatorname{st}(0, K^{(r)})|\} \cup \{0\},$$
$$A_2 := \{b \in A \setminus A_1\colon\ b \in \operatorname{Bd}|\operatorname{st}(a, K^{(r)})|,\ a \in A_1\},$$
$$A_3 := A \setminus (A_1 \cup A_2).$$

The sets A, A_1, A_2, A_3 are finite and symmetric. Since the map F is antipodal, so, in particular, the set $E := A_2 \cup F(A_3)$ is also symmetric. Let Z be the union of all k-simplexes, $k < n$, with vertices belonging to the set E. The set Z is a compact nowhere dense symmetric subset of $\boldsymbol{R}^n$. Hence there exist points

(2) $$c_s \in B(0, \delta) \setminus Z,$$

where

$$B(0, \delta) := \{x \in \boldsymbol{R}^n\colon\ \|x\| \leqslant \delta\}, \quad B(0, \delta) \subset |\operatorname{st}(0, K^{(r)})|, \quad 0 < \delta < \frac{\varepsilon}{36}.$$

Now, let us define an antipodal map $\varphi\colon A \to \boldsymbol{R}^n$ in the following way: For each $a = (a_1, \ldots, a_n) \in A_2 \cup A_3$ put

$$k := \max\{i \leqslant n\colon\ a_i \neq 0\},$$

and then define

(3) $$\varphi(a) := \begin{cases} a & \text{if } a \in A_1, \\ a + c_i & \text{if } a \in A_2 \text{ and } a_k > 0, \\ a - c_i & \text{if } a \in A_2 \text{ and } a_k < 0, \\ F(a) + c_j & \text{if } a \in A_3 \text{ and } a_k > 0, \\ F(a) - c_j & \text{if } a \in A_3 \text{ and } a_k < 0. \end{cases}$$

Let $f_\varepsilon\colon P \to \boldsymbol{R}^n$, $P := |K^{(r)}| = |K|$, be a simplicial map induced by the map φ, i.e.,

(4) $$f_\varepsilon(x) := \sum_{i=0}^{j} \lambda_i \varphi(a_i),$$

where

$$x = \sum_{i=0}^{j} \lambda_i a_i \in s = [a_0, \ldots, a_j] \in K^{(r)}, \quad 0 \leqslant \lambda_i, \quad \sum_{i=0}^{j} \lambda_i = 1.$$

We verify that

$$\|F(x) - f_\varepsilon(x)\| < \varepsilon \quad \text{for each } x \in P.$$

Indeed, first observe that

$$\varphi(a) = F(a)+\eta(a), \tag{5}$$

where $\|\eta(a)\| < \varepsilon/6$ for each $a \in A$.

From (1)–(5) we get, for $x \in [a_0, \ldots, a_j] \in K^{(r)}$,

$$\begin{aligned}\|F(x)-f_\varepsilon(x)\| &\leqslant \|F(x)-\varphi(a_0)\| + \|\varphi(a_0)-f_\varepsilon(x)\| \\ &\leqslant \|F(x)-F(a_0)\| + \|\eta(a_0)\| + \max\{\|\varphi(a_0)-\varphi(a_i)\|:\ i \leqslant j\} \\ &\leqslant \frac{\varepsilon}{6}+\frac{\varepsilon}{6}+\max\{\|F(a_0)-F(a_i)\|:\ i \leqslant j\} + \|\eta(a_0)\| \\ &\quad + \max\{\|\eta(a_i)\|:\ i \leqslant j\} \leqslant 5\cdot\frac{\varepsilon}{6} < \varepsilon.\end{aligned}$$

Since the map φ is antipodal, so is the map f_ε. The proof will be completed if we show that 0 is a regular value of the map f_ε. First, observe that f_ε is of class C^∞ on the open set

$$U := |\mathrm{st}(0, K^{(r)})| \cup [|K^{(r)}|\backslash S(K^{(r)})].$$

Next, consider a point $x \in f_\varepsilon^{-1}(0)$. If $x = 0$, then x is not a critical point of the map $f_\varepsilon|U$ because $f_\varepsilon|\mathrm{st}(0, K^{(r)})|$ is the identity map. If $x \neq 0$ and $f_\varepsilon(x) = 0$, then in view of the choice of the point $c \in \boldsymbol{R}^n$ we get $x \in \mathrm{Int}\, s$ for some n-simplex $s = [a_0, \ldots, a_n] \in K^{(r)}$ such that the set $\{\varphi(a_0), \ldots, \varphi(a_n)\}$ is affinely independent. Hence $\det f'_\varepsilon(x) \neq 0$. The proof that 0 is a regular value is completed.

3. On a proof of the Borsuk antipodal theorem. In this part we would like to explain a role which the approximation theorem plays in the proof of the Borsuk theorem suggested by Nirenberg [4].

The *classical degree function* is an integer-value function $\deg(f, X, a)$ defined for all continuous maps $f\colon X \to \boldsymbol{R}^n$, where X is a compact subset of $\boldsymbol{R}^n$ and $a \notin f(\mathrm{Bd}\, X)$, satisfying the following conditions:

(a) If $\deg(f, X, a) \neq 0$, then $a \in \mathrm{Int} f(X)$.

(b) If $f\colon X \to \boldsymbol{R}^n$ is a map of class C^1 and the point

$$a \in f(X)\backslash f(\mathrm{Bd}\, X)$$

is a regular value, then

$$\deg(f, X, a) = \sum\{\mathrm{sgn}\det f'(x):\ x \in f^{-1}(a)\}.$$

(c) If $H \subset X$ is a closed subset and $a \notin f(H \cup \mathrm{Bd}\, X)$, then

$$\deg(f, X, a) = \deg(f, \mathrm{Cl}(X\backslash H), a).$$

(d) For each continuous map $f\colon X \to \boldsymbol{R}^n$ and a point $a \notin f(\mathrm{Bd}\, X)$

there exists an $\varepsilon > 0$ such that, for every continuous map $g\colon X \to \boldsymbol{R}^n$, if $\|f(x)-g(x)\| < \varepsilon$ for each $x \in \operatorname{Bd} X$, then

$$\deg(f, X, a) = \deg(g, X, a).$$

From (a)–(d) we get further properties:

(e) $F\colon X \times [0, 1] \to \boldsymbol{R}^n$ is a continuous map such that for each $t \in [0, 1]$ and $x \in \operatorname{Bd} X$ we have $a \neq F(x, t)$, then

$$\deg(f_0, X, a) = \deg(f_1, X, a),$$

where $f_0(x) = F(x, 0)$ and $f_1(x) = F(x, 1)$.

(f) For any continuous maps $f, g\colon X \to \boldsymbol{R}^n$ and a point $a \notin f(\operatorname{Bd} X)$,

$$f|\operatorname{Bd} X = g|\operatorname{Bd} X \text{ implies } \deg(f, X, a) = \deg(g, X, a).$$

(g) If $f\colon X \to \boldsymbol{R}^n$ is a map of class C^1 and a point $a \in \boldsymbol{R}^n \setminus f(\operatorname{Bd} X)$ is a regular value of f, then $\deg(f, X, a)$ is an odd integer if and only if the cardinality of $f^{-1}(a)$ is an odd number.

Notice that if $f\colon X \to \boldsymbol{R}^n$ is a simplicial map and a point $a \in f(X) \setminus f(\operatorname{Bd} X)$ is a regular value of f, then there exists a closed subset $H \subset X$ such that $a \notin f(H)$. Then the map

$$g = f|\operatorname{Cl}(X \setminus H)$$

is of class C^1 and the point a is a regular value of g. The property (c) yields

$$\deg(f, X, a) = \deg(g, \operatorname{Cl}(X \setminus H), a).$$

But from the above and the property (g) we infer that:

(g′) If $f\colon X \to \boldsymbol{R}^n$ is a simplicial map and a point $a \in \boldsymbol{R}^n \setminus f(\operatorname{Bd} X)$ is a regular value of f, then $\deg(f, X, a)$ is an odd integer if and only if the cardinality of $f^{-1}(a)$ is an odd number.

THE BORSUK THEOREM (see [4]). *If $f\colon \operatorname{Bd} X \to \boldsymbol{R}^n \setminus \{0\}$ is a continuous antipodal map and $X \subset \boldsymbol{R}^n$ is a compact symmetric set such that $0 \in X$, then, for each continuous extension $f^*\colon X \to \boldsymbol{R}^n$ of the map f, $\deg(f^*, X, 0)$ is an odd integer.*

Proof. According to the property (f) and the Lemma we may assume that f^* is an antipodal map. The property (d) and the approximation theorem imply that there exists a simplicial antipodal map $f_\varepsilon\colon X \to \boldsymbol{R}^n$ such that 0 is a regular value of f_ε and, by (e),

$$\deg(f_\varepsilon, X, 0) = \deg(f^*, X, 0).$$

To see that $\deg(f_\varepsilon, X, 0)$ is an odd integer it suffices to observe, in view of the property (g′), that the cardinality of the set $f_\varepsilon^{-1}(0)$ is an odd number. But this is obvious because, since f_ε is an antipodal map and 0 is a regular value of f_ε, so $f_\varepsilon^{-1}(0)$ is a finite symmetric set which contains 0. It is clear that such a set has an odd number of elements.

THE BORSUK–ULAM THEOREM. *If g: $\operatorname{Bd} X \to \mathbf{R}^m \subset \mathbf{R}^n$, $n > m$, is a continuous map defined on the boundary of a compact symmetric set $X \subset \mathbf{R}^n$ such that $0 \in X$, then, for some point $x \in \operatorname{Bd} X$, $g(x) = g(-x)$.*

Proof. Suppose that, for each $x \in \operatorname{Bd} X$, $g(x) \neq g(-x)$. Define

$$f(x) := g(x) - g(-x).$$

The map f: $\operatorname{Bd} X \to \mathbf{R}^m \setminus \{0\}$ is antipodal. Hence, for an arbitrary continuous antipodal extension f^*: $X \to \mathbf{R}^m \subset \mathbf{R}^n$, $\deg(f^*, X, 0)$ is an odd integer (see the Lemma, property (f) and the Borsuk theorem). But from the property (a) we infer that

$$0 \in \operatorname{Int}_{\mathbf{R}^n} f^*(X) \subset \operatorname{Int}_{\mathbf{R}^n} \mathbf{R}^m = \emptyset,$$

a contradiction.

REFERENCES

[1] J. Dugundji and A. Granas, *Fixed Point Theory*, PWN, Warszawa 1982.

[2] P. J. Hilton and S. Wylie, *Homology Theory. An Introduction to Algebraic Topology*, University Press, Cambridge 1960.

[3] V. V. Ivanov, *An approximation of odd mappings* (in Russian), Mat. Zametki 38 (1985), pp. 417–421. English translation: Math. Notes 38 (1985), pp. 738–740.

[4] L. Nirenberg, *Topics in Nonlinear Functional Analysis*, Courant Institute of Mathematical Sciences, New York University 1974.

UNIWERSYTET ŚLĄSKI
INSTYTUT MATEMATYKI
UL. BANKOWA 14
40-007 KATOWICE, POLAND

Reçu par la Rédaction le 5.2.1988