COLLOQUIUM MATHEMATICUM

VOL. XL 1979 FASC. 2

CERTAIN PROPERTY OF THE RICCI TENSOR ON SASAKIAN MANIFOLDS

BY

ZBIGNIEW OLSZAK (WROCŁAW)

In the present paper we give some necessary and sufficient condition for η-parallelity of the Ricci tensor on Sasakian manifolds.

1. Preliminaries. Let M be a $(2n+1)$-dimensional Sasakian manifold [2]. Thus M is a Riemannian manifold (with not necessarily positive-definite metric tensor g_{ji}) together with tensor fields $F_j{}^i$ and v^i, which satisfy [2]

$$F_a{}^i F_j{}^a = -\delta_j^i + v_j v^i, \quad v_a v^a = 1, \quad F_a{}^i v^a = 0, \tag{1}$$

$$F_j{}^b F_k{}^a g_{ba} = g_{jk} - v_j v_k, \tag{2}$$

$$\nabla_k F_j{}^i = -g_{kj} v^i + \delta_k^i v_j, \tag{3}$$

$$\nabla_j v_k = F_{jk} \ (= -F_{kj}), \tag{4}$$

where ∇ denotes the covariant differentiation with respect to g_{ji} and $v_j = g_{ja} v^a$.

It is known (cf. [3]) that the relations

$$R_{kjia} v^a = v_k g_{ji} - v_j g_{ki}, \tag{5}$$

$$R_{ka} v^a = 2n v_k, \tag{6}$$

$$R_{ka} F_j{}^a + R_{bkja} F^{ba} = -(2n-1) F_{kj}, \tag{7}$$

$$R_{ka} F_j{}^a + R_{ja} F_k{}^a = 0, \tag{8}$$

$$R_{ba} F_k{}^b F_j{}^a = R_{kj} - 2n v_k v_j \tag{9}$$

can be obtained as consequences of (1)-(4), R_{lkji} and R_{kj} being the curvature and Ricci tensors of M, respectively, and $F^{ji} = F_a{}^i g^{ja}$.

The Ricci tensor of a Sasakian manifold is called *η-parallel* if it satisfies

$$(\nabla_k R_{ba}) F_j{}^b F_i{}^a = 0. \tag{10}$$

The notion of Ricci-η-parallelity for Sasakian manifolds was introduced by Kon in [1].

2. THEOREM. *The Ricci tensor of a Sasakian manifold is η-parallel if and only if it satisfies*

$$\nabla_k R_{ji} + \nabla_j R_{ik} + \nabla_i R_{kj} = 0. \tag{11}$$

Proof. Differentiating (9) covariantly and applying (3), (4), (6) and (1), we have

$$(\nabla_k R_{ba}) F_i^{\ b} F_j^{\ a} = \nabla_k R_{ij} - R_{ka} F_i^{\ a} v_j - R_{ka} F_j^{\ a} v_i - 2n F_{ki} v_j - 2n F_{kj} v_i. \tag{12}$$

If the Ricci tensor is η-parallel, then from (10) and (12) we get

$$\nabla_k R_{ji} = R_{ka} F_j^{\ a} v_i + R_{ka} F_i^{\ a} v_j + 2n F_{kj} v_i + 2n F_{ki} v_j.$$

Hence, using (8) and the skew-symmetry of F_{ji}, we obtain (11).

Conversely, assume that (11) is satisfied. Kon has proved ([1], Lemma 1.3) that any Sasakian manifold satisfies the equality

$$(\nabla_b R_{ka}) F_i^{\ b} F_j^{\ a} = \nabla_k R_{ji} - \nabla_j R_{ik} - R_{ka} F_i^{\ a} v_j - \\ - 2R_{ka} F_j^{\ a} v_i + 2n (F_{ik} v_j + 2F_{jk} v_i). \tag{13}$$

This, together with (11), gives

$$(\nabla_b R_{ka}) F_i^{\ b} F_j^{\ a} + 2\nabla_j R_{ik} + \nabla_i R_{kj} \\ = -R_{ka} F_i^{\ a} v_j - 2R_{ka} F_j^{\ a} v_i + 2n (F_{ik} v_j + 2F_{jk} v_i). \tag{14}$$

Relations (6) and (4) yield

$$(\nabla_k R_{ja}) v^a = -R_{ja} F_k^{\ a} + 2n F_{kj}. \tag{15}$$

By the transvection of (14) with $F_l^{\ i} F_m^{\ j}$ and the application of (1), (6) and (15), we find

$$\nabla_l R_{km} - (\nabla_a R_{km}) v^a v_l + 2(\nabla_b R_{ka}) F_m^{\ b} F_l^{\ a} + \\ + (\nabla_a R_{kb}) F_l^{\ a} F_m^{\ b} + R_{ka} F_l^{\ a} v_m - 2n F_{lk} v_m = 0. \tag{16}$$

Every Sasakian manifold satisfies the condition

$$v^a \nabla_a R_{km} = 0. \tag{17}$$

Indeed, (5) and (4) give

$$(\nabla_l R_{kjia}) v^a + R_{kjia} F_l^{\ a} = F_{lk} g_{ji} - F_{lj} g_{ki}. \tag{18}$$

Since $\nabla_l R_{aij}^{\ \ \ \ l} = \nabla_a R_{ij} - \nabla_i R_{aj}$, we obtain, contracting (18),

$$v^a \nabla_a R_{ij} = v^a \nabla_i R_{ja} - R_{bjia} F^{ba} - F_{ij},$$

which implies (17) by virtue of (15) and (7).

Considering (13) and (17) we find from (16)

$$3\nabla_k R_{lm} - \nabla_l R_{mk} - \nabla_m R_{kl} - 4R_{ka} F_l^{\ a} v_m - 4R_{ka} F_m^{\ a} v_l - 8n (F_{kl} v_m + F_{km} v_l) = 0,$$

whence, by (11), we get

$$\nabla_k R_{lm} - R_{ka} F_l{}^a v_m - R_{ka} F_m{}^a v_l - 2n(F_{kl} v_m + F_{km} v_l) = 0,$$

which in view of (12) completes the proof.

REFERENCES

[1] M. Kon, *Invariant submanifolds in Sasakian manifolds*, Mathematische Annalen 219 (1976), p. 277-290.

[2] T. Takahashi, *Sasakian manifold with pseudo-Riemannian metric*, Tôhoku Mathematical Journal 21 (1969), p. 271-290.

[3] K. Yano, *Anti-invariant submanifolds of a Sasakian manifold with vanishing contact Bochner curvature tensor* (to appear).

INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY, WROCŁAW

Reçu par la Rédaction le 21. 4. 1977