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IRREDUCIBLE CONTINUA OF HIGHER DIMENSION

BY

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Let $\mathscr{K}$ denote the class of all compact metric continua K such that there exists an upper semi-continuous decomposition G of a compact metric irreducible continuum M with each element of G homeomorphic to K and with decomposition space M/G an arc. In 1935, Knaster [3] showed that an arc is in $\mathscr{K}$. Recently, Mahavier [5] has given a construction from which it follows that, if the definition of $\mathscr{K}$ were changed by allowing M to be non-metrizable, then every compact metric continuum would be in $\mathscr{K}$. He raised the question of whether a simple closed curve is in $\mathscr{K}$. In [6] it is shown that the arc is the only connected finite 1-polyhedron in $\mathscr{K}$. In this paper it is shown that if n is a positive integer, there exists an n-dimensional continuum in $\mathscr{K}$, and that the Hilbert cube is in $\mathscr{K}$.

THEOREM 1. *The universal plane curve is in* $\mathscr{K}$.

Proof. Let S denote the universal Sierpiński curve in the plane (cf. [4], p. 275 and 276) bounded by the unit circle. Let T denote the Cantor middle-third set on $[0, 1]$. Let $c_1, c_2, c_3, \ldots$ denote the components of $[0, 1]\setminus T$. Let E denote the set of all end points of components of $[0, 1]\setminus T$. For each positive integer i, let t_i and r_i denote the left and right end points of c_i, respectively. Let $J_1, J_2, J_3, \ldots$ denote a sequence of simple closed curves such that (1), for each i, J_i is the boundary of a compact complementary domain of S, (2) if, for each i, $d(J_i)$ is the diameter of J_i, then $\lim\limits_{i\to\infty} d(J_i) = 0$, and (3) $\bigcup\limits_{i=1}^{\infty} (t_i \times J_i)$ is dense on $T\times S$.

Let G denote the collection to which g belongs if and only if (1), for some point t of $T\setminus E$, g is a point of $t\times S$, or (2), for some i, g is a point of $(t_i \cup r_i)\times(S\setminus J_i)$, or (3), for some i and some point P of J_i, g is the pair $(t_i\times P, r_i\times P)$. Let H denote the collection to which h belongs if and only if (1), for some t in $T\setminus E$, h is $t\times S$, or (2), for some i, h is equal to $[(t_i\cup r_i)\times S]/G$.

Clearly, $[(T\times S)/G]/H$ is an arc. $(T\times S)/G$ is irreducible since if A and B are points of $0\times S$ and $1\times S$, respectively, P is a point of $(T\times S)/G$

that is neither A nor B, and D is a domain containing P but neither A nor B, then there is a positive integer i such that $[(t_i \cup r_i) \times J_i]/G$ is a subset of D. $[(t_i \cup r_i) \times J_i]/G$ separates A from B in $(T \times S)/G$.

In order to verify that the elements of H are homeomorphic to S, first observe that, by construction, the elements h of H such that h is $t \times S$ for some t in $T \setminus E$ are homeomorphic to S. Suppose h is an element of H such that, for some positive integer i, h is $[(t_i \cup r_i) \times S]/G$. Then $[(t_i \cup r_i) \times J_i]/G$ is a simple closed curve in h which separates h into two mutually separated connected point sets k_1 and k_2. Let J denote a simple closed curve in S that does not intersect the boundary of any complementary domain of S. J separates S into two mutually separated connected sets h_1 and h_2. It is clear that the roles of $[(t_i \cup r_i) \times J_i]/G$, k_1 and k_2 in h are played by J, h_1 and h_2, respectively, in S.

Remarks. Anderson has shown in [1] that a continuum X which is the sum of any finite number of universal curves such that the intersection of any pair contains no isolated point of itself is a universal curve. By a construction similar to that used in Theorem 1, it can also be shown that the universal curve is also a member of $\mathscr{K}$.

THEOREM 2. *If n is a positive integer greater than* 1, *there is an n-dimensional continuum in $\mathscr{K}$.*

Proof. Let K denote an n-cell and T the Cantor middle-third set on $[0, 1]$. Again, let $c_1, c_2, c_3, \ldots$ denote the components of $[0, 1] \setminus T$, but arrange them so that the end points of $c_1, c_3, c_5, \ldots$ are dense in T and the end points of $c_2, c_4, c_6, \ldots$ are dense in T. For each i, let t_i and r_i denote the left and right end points of c_i, respectively. There exists a sequence $k_1, k_2, k_3, \ldots$ of n-cells such that (1), for each i, k_i is a subset of the interior of K, (2) if $d(k_i)$ denotes the diameter of k_i, then $\lim_{i\to\infty} d(k_i) = 0$, (3) if J_i denotes the boundary of k_i, then every point of $T \times K$ is a limit point of $\bigcup_{i=1}^{\infty}(t_i \times J_i)$. Let G denote the collection to which g belongs if and only if (1), for some t in T that is not an end point of a component of $[0, 1] \setminus T$, g is a point of $t \times K$, or (2), for some i, g is a point of $(t_i \cup r_i) \times \times (K \setminus k_i)$, or (3), for some i and some point P of k_i, g is the pair of points $(t_i \times P, r_i \times P)$. $(T \times K)/G$ is a compact n-dimensional continuum which is irreducible.

Let H denote the collection to which h belongs if and only if (1), for some t in T that is not an end point of a component of $[0, 1] \setminus T$, h is a point of $t \times (T \times K)/G$, or (2), for some i, h is a point of $(t_i \cup r_i) \times [(T \setminus z_i) \times \times K]/G$, where $z_i = 0$ if i is even and $z_i = 1$ if i is odd, or (3), for some i and some point P of $z_i \times K$, h is the pair of points $(t_i \times P, r_i \times P)$. Then $[T \times (T \times K)/G]/H$ is an irreducible continuum.

Let U denote the collection to which u belongs if and only if, for some t in T that is not an end point of a component of $[0,1]\setminus T$, u is $t\times(T\times K)/G$, or (2), for some i, u is the set of all points in $[(t_i\cup r_i)\times\times(T\times K)/G]/H$. U is an upper semi-continuous collection of mutually exclusive continua filling up $[T\times(T\times K)/G]/H$ such that U is an arc with respect to its elements, and the elements of U are all homeomorphic to $(T\times K)/G$.

THEOREM 3. *The Hilbert cube is in $\mathscr{K}$.*

Proof. Let X denote the Hilbert cube. Using Theorem 8.1 of [2], it is clear there exists a sequence $C_1, C_2, C_3, \ldots$ of subsets of X such that (1) $\bigcup_{i=1}^{\infty} C_i$ is dense in X, (2) $\lim_{i\to\infty} d(C_i) = 0$ and (3) if i is a positive integer, and X' denotes the upper semi-continuous decomposition of $(0\times X)\cup\cup(1\times X)$ such that P is in X' if and only if (1) P is a point of $(j\times X)\setminus\setminus(j\times C_i)$, $j = 0, 1$, or (2), for some point c of C_i, P is the pair $(0\times C, 1\times C)$, then X' is homemorphic to X. Now, by an argument similar to that of Theorem 1, it follows that X is in $\mathscr{K}$.

It is still not known if the 2-cell belongs to $\mathscr{K}$. (**P 832**)

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