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ON MIXED PRODUCT OF BOOLEAN ALGEBRAS

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0. Let $\langle L_i; \vee, \wedge\rangle$, $i = 1, 2, 3$, be three distributive lattices and let L be the cartesian product of the sets L_i. We define a new algebra—called *mixed product of distributive lattices* (1) — $\langle L; \mathrm{O}_1, \mathrm{O}_2, \mathrm{O}_3\rangle$ with three binary operations $\mathrm{O}_1, \mathrm{O}_2, \mathrm{O}_3$ on L as follows (2):

$$
\begin{aligned}
(x_1, x_2, x_3)\,\mathrm{O}_1(y_1, y_2, y_3) &= (x_1\vee y_1, x_2\wedge y_2, x_3\wedge y_3),\\
(x_1, x_2, x_3)\,\mathrm{O}_2(y_1, y_2, y_3) &= (x_1\wedge y_1, x_2\vee y_2, x_3\wedge y_3),\\
(x_1, x_2, x_3)\,\mathrm{O}_3(y_1, y_2, y_3) &= (x_1\wedge y_1, x_2\wedge y_2, x_3\vee y_3).
\end{aligned}
\tag{I}
$$

The class of all mixed products of distributive lattices is a generalization of the class of all distributive lattices. In fact, by taking L_2 and L_3 as one-element lattices one arrives at the distributive lattice L_1. The equational class of algebras generated by such algebras was described in [2].

The situation becomes more complicated if, instead of distributive lattices, we take Boolean algebras $\langle B_i; \vee, \wedge, '\rangle$, $i = 1, 2, 3$. We consider the set $B = B_1\times B_2\times B_3$ and want to introduce Boolean operations on B analogous to that of (I).

In Boolean algebras the complementation is connected with the two binary operations by the crucial de Morgan laws and it is but natural to have its analogue in our set up. Accordingly we define three binary operations $\mathrm{O}_1, \mathrm{O}_2, \mathrm{O}_3$ as above and three unary operations (one with each pair of binary operations) $x\to x^{(12)}$, $x\to x^{(13)}$, $x\to x^{(23)}$ as follows:

$$
\begin{aligned}
(x_1, x_2, x_3)^{(12)} &= (x_1', x_2', x_3),\\
(x_1, x_2, x_3)^{(13)} &= (x_1', x_2, x_3'),\\
(x_1, x_2, x_3)^{(23)} &= (x_1, x_2', x_3').
\end{aligned}
\tag{II}
$$

(1) This terminology was suggested to me by Professor G. Grätzer.

(2) B. H. Arnold examines in [1] similar algebras.

Definition. An algebra $\langle B; O_1, O_2, O_3, {}^{(12)}, {}^{(13)}, {}^{(23)}\rangle$ with three binary operations O_1, O_2, O_3 and three unary operations ${}^{(12)}, {}^{(13)}, {}^{(23)}$ defined by (I) and (II) is called a *mixed product of Boolean algebras* (MPBA for brevity).

It is clear that every Boolean algebra can be thought of as an MPBA. One can easily verify that the following identities ([3]) are true in an MPBA $\langle B; O_1, O_2, O_3, {}^{(12)}, {}^{(13)}, {}^{(23)}\rangle$:

(1) $\quad x O_i x = x, \qquad i = 1, 2, 3,$

(2) $\quad x O_i y = y O_i x, \qquad i = 1, 2, 3,$

(3) $\quad x O_i (y O_i z) = (x O_i y) O_i z, \qquad i = 1, 2, 3,$

(4) $\quad (x O_i y) O_j z = (x O_j z) O_i (y O_j z), \qquad i, j \in \{1, 2, 3\},$

(5) $\quad x O_i \big(x O_j (x O_k y)\big) = x, \qquad i \neq j \neq k \neq i,$

(6) $\quad (x O_i y)^{(ij)} = x^{(ij)} O_j y^{(ij)},$

(7) $\quad x O_i (x O_j x^{(ij)}) = x,$

(8) $\quad (x O_i x^{(ij)}) O_j (y O_i y^{(ij)}) = x O_i x^{(ij)} O_i y O_i y^{(ij)},$

(9) $\quad x^{(ik)} O_i [(x^{(ik)} O_j y^{(ik)}) O_k (x O_j y)^{(ik)}] = x^{(ik)},$

(10) $\quad (x^{(ij)})^{(ij)} = x,$

(11) $\quad (x^{(ij)})^{(ik)} = x^{(jk)}, \qquad j \neq k.$

So the identities (1) to (11) hold in the equational class Σ_B of algebras generated by all MPBA's (i.e., the smallest class of algebras closed under the formation of homomorphic images, subalgebras and direct products). The purpose of this paper is to prove the converse of this statement, namely, that Σ_B is characterized by the identities (1) to (11).

1. We need the following

LEMMA. *Let* $\mathfrak{A} = \langle X; O_1, O_2, O_3, {}^{(12)}, {}^{(13)}, {}^{(23)}\rangle$ *be an algebra in which the identities* (1) — (11) *hold. If* $x O_i y = x O_j y$ *holds in* $\mathfrak{A}$ *for some* $i \neq j$, *then we have*

(a) $x^{(ij)} = x$,

(b) $x^{(ik)} = x^{(jk)}$,

(c) $x O_i x^{ik} = y O_i y^{ik}$.

Proof. (a) We have

$$x = x O_i (x O_j x^{(ij)}) = x O_i (x O_i x^{(ij)}) = x O_i x^{(ij)},$$

([3]) Obviously, this list of axioms can be replaced by a smaller one.

where the first equality is by virtue of (7), and so

$$x^{(ij)} = (x \mathrm{O}_i x^{(ij)})^{(ij)} = x^{(ij)} \mathrm{O}_j x = x^{(ij)} \mathrm{O}_i x = x,$$

where the second equality follows by (10).

(b) follows immediately from (a) and (11).

(c) We have

$$\begin{aligned} x \mathrm{O}_i x^{(ik)} &= (x \mathrm{O}_i x^{(ik)}) \mathrm{O}_i \big((x \mathrm{O}_i x^{(ik)}) \mathrm{O}_j \big((x \mathrm{O}_i x^{(ik)}) \mathrm{O}_k (y \mathrm{O}_i y^{(ik)})\big)\big) \\ &= (x \mathrm{O}_i x^{(ik)}) \mathrm{O}_i \big((x \mathrm{O}_i x^{(ik)}) \mathrm{O}_k (y \mathrm{O}_i y^{(ik)})\big) \\ &= (x \mathrm{O}_i x^{(ik)}) \mathrm{O}_i \big((x \mathrm{O}_i x^{(ik)}) \mathrm{O}_i (y \mathrm{O}_i y^{(ik)})\big) \\ &= (x \mathrm{O}_i x^{(ik)}) \mathrm{O}_i (y \mathrm{O}_i y^{ik}) \end{aligned}$$

where the first equality follows by (5), the second by $\mathrm{O}_j \equiv \mathrm{O}_i$, and the third by (8). Since the last expression is symmetric in x and y, we have (c).

COROLLARY. *Under assumptions of the lemma the algebra $\mathfrak{A}$ becomes a Boolean algebra if we denote $\mathrm{O}_i = \wedge$, $\mathrm{O}_k = \vee$ and $x^{ik} = x'$.*

THEOREM. *An algebra $\mathfrak{A} = \langle X; \mathrm{O}_1, \mathrm{O}_2, \mathrm{O}_3, {}^{(12)}, {}^{(13)}, {}^{(23)} \rangle$ satisfies equations* (1)-(11) *if and only if it is a subalgebra of some MPBA.*

Proof. The "if" part being obvious we will prove only the "only if" part. Let us define three relations R_{12}, R_{13}, R_{23} in $\mathfrak{A}$ as follows:

$$a R_{ij} b \text{ iff } a \mathrm{O}_i (a \mathrm{O}_j b) = a \text{ and } b \mathrm{O}_i (b \mathrm{O}_j a) = b, \qquad (i, j) \in \{(1, 2), (1, 3), (2, 3)\}.$$

It follows from Lemma 5 of [2] that the R_{ij}'s are congruence relations in the reduct $\langle X; \mathrm{O}_1, \mathrm{O}_2, \mathrm{O}_3 \rangle$. We now prove that they are congruences in $\mathfrak{A}$ by verifying the substitution laws for the unary operations ${}^{(ij)}$. Let $a R_{ij} b$. Then

$$\begin{aligned} a^{(ij)} &= \big(a \mathrm{O}_i (a \mathrm{O}_j b)\big)^{(ij)} = a^{(ij)} \mathrm{O}_j (a \mathrm{O}_j b)^{(ij)} \\ &= a^{(ij)} \mathrm{O}_j (a^{(ij)} \mathrm{O}_i b^{(ij)}) = a^{(ij)} \mathrm{O}_i (a^{(ij)} \mathrm{O}_j b^{(ij)}), \end{aligned}$$

and, similarly,

$$b^{(ij)} = b^{(ij)} \mathrm{O}_i (b^{(ij)} \mathrm{O}_j a^{(ij)}),$$

which proves that $a^{(ij)} R_{ij} b^{(ij)}$.

Again,

$$\begin{aligned} a^{(ik)} \mathrm{O}_i (a^{(ik)} \mathrm{O}_j b^{(ik)}) &= a^{(ik)} \mathrm{O}_i \Big(\big(a \mathrm{O}_i (a \mathrm{O}_j b)\big)^{(ik)} \mathrm{O}_j \big(b \mathrm{O}_i (b \mathrm{O}_j a)^{(ik)}\big)\Big) \\ &= a^{(ik)} \mathrm{O}_i \Big(\big(a^{(ik)} \mathrm{O}_k (a \mathrm{O}_j b)^{(ik)}\big) \mathrm{O}_j \big(b^{(ik)} \mathrm{O}_k (b \mathrm{O}_j a)^{(ik)}\big)\Big) \\ &= a^{(ik)} \mathrm{O}_i \big((a^{(ik)} \mathrm{O}_j b^{(ik)}) \mathrm{O}_k (a \mathrm{O}_j b)^{(ik)}\big) = a^{(ik)}, \end{aligned}$$

where the first equality follows by a $R_{ij} b$, the second by (6), the third (2) and (4), and the last by (9).

Similarly, $b^{(ik)}O_i(b^{(ik)}O_j a^{(ik)}) = b^{(ik)}$, and hence $a^{(ik)}R_{ij}b^{(ik)}$.

From lemma 7 of [2] we have $(xO_iy)R_{ij}(xO_jy)$. Thus by the corollary of the previous lemma, the algebra $\mathfrak{A}/R_{ij}$ becomes a Boolean algebra if we denote, in $\mathfrak{A}/R_{ij}$, $xO_ky = x\vee y$, $xO_iy = xO_jy = x\wedge y$ and $x^{(ik)} = x'$. For $a\in x$, let $[a]_{ij}$ denote the equivalence class (of the relation R_{ij}) to which a belongs. It follows from Lemma 6 of [2] that the mapping

$$a \to \langle [a]_{12}, [a]_{13}, [a]_{23}\rangle$$

is one-one and thus the above correspondence sets an isomorphism between $\mathfrak{A}$ and a subalgebra of $M = \mathfrak{A}/R_{12}\times\mathfrak{A}/R_{13}\times\mathfrak{A}/R_{23}$. This completes the proof of the theorem.

Remark. In virtue of the above theorem one can omit the words "homomorphic images" and "direct products" in the definition of Σ_B. In other words, the class of all subalgebras of MPBA's is automatically closed with respect to the formation of homomorphic images and direct products.

REFERENCES

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