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ANOTHER GEOMETRIC PROOF OF SUPERCOMPACTNESS OF COMPACT METRIC SPACES

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In 1967, during the Berlin Symposium, J. de Groot [4] proposed to call a space *supercompact* if it has a binary subbase for closed subsets (see Section 2). De Groot conjectured that each compact metric space is supercompact. This conjecture was confirmed by Strok and Szymański [9], and the proof was based on the Freudenthal theorem on expansions of compact metric spaces into an inverse sequence of polyhedra with piecewise linear maps. The proof was rather complicated, and later Mills [5] and van Douwen [1] gave simpler proofs which were purely set-theoretical.

The aim of this note is to show that the geometric approach based on Freudenthal's theorem leads also to a simple proof of that theorem.

1. Preliminaries. Polyhedra will be considered with fixed triangulations consisting of closed simplices.

Let K be a compact polyhedron. If s is a simplex of K, then $b(s)$ denotes the *barycentrum* of s, i.e. a point whose all barycentric coordinates with respect to the vertices are equal. By $\mathrm{sd}\, K$ we denote the barycentric subdivision of K, and by $\mathrm{sd}^n K$ the n-th barycentric subdivision of K, defined inductively as follows:

$$\mathrm{sd}^0 K = K \quad \text{and} \quad \mathrm{sd}^{n+1} K = \mathrm{sd}(\mathrm{sd}^n K).$$

By $\mathrm{st}_K a$ we denote the (closed) *star of the vertex* a in K, i.e. the union of simplices of K having a as a vertex.

If a polyhedron L consists of some simplices of K, then L is said to be a *subpolyhedron* of K.

A simplex with the vertices $a_0, \ldots, a_n$ will be denoted by $\langle a_0, \ldots, a_n \rangle$.

A subpolyhedron L of K is said to be a *full subpolyhedron* of K if each simplex of K having vertices in L is a simplex of L. Easily,

LEMMA 1 (Spanier [7]; 3.3, Lemma 10). *If L is a subpolyhedron of K, then* $\mathrm{sd}\, L$ *is a full subpolyhedron of* $\mathrm{sd}\, K$.

A polyhedron K is said to be *complete* if each set of vertices of K such that each pair of vertices from that set spans a simplex is the set of vertices of a simplex of K.

LEMMA 2. *If K is a polyhedron, then* sd K *is a complete polyhedron.*

Proof. Let $b(s_1), \ldots, b(s_n)$ be vertices of sd K such that each pair of these vertices spans a simplex in sd K. Then s_i is a face of s_j for all i and j or conversely. Therefore, $b(s_1), \ldots, b(s_n)$ span a simplex in sd L.

2. Binary collections in polyhedra. A collection B of sets is said to be *binary* if every subcollection of B, each two members of which intersect, has non-empty intersection.

LEMMA 3. *Let K be a complete polyhedron and let B be a binary collection of full subpolyhedra of K. Then the collection*

$$(*) \qquad B \cup \{\mathrm{st}_{\mathrm{sd}K}\, a\colon a \text{ is a vertex of } K\}$$

is binary.

Proof. Let $\{L_1, \ldots, L_m, \mathrm{st}_{\mathrm{sd}K}\, a_0, \ldots, \mathrm{st}_{\mathrm{sd}K}\, a_n\}$ be a subcollection of the collection $(*)$ such that every two members of that collection intersect. Since $\mathrm{st}_{\mathrm{sd}K}\, a_i \cap \mathrm{st}_{\mathrm{sd}K}\, a_j \neq \emptyset$, a_i and a_j are vertices of a simplex of K for each pair of vertices appearing in symbols of sets of that subcollection. Since K is complete, $a_0, \ldots, a_n$ are vertices of a simplex of L and

$$b(\langle a_0, \ldots, a_n\rangle) \in \mathrm{st}_{\mathrm{sd}K}\, a_0 \cap \ldots \cap \mathrm{st}_{\mathrm{sd}K}\, a_n.$$

From $L_k \cap \mathrm{st}_{\mathrm{sd}K}\, a_i \neq \emptyset$ it follows that a_i is a vertex of L_k. Hence the simplex $\langle a_0, \ldots, a_n\rangle$ belongs to L_k, L_k being full in K, and, in consequence, $b(\langle a_0, \ldots, a_n\rangle) \in L_k$. Therefore, we obtain

$$L_1 \cap \ldots \cap L_m \cap \mathrm{st}_{\mathrm{sd}K}\, a_0 \cap \ldots \cap \mathrm{st}_{\mathrm{sd}K}\, a_n \neq \emptyset.$$

Let us note a general property of binary collections (not necessarily in polyhedra):

LEMMA 4. *If $f\colon X \to Y$ is a map onto Y and B is a binary collection in Y, then $f^{-1}(B) = \{f^{-1}(Z)\colon Z \in B\}$ is a binary collection.*

3. Proof of the theorem. We begin with recalling the theorem about inverse expansions of compact metric spaces into polyhedra.

THEOREM (Freudenthal [3]). *If X is a metric compact space, then X is the inverse limit of a sequence of maps of polyhedra*

$$(**) \qquad K_1 \xleftarrow{p_1^2} K_2 \leftarrow \ldots \leftarrow K_n \xleftarrow{p_n^{n+1}} K_{n+1} \leftarrow \ldots$$

such that

(1) p_n^{n+1} *are onto*;

(2) *there exist positive integers* r_n, $r_n \geq 2$, *such that* p_n^{n+1} *are simplicial if they are considered as maps* $p_n^{n+1}\colon K_{n+1} \to \operatorname{sd}^{r_n} K_n$;

(3) $\lim_{n\to\infty} \max\{\operatorname{diam} p_n^{-1}(s)\colon$ *s is simplex in* $K_n\} = 0$, p_n *being the projection from the limit* X *onto* K_n.

For a proof of Freudenthal's theorem, see Pasynkov [6] or Problem 1.13.G(a) in [2] (cf. also Strok [8]).

THEOREM (Strok and Szymański [9]). *Compact metric spaces are supercompact, i.e. compact metric spaces have binary subbases for closed subsets.*

Proof. Let X be a compact metric space. According to Freudenthal's theorem, X is the inverse limit of sequence (**) of polyhedra having properties (1)–(3).

By (3), the collections

$$B_n = \{p_n^{-1}(\operatorname{st}_{\operatorname{sd}^{r_n}K_n} a)\colon a \text{ is a vertex of } \operatorname{sd}^{r_n-1} K_n\}, \quad n = 1, 2, \ldots,$$

are finite closed coverings of X such that the maximum diameter of members of B_n tends to 0 as $n \to \infty$. The union $B = \bigcup\{B_n\colon n = 1, 2, \ldots\}$ is a subbase for closed subsets of X.

We shall show that B is a binary collection.

Observe that, by Lemmas 1 and 2, the assumptions of Lemma 3 are satisfied. Therefore, applying successively Lemma 3, we see that the collection

$$D_n = \{(p_j^n)^{-1}(st_{\operatorname{sd}^{r_j}K_j} a)\colon a \text{ is a vertex of } sd^{r_j-1} K_j \text{ and } j \leq n\}$$

is binary for any given n. Since $p_n\colon X \to K_n$ are onto, the collection $p_n^{-1}(D_n)$ is binary by Lemma 4. From the equality $p_j = p_j^n \circ p_n$ for $j \leq n$, we obtain

$$p_n^{-1}(D_n) = \bigcup\{B_j\colon j \leq n\}.$$

Thus the collection B, being the union of an increasing sequence of binary collections $p_n^{-1}(D_n)$ of closed subsets of a compact space, is binary.

4. Remark. If, in particular, the space is a polyhedron K, then the union of the collections

$$B_n = \{\operatorname{st}_{\operatorname{sd}^{2n}K} a\colon a \text{ is a vertex of } \operatorname{sd}^{2n-1} K\}, \quad n = 1, 2, \ldots,$$

forms a binary subbase for closed subsets on K. However, it can be proved that the union of the collections $\{\operatorname{st}_{\operatorname{sd}^{n+1}K} a\colon a$ is a vertex of $\operatorname{sd}^n K\}$, $n = 1, 2, \ldots$, also forms such a subbase.

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